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Local and Global Clique Numbers

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Consider a graph G with the property that any set of p vertices in G contains a q-clique. Fairly tight lower bounds are proved on the clique number of G as a function of p, q and the number of vertices in G. © 1994 Academic Press, Inc.

1. THE PROBLEM

Let G be a graph and $p \geqslant q \geqslant 2$ integers. We say that G has the (p, q)-property if for every set P of p vertices in G, the graph induced by P contains a clique of cardinality q. If every graph of order $\geqslant n \geqslant p$ with the (p, q)-property necessarily also has the (n, s)-property, we say that (p, q) implies (n, s) and write $(p, q) \rightarrow (n, s)$. Thus $\rightarrow$ defines a partial order relation on ordered pairs of integers. This relation is the subject of the present paper, and our main objective is to determine, or estimate, for given p, q, and n the least possible clique number of a graph of order n with the (p, q) property.

This specific question is an instance of the meta-problem: How does the local structure of a graph affect its global properties? A number of existing results in graph theory fall in the same category. Consider, for example, the classical question (Erdős [4]) asking how large the chromatic number $\chi(G)$ of an n-vertex graph G may be, given its girth girth(G). In our framework this question reads: provided that the $\frac{1}{2}$girth(G)-neighbourhood of every vertex in G is a tree, how large can (the global parameter) $\chi(G)$ be? In this specific question "local" is taken in the metric sense; i.e., assumptions are made on the nature of sets of small diameter in G. In the present article "local" is taken in the sense of cardinality. Assumptions are made on the structure of subgraphs induced by small sets of vertices (they

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must contain a sizable clique) and questions are posed concerning the whole graph (its clique number $\omega(G)$).

Many questions in this general class of problems come up in the theory of distributed processing (Linial [8]; Linial and Saks [9]), and many are still expected to come up from both computational and purely combinatorial considerations. As it turns out, the study of the $\rightarrow$ relation has to do with Ramsey type problems, with Turan's theorem, as well as with some aspects of matching theory. The earliest mention of a problem of this type that we are aware of, is in Erdős and Rogers [7], who, in the present terminology, studied the behavior of the smallest x such that $(p, q) \rightarrow (x, q+1)$. A recent paper of Bollobás and Hind [3] further develops this line of study. In the present terminology, that paper concentrates on estimating the least n for which $(p, q) \rightarrow (n, s)$, where p is large and q, s are small. Their main results are: for $s > q \geqslant 3$ there is a constant $c \leqslant 1$, with

$$(p, q) \rightarrow (cp^{s-q+1}, s)$$

while for $q \geqslant 3$ and p big enough,

$$(p, q) \not\rightarrow (p^{1+q/(q^2-2)-\varepsilon}, q+1).$$

For purposes of comparison, the reader should note that unlike [3] the main focus of the present paper is on small p, q and a large n. Local and global conditions on coloring numbers are studied in Erdős [5] as well.

In combinatorial geometry, the notion of (p, q)-properties first (implicitly) occurs in Helly's fundamental work in convexity and is explicitly defined and studied by Hadwiger and Debruner. Further developments were obtained by numerous investigators in combinatorial and computational geometry. For an overview and some recent advances, see [6, 12, 1, 10].

In what follows it will be convenient to have a notation for *exact implication*: $(p, q) \Rightarrow (n, s)$ if $(p, q) \rightarrow (n, s)$ but $(p, q) \not\rightarrow (n-1, s)$. Our considerations bring about also the notion of *weak implication*: Property (p, q) is said to k-weakly imply (n, s), denoted $(p, q) \xrightarrow{k} (n, s)$, if from every G satisfying condition (p, q) it is possible to delete k vertices, so the remaining graph satisfies (n, s).

For a graph G, and for q not exceeding $\omega(G)$, the clique number of G, let $w_q(G)$ be the least p such that condition (p, q) holds in G. For instance, $w_2(G) = \alpha(G) + 1$, i.e., one more than the largest cardinality of an independent set in G. The characterization of sequences w_q which arise in this way, includes our problem as a special case, and seems interesting. However, in this paper we confine ourselves to the aforementioned more restricted problem.

