Exercise 2 – Topics on the Border of Computation and Economics

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1. (Non-Arrovian domains) In this question you are asked to design social choice functions in the non-quasi-linear case (i.e. no money exists). The social choice function gets as input the preferences of all participants and returns a single alternative. In each case find a social choice function that satisfies the following two conditions (and prove it): (A) Truthfulness: no participant can ever gain from mis-stating his preferences (B) Condorcet property: if there exists an alternative that is prefered to every other alternative by more than half of the participants, then this alternative will be chosen (note that this implies unanimity and rules out dictators). This will be possible despite the Gibbard-Satterthwaite theorem since the domains of preferences will be restricted. For simplicity you may assume the the number of participants is odd.

   (a) (Single-Peaked Preferences) In the land of Oz all political debate is centered around a single political question: what fraction of the bricks used for road construction should be colored yellow. Each citizen i in Oz has an ideology: his desired fraction, $0 \leq x_i \leq 1$. Citizen i's preference between any two alternatives y and z is completely determined by the distance from x_i: i prefers y to z if $|x_i - y| < |x_i - z|$. The social choice should be a point $0 \leq x \leq 1$.

   (b) (Spatial Preferences on a Tree) The Road system in the Land of Etz has the structure of a tree in the combinatorial sense – V is the set of vertices=junctions and E is the set of edges=roads. Each edge e has a length $l_e > 0$. The current debate concerns which junction to locate the newly acquired candy machine. Each citizen i has a preferred junction v_i, and prefers u to w if $d(v_i, u) < d(v_i, w)$, where $d(x, y)$ is the distance in the tree, i.e. the sum of lengths of the edges on the unique path in the tree between x and y. The social choice should be a vertex $v \in V$.

2. (Voting rule manipulation) This question concerns voting protocols that aggregate the preferences of n voters over m candidates and choose a single candidate. The constructive manipulation problem for a given voting rule

recieves as input the preference orders of $n-1$ voters, and a candidate c and answers whether the remaining voter can choose an order on the candidates such that the candidate chosen by the voting rule will be c.

(a) Prove that the **CUP** voting protocol allows constructive manipulation in polynomial time.

(b) Find a voting protocol (other than the one shown in class) for which the constructive manipulation problem is NP-hard (and prove it, of course.) Suggestion (maybe not easiest): **STV**.

3. (VCG examples) Describe a $truthful$ mechanism for each of the following scenarios by deriving a *simple* description for the VCG prices in each case. The description of the VCG prices should be natural from the point of view of the scenario.

   (a) (Bilateral Trade) A "seller" has an item and values this item at s_i. A potential buyer values this item at b_i. They should trade iff $s_i < b_i$. How much should each player pay or receive?

   (b) (Multi-unit auction with unit demand) There are k identical items for sale (k is publically known). n bidders are competing for them, and bidder i values an item at v_i – each wants only a single item. The items should be allocated to those with highest values. How much should each winner pay?

   (c) (General Multi-unit auction) There are k identical items for sale (k is publically known), and n bidders are competing for them. Bidder i wants at most q_i items, and values each item at v_i. I.e. player i's value for any q items is given by $min(q, q_i) \cdot v_i$. The items should be allocated to those with highest values, but no more than q_i units to bidder i. How much should each winner pay?

4. (Approximation mechanism for procurement auctions)

   This question deals with procurement auctions among *single minded bidders*. I.e. a single buyer needs to buy a set of m items from n possible suppliers. Each supplier i can provide a single set of items S_i for price v_i. The buyers needs to buy all items, and aims to minimize the total price paid. Denote by OPT the total price paid by the optimal (cheapest) procurement.

   (a) Prove that computing OPT is NP-complete.

   (b) Prove that for every set R of items there exists i such that $v_i/|R \cap S_i| \leq OPT/|R|$ (Hint: Look first at the set R of all items).

   (c) Prove that the following process finds a $(1 + lnm)$-approximation to the optimal procurement: Let R_1 be the initial set of m items; for each i, find the set S_i that minimizes $v_i/|R_i \cap S_i|$, let player i win (i.e. supply S_i), and let $R_{i+1} = R_i - S_i$, until $R_{k+1} = \emptyset$. Prove that

$\sum_{i=1}^{k} v_i \leq OPT \cdot (1 + \ln m)$. (Hint: you may want to prove first the following simple combinatorial lemma: Let $R_1 \supset R_2 \supset ... \supset R_{k+1} = \emptyset$ be a decreasing family of finite subsets, where $|R_1| = m$. Then $\sum_{i=1}^{k} |R_i - R_{i+1}|/|R_i| \leq 1 + \ln m$.)

(d) Prove the the process above is monotone in the following sense: if a player i wins with a bid (v_i, S_i), then he will also win with a bid (v', S') for every $v' \leq v_i$ and $S' \supseteq S_i$ (with all other bids fixed).

(e) Describe a polynomial time algorithm to compute the maximum value v_i^* that will win with a set S_i (this value depends, of course, on the other bids v_{-i} and S_{-i}).

(f) Deduce an incentive compatible polynomial time $1+lnm$-approximation mechanism for procurement auctions among single minded bidders.