Asymptotically Almost Every 2r-Regular Graph Has an Internal Partition

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Abstract

An *internal partition* of a graph $G = (V, E)$ is a partitioning of V into two parts such that every vertex has at least a half of its neighbors on its own side. We prove that for every positive integer r, asymptotically almost every $2r$-regular graph has an internal partition. Whereas previous results in this area apply only to a small fraction of all $2r$-regular graphs, ours applies to almost all of them.

Keywords Graph partitions · Internal partition · Satisfactory partition · Asymptotic · Vertex degree · Optimization

1 Introduction

1.1 Notations

Let x be a vertex in a graph $G = (V, E)$. We denote its neighbor set by $N(x)$ and its degree by $d(x)$. The number of neighbors that x has in a subset $A \subseteq V$ is denoted by $d_A(x) = |N(x) \cap A|$. We denote by $\mathbb{N}_{\geq k}$ the set $\{k, k + 1, \ldots\}$. For a vertex x and a set A we use the shorthand notation $A \cup x$ and $A \backslash x$ rather than $A \cup \{x\}$ and $A \backslash \{x\}$ respectively.

Our main concern is with *partitions* of V into two parts $\langle A, B \rangle$. We denote by $e(A, B)$ the *cut size* of this partition, i.e., the number of edges $e = (x, y)$ with $x \in A$ and $y \in B$.

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1.2 Internal Partitions

Let $G = (V, E)$ be a simple graph. A partition $\langle A, B \rangle$ of V is called *external* if every vertex has at least as many neighbors on the other side as it has on its own side. Since the partition that maximizes the cut size $e(A, B)$ is external, every graph has an external partition. Likewise, in an *internal partition* every vertex has at least as many neighbors on its own side as on the other side. This requirement is clearly satisfied by the trivial partition $\langle \emptyset, V \rangle$, but we insist on a *non-trivial* internal partition where both parts are non-empty.

As it turns out, it is worthwhile to consider a more general class of problems:

Definition 1.1 Let $G = (V, E)$ be a simple graph and let $a, b : V \rightarrow \mathbb{N}$ be two functions. We say that a partition $\langle A, B \rangle$ of V is (a, b)-*internal*, if:

1. $d_A(x) \geq a(x)$ for every $x \in A$, and
2. $d_B(x) \geq b(x)$ for every $x \in B$.

In these terms, an *internal partition* is synonymous with a $\left(\lceil \frac{d(x)}{2} \rceil, \lceil \frac{d(x)}{2} \rceil \right)$-internal partition.

A vertex in A (resp. B) for which condition (1) (resp. (2)) holds, is said to be *satisfied*. Otherwise, we say that it is *unsatisfied*. Clearly, a partition is internal if and only if all vertices are satisfied.

The problem whether an internal partition exists, and the algorithmic problem of efficiently finding such partitions, appear in the literature under various names: *Decomposition under Degree Constraints* [23], *Cohesive Subsets* [19], *q-internal Partition* [2] and *Satisfactory Graph Partition* [13]. A related subject is *Defensive Alliance Partition Number* of a graph. A short survey about *alliance* and *friendly* partitions, and the link to the internal partitions, can be found in [11]. A generalization (and a survey) of *Alliance Partitions* can be found in [12].

Note that $\langle A, B \rangle$ is an (a, b)-internal partition if and only if $d_B(x) \leq d(x) - a(x)$ for every $x \in A$ and $d_A(y) \leq d(y) - b(y)$ for every $y \in B$. This perspective suggests that an (a, b)-internal partition can be viewed as a strong *isoperimetric inequality* for G. The isoperimetric number (or Cheeger constant) of a graph is defined as $\min \frac{e(A,B)}{|A|}$ over all partitions $\langle A, B \rangle$ where $|A| \leq |B|$. Namely, it is determined by a set A with at most a half of the vertices which minimizes the *average* $\frac{1}{|A|} \sum_{x \in A} d_B(x)$, whereas an (a, b)-internal partition calls for an upper bound $d_B(x)$ on *each* vertex in A. We return to this perspective in the open problems section.