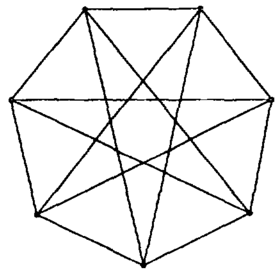

FIG. 1. A graph with the (5, 3) property lacking a four-clique.

2. AN EXAMPLE: $(5, 3) \Rightarrow (8, 4)$

Since a complete determination of the arrow relation determines the values of all Ramsey numbers, this problem is certainly very hard. As an illustration we show one small case which we managed to resolve completely, viz., $(5, 3) \Rightarrow (8, 4)$.

The counterexample for $(5, 3) \nrightarrow (7, 4)$ is given in Fig. 1. It remains to show that a graph G of order 8 with the (5, 3) property must contain a four-clique.

Let G have order 8 and satisfy the (5, 3) property. If G may be covered by two cliques it obviously contains a four-clique. Otherwise, the complementary graph $\bar{G}$ is not bipartite and must contain an odd circuit. Let g be the length of the shortest odd circuit in $\bar{G}$. If $g = 3$, the remaining five vertices must form a clique, for any two nonadjacent vertices can be added to the anti-triangle for a five-set with no three-clique, contradicting the (5, 3) condition. Also, $g = 5$ is out of the question, because a pentagon fails to have property (5, 3). The last case, $g = 7$, requires a simple case analysis, according to the set of neighbours of the remaining vertex. It is not hard to find a four-clique in this case as well.

3. STATEMENT OF THE THEOREM

The theorem has three parts, which provide the best bounds known to us for the ranges, where $(p-1)/(q-1)$ is, respectively, less than, equal and bigger than two. The negative result in the third part yields nothing when $3q - 4 > p$. However, in this case the negative result in the second part can be employed. The reader should note that, obviously, x which is defined by $(p, q) \Rightarrow (n, x)$ increases with q and n and decreases with p. Also, it is easy to verify that for all $p \geqslant q \geqslant k+2$, $(p, q) \Rightarrow (p-k, q-k)$. In what follows we concentrate mainly on $n \geqslant p$.

THEOREM 3.1. *The following implications hold:*

- *For* $p \leqslant 2q-2$ *and all* $n \geqslant p-q-2$

$$(p, q) \Rightarrow (n, n-p+q).$$

- *For* $p = 2q-1$ ($q > 3$ *in the positive implication*) *and all* $n \geqslant p$

$$(p, q) \rightarrow (n, n^{1-2/(q-1)+o(1)}) \qquad \text{but} \qquad (p, q) \not\rightarrow (n, n^{1-1/(8q-4)}).$$

- *For all* $s \geqslant 2$,

$$(p, q) \rightarrow (R(r, s) + p - 1, s) \rightarrow (cs^{r-1}, s), \qquad r = \left\lceil \frac{p}{q-1} \right\rceil,$$

where $R(r, s)$ *is the off-diagonal Ramsey number and* c *is an absolute constant. On the other hand,*

$$(p, q) \not\rightarrow (s^{(T-1)/(p-2)+o(1)}, s),$$

where T *is Turan's number,* $T = (p-b)(p+b-q+1)/2(q-1)$, *with* $b \equiv p \pmod{q-1}$. *For large* r *the bounds become*

$$(p, q) \rightarrow (cs^{r-1}, s), \qquad (p, q) \not\rightarrow (s^{r/2+o(1)}, s).$$

All $o(1)$ *terms are taken for fixed* p, q *and large* s.

Note that T appearing in the theorem is the largest number of edges in a p-vertex graph with no independent set of q vertices, as shown by Turan's theorem. The proof of the three parts of the theorem is the subject of the following sections.

4. THE CASE $2q-2 \geqslant p$

Let G be a graph of order n with the (p, q) property, where $2q-2 \geqslant p$. We wish to estimate the size of a maximum clique in G.

