Exercise 1 – Issues on the Border of Economics and Computation

December 13, 2005

1. (Hoteling Model) In the land of Oz all political debate is centered around a single political question: what fraction of the bricks used for road construction should be colored yellow. Each citizen in Oz has an ideology: his desired fraction, $0 \leq x \leq 1$. Citizens are uniformly distributed in $[0, 1]$. Each political party i announces its platform: a single real number $0 \leq s_i \leq 1$. Each citizen votes for the closest party, i.e. for the party i that minimizes $|x - s_i|$. E.g. if there are two parties, with $s_1 = 0.2$, and $s_2 = 0.6$, then 40% of the vote would go to party 1 (all those with $x < 0.4$). (In case of equality, you may assume that the vote is evenly split between the 2 (or more) closest parties – this can happen in particular if two parties choose the same platform.) Parties have no ideology: the only consideration that political parties have is to maximize the number of citizens who vote for them. Model the parties' choice of platform as a game and:

   (a) Find a Nash equilibrium when there are exactly 2 parties.

   (b) Find a Nash equilibrium when there are exactly 4 parties.

   (c) Prove that there exists no pure Nash equilibrium when there are exactly 3 parties.

2. (randomized game-tree evaluation) Analyze exactly the running time of the randomized game-tree evaluation algorithm shown in class on the balanced alternating and/or tree with n leaves. The analysis should give the correct running time, upto $\Theta()$ notation, which will be better than the $n^{log_3 4}$ upper bound shown in class.

3. (Perceptrons) This question deals with boolean functions over a fixed input length n, from $\{0, 1\}^n$ to $\{0, 1\}$.

   (a) Let B be a family of boolean functions that for every $b \in B$ also contains its negation $b^c(x) = not(b(x))$. f is called a B-threshold function if there exist constants $w_1, ...w_m \geq 0$, and functions $b_1...b_m \in B$ such that for all $x \in \{0, 1\}^n$, $f(x) = 1$ iff $\sum_i w_i b_i(x) > \sum_i w_i/2$

(we are idenfiying here the boolean 0 and 1 with the real numbers 0 and 1). Prove that f is *not* a B-thershold function iff there exists a distribution D on $\{0,1\}^n$ such that for all $b \in B$, $Pr_D[b(x) = f(x)] = 1/2$. (Hint: min-max theorem.)

(b) Let B_* be the set of functions on $\{0,1\}^n$ that depend on at most $n-1$ of the input bits. Prove that the parity function on n bits is not a B_*-threshold function. (Hint: uniform distribution.)

(c) Fix B as in the first item, and a function f that is a B-threshold function. For a two functions b and f, and a probability distribution D on $\{0,1\}^n$, we define the bias of b on f with respect to D as $\delta_D(b,f) = |Pr_D[b(x) = f(x)] - 1/2|$; we define the bias of the whole set B on f with respect to D as $\delta_D(f) = max_{b \in B} \delta_D(b,f)$; and we define the bias of B on f as $\delta(f) = \inf_D \delta_D(f)$, where the infimum is over all distributions D on $\{0,1\}^n$. We define the weight of f to be the minimum sum of weights in a representation of f as a D-threshold: $W(f) = \inf \sum_i w_i$, where the infimum is taken over all representations $b_1...b_n \in B$ and natural numbers $w_1...w_n$ such that for all $x \in \{0,1\}^n$, $f(x) = 1$ iff $\sum_i w_i b_i(x) > \sum_i w_i/2$. (This is the "complexity" of f when represented as a majority of functions from B.) Prove that δ nearly completely characterizes W: $1/(2\delta(f)) \le W(f) \le O(n/\delta(f)^2)$. (Hint: for the upper bound, use Chernoff with the following parameters: if one has a coin that gives heads with probability $1/2 + \delta$, and flips it $2n/\delta^2$ times, then the probability that less than half of the times turn up heads is less than 2^{-n}.)

4. (Load Balancing Game) This question deals with the load balancing game with exactly two identical processors. (I.e. there are n players, where each i has a job of weight w_i that can be run on any one of the two availabe processors. The cost for player i is exactly $\sum w_j$, where the sum is over all j's that are assigned to the same processor as player i.)

   (a) What is the exact price of Anarchy for the measure of the maximum cost of a player?

   (b) Show that a Nash equilibrium can be found in polynomial time (even when the w_i's are given in binary – i.e. the running time can not be polynomial in w_i but rather only in the length in bits of w_i). (Comment: finding the exact optimum is NP-complete.)