While every graph has an external partition, there are simple examples of graphs with no *internal* partitions, e.g., cliques or complete bipartite graphs in which at least one part has odd cardinality. On the other hand, it is not easy to find large sparse graphs that have no internal partition. Also, as some of the theorems mentioned next show, *nearly* internal partitions (the exact meaning of this is clarified below) always exist. Stiebitz [21], responding to a problem of Thomassen [23], made a breakthrough in this area. His results and later work by others in a similar vein are summarized in the following theorem.

Theorem 1.2 *Let $G = (V, E)$ be a graph and let $a, b : V \rightarrow \mathbb{N}$. Each of the following conditions implies the existence of an (a, b)-internal partition:*

1. [21] $d(x) \geq a(x) + b(x) + 1$ *for every $x \in V$.*
2. [16] $d(x) \geq a(x) + b(x)$ *and $a(x), b(x) \geq 1$ for every $x \in V$ and G is triangle-free.*
3. [10] $d(x) \geq a(x) + b(x) - 1$ *and $a(x), b(x) \geq 2$ for every $x \in V$ and $girth(G) \geq 5$.*
4. [18] $d(x) \geq a(x) + b(x) - 1$ *and $a(x), b(x) \geq 1$ for every $x \in V$ and G is C_4-free.*

Note that the last three conditions asymptotically hold only for a small fraction of all regular graphs.

Further work in this area falls into several main categories:

- **The decision problem: Does a given graph have an (a, b)-internal partition?** This issue is investigated in a series of papers by Bazgan et al., surveyed in [7]. Each existence statement in Theorem 1.2 comes with a polynomial time algorithm to find the promised partition. For large a and b the problem seems to become difficult. For example, a theorem of Chvátal [8] says that the case $a(x) = b(x) = d(x) - 1$ is NP-hard for graphs in which all vertex degrees are 3 and 4.
- **Generalizations and variations:** Gerber and Kobler [13] introduced vertex- and edge-weighted versions of the problem and showed that these are NP-complete. Recent works by Ban [3] and by Schweser and Stiebitz [22] extend Theorem 1.2 to edge-weighted graphs. It is NP-hard to decide the existence of an internal *bisection* i.e., an internal partition with $|A| = |B|$ [6]. There is literature concerning approximate internal partitions, partitions into more than two parts etc. See [4,5,7,14,15].
- **Sufficient conditions:** In [15,20] one finds several sufficient conditions for the existence of an internal partition in general graphs, and more specific conditions for line-graphs and triangle-free graphs.
- **Necessary conditions:** It is proved in [20] that there is no forbidden subgraph characterization for the existence or non-existence of an internal partition. Given a graph's edge-density it is possible to bound the cardinality of the parts of an internal partition, if it exists [13].
- **Regular graphs:** For $d = 3, 4$ the only d-regular graphs with no internal partition are $K_{3,3}$, K_4, K_5 [20]. As shown by Ban and Linial [2], every 6-regular graph with 12 or more vertices has an internal partition. The case of 5-regular graphs remains open.

The most comprehensive survey of the subject of which we know is [7].

2 The Theorem: 2r-Regular Graphs

As mentioned, the repertoire of known graphs with no internal partitions seems rather limited, and for $d \in \{3, 4, 6\}$ there are only finitely many d-regular graphs with no internal partition. This has led to the following conjecture [2]:

Conjecture 2.1 *For every d only finitely many d-regular graphs have no internal partitions.*

The main theorem of this paper is a weaker version of this conjecture, namely, an asymptotic result for even d.

We say that a graph G is 4-*sparse* if every set of four vertices spans at most four edges (i.e., G contains no 4-clique and has no diamond-graph subgraph). We prove:

Theorem 2.2 *Let $G = (V, E)$ be a graph and let $a, b : V \to \mathbb{N}_{\geq 1}$ be such that $d(x) \geq a(x) + b(x)$ for every vertex $x \in V$. If G is 4-sparse then it has an (a, b)-internal partition.*

Corollary 2.3 *If G is a 4-sparse graph and all its vertices have even degrees, then G has an internal partition.*

We note the following simple fact about random regular graphs:

Proposition 2.4 *For every $d \geq 3$ asymptotically almost every d-regular graph is 4-sparse.*

Proof We work with the configuration model of random n-vertex d-regular graphs. Let X be the random variable that counts the number of sets of four vertices that span five or six edges. Then $\mathbb{E}(X) \leq O(n^4) \cdot \frac{(dn-11)!!}{(dn-1)!!} = O(\frac{1}{n})$. $\qquad\qquad$ □

We can now conclude the theorem in the title of this paper, namely:

Corollary 2.5 *Asymptotically almost every $2r$-regular graph has an internal partition.*

Before we prove Theorem 2.2, we need to introduce several definitions.