A simple argument shows that G has a clique of size $n-2(p-q)$. Indeed, the (p, q) property implies that no matching in $\bar{G}$ has more than $(p-q)$ edges. For any maximal matching in $\bar{G}$, the vertices outside it form the desired clique. A stronger and, in fact, best possible result is given by the following theorem.

THEOREM 4.1. *A graph of order* n *has property* (p, q) *for* $2q-2 \geqslant p$ *if and only if it has a clique of size* $(n-p+q)$.

The sufficiently part is obvious. For the necessity part we need two preparatory lemmas.

Let A, B be disjoint sets of vertices in a graph H. We say that A is *matchable* to B if H contains a matching M of size $|A|$, every edge of which has one endpoint in A and the other in B. A is *anti-matchable* to B if A is matchable to B in $\bar{H}$.

LEMMA 4.1. *A clique Q of a graph G is of maximal cardinality if and only if every clique in $G - Q$ is anti-matchable to Q.*

Proof. (Sufficiency) Suppose that every clique in $G - Q$ is anti-matchable to the clique Q. Let Q_1 be any clique in G. We show $|Q| \geqslant |Q_1|$. By assumption, the clique $Q_1 - Q$ is anti-matchable to Q. Since there are no anti-edges between $Q_1 - Q$ and $Q_1 \cap Q$, the set $Q_1 - Q$ is anti-matchable to $Q - Q_1$, whence

$$|Q_1| = |Q_1 \cap Q| + |Q_1 - Q| \leqslant |Q_1 \cap Q| + |Q - Q_1| = |Q|$$

as claimed.

(Necessity) Let Q be a maximum size clique in G, Q_1 a clique in G so $|Q| \geqslant |Q_1|$. Let A be any subset of $Q_1 - Q$ and let B be the set of members in Q having an anti-neighbour in A. Now $Q \cup A - B$ is a clique in G, and so its cardinality cannot exceed $|Q|$. Therefore $|B| \geqslant |A|$, and by Hall's marriage theorem the proof is complete. ∎

LEMMA 4.2. *Let $G = (A, B, E)$ be a bipartite graph, and let $B' \subseteq B$, be matchable to A and have the largest cardinality under this condition. Then any subset of A which is matchable to B is matchable to B' as well.*

Proof. Let M be a matching of B' to A in G and let $K \subseteq A$ be matchable to B. We show that K is matchable to B' as well. Let M_K be a matching of K into B. The symmetric difference $M \triangle M_K$ is a union of disjoint alternating paths and cycles. The alternating paths come in three types:

1. Those starting and ending in A.
2. Those starting and ending in B.
3. Those with one end in A and one in B.

Note that in paths of the first two classes, one terminal edge is from M and the other from M_K. Paths of the third class must start and end with edges from M, for a path which starts and ends with an M_K edge is an augmenting path for M which is, however, of maximum cardinality.

Let Z be the matching obtained from M_K upon exchanging its edges in paths of type (2) with the corresponding M-edges. We claim that Z

matches K into B'. Like M_K, the matching Z matches K and we only have to show that all B-vertices in Z are, in fact, in B'. A Z-edge whose B-vertex meets also an M-edge is certainly in B', so the only edges in Z that need to examined are those which are either first (or last) in their path and where the path starts (ends) in B. In particular, edges in Z which are in an alternating cycle are taken care of. Type (1) paths have no endpoint in B, so they create no problem. The switching performed on type (2) paths makes the terminal Z-edge of such paths identical to an M-edge, so incidence with B' is guaranteed. Finally, as we pointed out, paths of type (3) do not have Z-edges at their ends. This concludes the argument. ∎

We return to the proof of the theorem. Let G be a graph of order $\geqslant p$, Q a maximum-size clique of G, and suppose by contradiction that $|G| - |Q| > p - q$. We claim that G fails to have the (p, q) property (recall that $2q - 2 \geqslant p \geqslant q$ is assumed).

By the assumption, $|Q| \geqslant q$ must hold. Let U be any set of $p - q + 1$ vertices in $G - Q$, and let G_1 be the subgraph of G induced on $U \cup Q$.

Q is a clique of maximum size in G_1 as well, so Lemma 4.1 implies that every clique contained in U is anti-matchable to Q. Consider the bipartite graph $H = (U, Q, F)$, where F is the set of anti-edges of G with one vertex in U and one in Q. By Lemma 4.2 there is a subset R of Q of cardinality at most $|U| = p - q + 1$ such that each subset of U matchable in H may also be matched to R. But, since $p \leqslant 2q - 2$, $|Q| > q - 1 \geqslant p - q + 1$, so there is $Q_1 \subset Q$, $|Q_1| = q - 1$ with the same property as R.

Let G_2 be be the subgraph of G spanned by $Q_1 \cup U$, it has order $(p - q + 1) + (q - 1) = p$. Since every clique of U is anti-matchable to Q_1, an application of the sufficiency part of Lemma 4.1 shows that Q_1 is a maximum size clique in G_2. But $|Q_1| = q - 1$ and we found an induced subgraph of G of p vertices without a q-clique. ∎

Remark. The statement of this section strengthens a corollary due to Wegner (see [12]) that for any p, q such that $p \leqslant 2q - 2$, a graph with (p, q) property can be covered by $p - q + 1$ cliques.

5. The Case $(2q - 1, q)$

Let G be a graph of size n with the $(2q - 1, q)$ property. We wish to estimate the size of a maximum clique in G.

First we note that $(2q - 1, q)$ property either implies the property $\bigwedge_{i=2}^{q} (2i - 1, i)$ or weakly implies the property $(2k - 2, k)$ for some $k \leqslant q$. Indeed, assume that G does not have the $(2i - 1, i)$ property for some $i \leqslant q$. This means that G has an induced subgraph G' on $2i - 1$ vertices, where the size of a maximum clique is at most $i - 1$. Let $G_1 = G - G'$. Since G has property $(2q-1, q)$, G_1 must have the $(2q-1-2i+1, q-i+1)=(2q-2i, q-i+1)$ property.

In what follows we pursue only the first possibility and derive certain bounds in that case. Note that the second possibility has already been studied in the previous section and shown to yield bounds much better than those obtained for the first case, which we now proceed to study. Thus no generality is lost in considering only the first alternative.

An equivalent, more convenient statement of the property $\bigwedge_{i=2}^{q}(2i-1, i)$ is that G contains no odd anti-circuits of length $\leqslant 2q-1$. Clearly, the existence of such an anti-circuit would contradict some $(2i-1, i)$ property. On the other hand, the absence of such circuits implies that $\bar{G}$ is locally bipartite, so any subset of $2q$ vertices in G can be covered by two cliques, which implies the $\bigwedge_{i=2}^{q}(2i-1, i)$ property.

To establish a lower bound, we strengthen the assumption (incurring, perhaps, a loss in the coefficient of $1/g$ in the next proposition) and suppose that G contains no short cycles (even or odd) at all, i.e., $\mathrm{girth}(G) \geqslant 2q-1$. We need the fact that the there exist graphs with a large girth and no large independent sets. It is a classical result, of course (Erdős [4]), that graphs of large girth and large *chromatic number* exist. A more careful reading of the proof (e.g., in Bollobás [2, p. 256]) reveals that the result we need is proved as well.

PROPOSITION 5.1. *For any integers n, g there exists a graph H of order n, with $\mathrm{girth}(\bar{H}) \geqslant g$, and $\omega(H) \leqslant n^{1-1/4g}$. ($\omega(H)$ is a size of a maximum clique in H.)*

The existence of G whose complement has the above-mentioned properties thus implies:

THEOREM 5.1. $(2k-1, k) \not\rightarrow (n, n^{1-1/(8k-4)})$.

We turn now to the upper bound employing the following recent result of Linial and Saks [9]: The vertices of any graph of order n can be colored with at most $\log n/\log(1-p)^{-1}$ colors so that any monochromatic connected subgraph has diameter $\leqslant 2\log n/\log p^{-1}$. The parameter p can vary anywhere between zero and one. Note that the said coloring is not necessarily a proper coloring.