Definition 2.6 Let $G = (V, E)$ be a graph, and let $f : V \to \mathbb{N}$.

- We say that $A \subseteq V$ is f-*internal* if $d_A(x) \geq f(x)$ for every $x \in A$.
- We say that $A \subseteq V$ is f-*degenerate* if every non-empty subset $K \subseteq A$ has a vertex $x \in K$ such that $d_K(x) \leq f(x)$.

Remark 2.7 Clearly, a set is not $(a-1)$-degenerate if and only if it contains a non-empty a-internal subset.

Proof of Theorem 2.2

For the proof of Theorem 2.2 it is clearly sufficient to consider the case where $d(x) = a(x) + b(x)$ for every $x \in V$. Our proof is based on the methodology initiated by Stiebitz [21]. We assume throughout that G is 4-sparse.

For a function $f : V \to \mathbb{N}$ and $S \subseteq V$, we denote $f(S) := \sum_{x \in S} f(x)$. We next associate a potential with every vertex partition $\langle A, B \rangle$ of V. Namely:

$$w(A, B) = a(B) + b(A) - e(A, B)$$

A good feature of the functional w is the way it changes when some vertex switches sides. E.g., it is easy to verify that if $\langle A', B' \rangle = \langle A \cup x, B \backslash x \rangle$, then

$$\Delta w = w(A', B') - w(A, B) = 2\left(b(x) - d_B(x)\right). \tag{1}$$

Namely, the change in w as a single vertex changes sides is the local improvement towards internal partition. Therefore w is a good measure for how internal the partition at hand is. In this view, we formulate the rest of the proof and the algorithmic part of this paper as optimization problems. We define $\mathcal{F}$ as the family of all non-empty sets $A \subsetneq V$ that are a-degenerate, but not $(a - 1)$-degenerate. We first note that $\mathcal{F}$ is non-empty, i.e., such sets A exist. E.g., let A be an inclusion-minimal a-internal subset. By Remark 2.7 it is not $(a - 1)$-degenerate. If A is not a-degenerate, then by the same argument it has a non-empty $(a + 1)$-internal subset, and therefore also a proper a-internal subset contrary to the assumed minimality.

Proposition 2.8 *For every A in $\mathcal{F}$ there holds $|A| \geq 2$.*

Proof Since $A \in \mathcal{F}$, it is not $(a - 1)$-degenerate, and therefore it has a non-empty a-internal subset A'. But then, for every $x \in A'$ there holds $|A| \geq d_A(x) + 1 \geq d_{A'}(x) + 1 \geq a(x) + 1 \geq 2$. □

Let A be a member of $\mathcal{F}$ that maximizes $w(A, V \backslash A)$ and minimizes $|A|$ under this condition.[1] Namely, we assume:

$$\text{For every } A' \in \mathcal{F} \text{ there holds } w(A) \geq w(A'), \text{ and if}$$
$$w(A) = w(A'), \text{ then } |A'| \geq |A|. \tag{2}$$

Proposition 2.9 *The set A thus chosen is a-internal.*

Proof Suppose there is $v \in A$ with $d_A(v) \leq a(v) - 1$. The set $A \backslash v$ is in $\mathcal{F}$, since it is non-empty (by Proposition 2.8), a-degenerate and not $(a - 1)$-degenerate. In addition, $\Delta w \geq 2$, contradicting the maximality of w. □

We denote throughout $B = V \backslash A$.