For $p=n^{-2/(k-1)}$ the above result implies that $V(G)$ can be colored with $n^{2/(k-1)}\log n$ colors so that each monochromatic connected component has diameter at most $k-1$. Since $\bar{G}$ has no odd circuits of length $\leqslant 2k-1$, each monochromatic connected component is in fact bipartite. Indeed, consider, for contradiction, a shortest odd circuit C contained in a monochromatic connected component K, and denote its length by $2l+1$. By assumption $l \geqslant k$. Let x_1, x_2 be vertices on C whose distance along C is l. Since $\mathrm{diam}(K) \leqslant k-1$, there is a path P connecting x_1 to x_2 of length at most

$k-1$. Now P, together with one of the two arcs of C connecting x_1 to x_2, constitutes an odd circuit, shorter than C, a contradiction.

It follows that each of the original (generally improper) color classes constitutes a two-colorable subgraph. Dedicate two separate colors for each such class to obtain a *proper* coloring of $\bar{G}$ by $2 \log n\, n^{2/(k-1)}$ colors. The largest color class in this coloring must have cardinality at least $(1/2 \log n)\, n^{1-2/(k-1)}$, and this set is a clique in G. Therefore

THEOREM 5.2. $(2k-1, k) \rightarrow (n, 1/2 \log n\, n^{1-2/(k-1)})$.

6. GENERAL CASE

In this section we establish general bounds on x defined by $(p, q) \Rightarrow (x, s)$. Property $(p, 2)$ for a graph means that it contains no independent set of cardinality p. Thus $(p, 2)$ implies $(g(s), s)$ if and only if each graph H of order $g(s)$ has either an independent set of size p or a clique of size s. But this is exactly the definition of the Ramsey number $R(p, s)$, whence $(p, 2) \Rightarrow (R(p, s), s)$. Using a well-known bound for Ramsey numbers, we obtain

$$g(s) = R(p, s) \leqslant \binom{s+p-2}{p-1} = O(s^{p-1})$$

for a fixed p.

We turn to the general case. Note that the (p, q) property $(p-1)$-weakly implies property $(r, 2)$, where $r = \lceil p/(q-1) \rceil$. Indeed, suppose that G has the (p, q) property and contains an independent set of size r. Delete this set and proceed to look for another such set. This cannot go on for $q-1$ times: Since $r(q-1) \geqslant p$, the union of these independent sets must contain a q-clique which is, of course, impossible. Therefore, this union is a set of fewer than p vertices, whose removal eliminates all independent sets of size r from the graph.

Combining the two remarks, we obtain an upper bound,

$$(p, q) \xrightarrow{p-1} (r, 2),$$

whence, for some constant c,

$$(p, q) \rightarrow (R(r, s) + p - 1, s) \rightarrow (cs^{r-1}, s).$$

We turn now to the lower bound. Our aim is to show that given p and q the (p, q) property does not imply property $(s^{(T-1)/(p-2)+o(1)}, s)$. In other words, there exists a graph G satisfying property (p, q) of order $N = s^{(T-1)/(p-2)+o(1)}$ with no s-clique. Here T is the largest number of edges a graph of order p can have, if it contains no clique of size q.

Turan's theorem shows $T=(p-b)(p+b-q+1)/2(q-1)$, where $b\equiv p \pmod{q-1}$. The existence of such G is shown using a variant of the Lovász local lemma, much in the spirit of Spencer [11].

Consider a random graph on N vertices with edge probability $(1-\varepsilon)$, where $\frac{1}{2}\geqslant\varepsilon\geqslant 0$. For a set P of p vertices, let A_P be the event that the subgraph induced by P contains no q-clique. Also, B_S is the event that S, a set of s vertices, forms a clique.

An easy check shows that $\Pr(B_S)=(1-\varepsilon)^{\binom{s}{2}}<e^{-\varepsilon\binom{s}{2}}$. To estimate $\Pr(A_P)$ we argue as follows: Let $\mathscr{F}$ be the set of all labeled graphs on p vertices containing no q-clique. Then

$$\Pr(A_P)=\sum_{J\in\mathscr{F}}(1-\varepsilon)^{e(J)}\,\varepsilon^{\bar{e}(J)},$$

where $\bar{e}(J)$ stands for the number of edges of the complement graph $\bar{J}$. Turan's theorem implies $\bar{e}(J)\geqslant T$. According to our choice, $\varepsilon<\frac{1}{2}$, whence

$$\Pr(A_P)\leqslant c_{p,q}(1-\varepsilon)^{\binom{p}{2}-T}\varepsilon^T\leqslant c_{p,q}\varepsilon^T,$$

$c_{p,q}$ being the cardinality of $\mathscr{F}$.

In order to apply the Lovász local lemma we need to set up a dependency graph of the events. A_P and $A_{P'}$ are independent except if P and P' have at least two vertices in common. The same is true for A_P and B_S or B_S and $B_{S'}$. Consider, then, the graph whose vertices correspond to all A_P with P ranging over all p-sets, and all B_S, where S is an s-set and where two such vertices are adjacent iff the corresponding sets have at least two vertices in common. Each A_P has $\sum_{j\geqslant 2}\binom{p}{j}\binom{N}{p-j}=(1+o(1))\binom{p}{2}\binom{N}{p-2}$ neighbouring $A_{P'}$. Similarly, each B_S is adjacent to $\sum_{j\geqslant 2}\binom{s}{j}\binom{N}{p-j}=(1+o(1))\binom{s}{2}\binom{N}{p-2}$ of the A_P. (Both $o(1)$ terms are for fixed p and large N.) Each A_P or B_S is adjacent to at most $\binom{N}{n}$ $B_{S'}$ (i.e., all of them).

To apply the local lemma (the statement used here is that of [11, p. 62]; it is not the strongest version) associate the same y with every A_P and the same z with every B_S. Suppose there exist positive y, z and $0<\varepsilon<\frac{1}{2}$ such that

$$c_{p,q}\,y\varepsilon^T<1;\qquad ze^{-\varepsilon\binom{s}{2}}<1;$$

$$\ln y>z\binom{N}{s}e^{-\varepsilon\binom{n}{2}}+(1+o(1))\,y\binom{p}{2}\binom{N}{p-2}\varepsilon^T;$$

$$\ln z>z\binom{N}{s}e^{-\varepsilon\binom{s}{2}}+(1+o(1))\,y\binom{s}{2}\binom{N}{p-2}\varepsilon^T.$$

Then there is an order N graph where none of the events A_P ($|P| = p$) and B_S ($|S| = s$) hold; i.e., this graph has property (p, q) but contains no s-clique. How big can N be for the above inequalities to be feasible? An elementary (although easy to become lost in) analysis shows that the optimum is achieved at

$$\varepsilon = s^{-1+o(1)}; \qquad N = s^{(T-1)/(p-2)+o(1)}; \qquad e = e^{s^{1+o(1)}}; \qquad y = 1 + o(1).$$

Thus $(p, q) \nrightarrow (s^{(T-1)/(p-2)+o(1)}, s)$ as claimed.

Open Questions

Our first comment is that the main problem of this article may be viewed as a question about set systems, rather than about graphs. Namely, a q-graph H (a hypergraph all edges of which have cardinality q) has property $[p]$ if every set of p vertices contains an edge of H. A set of vertices Y is said to be two-exhausted if every two-element subset of Y is contained in some edge of H. Our question, then, is for given $n \geqslant p \geqslant q \geqslant 2$ to find the largest s, such that any q-graph H on n vertices having property $[p]$ must have a two-exhausted set of s vertices.

It should be obvious what it means for a set of vertices to be d-exhausted by H, and one may ask for the largest s such that every n-vertex q-graph with property $[p]$ must d-exhaust some set of s vertices.

Our second comment is that the present theory can be extended to infinite sets as well. The relevance of the Erdős–Rado arrow relation should be obvious. We do not elaborate on this topic.

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