Proposition 2.10 *If B is not $(b-1)$-degenerate, then G has an (a, b)-internal partition.*

Proof If B is not $(b - 1)$-degenerate, then it contains a non-empty b-internal subset. Let $B' \subset B$ be such a subset that is inclusion-maximal. If $B' = B$, then $\langle A, B \rangle$ is an (a, b)-internal partition. By the maximality of B', every vertex $x \in B \backslash B'$ satisfies $d_B(x) \leq b(x) - 1$ and therefore $d_{V \backslash B}(x) \geq a(x) + 1$. It follows that $\langle V \backslash B, B \rangle$ is an (a, b)-internal partition. □

Lemma 2.11 *B is not $(b - 1)$-degenerate.*

Together with Propositions 2.10 and 2.9 this lemma implies that $\langle A, B \rangle$ is indeed an (a, b)-internal partition.

The proof of Lemma 2.11 is by contradiction and is comprised of several propositions. We assume B is $(b - 1)$-degenerate, and show that this contradicts our choice of A as in Condition (2): either the maximality of w or the minimality of $|A|$.

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[1] In Sect. 3 we present a polynomial time algorithm that finds a partition whose existence is stated in Theorem 2.2. For algorithmic purposes we cannot consider a *globally* optimal $A \in \mathcal{F}$ as described here, but as we show below, a properly chosen locally optimal version of such A will do.

Consider the vertices of "low internal-degree" in A and in B. Denote

$$C = \{v \in A \mid d_A(v) = a(v)\} \quad \text{and} \quad D = \{v \in B \mid d_B(v) \le b(v) - 1\}.$$

Note that $C \ne \emptyset$ since A is non-empty and a-degenerate. In addition, $D \ne \emptyset$, since B is assumed to be $(b-1)$-degenerate and by the definition of $\mathcal{F}$ it is non-empty.

Equation (1) implies that if any vertex is moved from C to B, then w stays unchanged. Also, if any vertex is moved from D to A, then w grows at least by 2.

Proposition 2.12 *For every $A' \subseteq A$, if A' is a-internal, then $C \subseteq A'$.*

Proof Suppose there is $y \in C \backslash A'$, and we show that $A \backslash y$ is a "better" member of $\mathcal{F}$ than A. Clearly, $A \backslash y$ is a-degenerate, but it is also not $(a-1)$-degenerate since it contains the a-internal subset A'. In addition $w(A \backslash y, B \cup y) = w(A, B)$, contradicting the minimality of $|A|$. □

Proposition 2.13 *For every $x \in D$ there is a subset $A_x \subseteq A$ such that $A_x \cup x$ is $(a+1)$-internal.*

Proof As mentioned, moving x from D to A yields $\Delta w \ge 2$. Also, $A \cup x$ is clearly not $(a-1)$-degenerate. Consequently, by the maximality of $w(A, V \backslash A)$, $A \cup x$ cannot be a-degenerate, whence it must contain an $(a+1)$-internal subset. This $(a+1)$-internal subset must contain x, as claimed. □

Note that for every $x \in D$, such a set A_x is necessarily a-internal, and hence, according to Proposition 2.12, $C \subseteq A_x$.

Proposition 2.14 *Every vertex in C is adjacent to every vertex in D.*

Proof Consider some $x \in D$ and $y \in C$. Then $d_A(y) = a(y)$. But y also belongs to the $(a+1)$-internal set $A_x \cup x$, so that $d_{A \cup x}(y) = a(y) + 1$. The conclusion follows. □

Proposition 2.15 *Every vertex in C has a neighbor in C.*

Proof We know already that $C \ne \emptyset$. Consider some $y \in C$. Clearly $A \backslash y$ is a-degenerate. Also $w(A \backslash y, B \cup y) = w(A, B)$, therefore, by the minimality of $|A|$, the set $A \backslash y$ must be $(a-1)$-degenerate. In particular there is $z \in A$ such that $d_{A \backslash y}(z) \le a(z) - 1$, whereas $d_A(z) \ge a(z)$, which implies that z is in C and a neighbor of y. □

As mentioned, D is not empty, so fix $x \in D$. If some vertex in C has two or more neighbors in C, then by Proposition 2.14 this yields a set of four vertices with five or more edges, contrary to the assumption that G is 4-sparse. Together with Proposition 2.15 this implies that the subgraph of G induced by C is a perfect matching.

We claim that $D = \{x\}$ is a singleton. Otherwise if $x_1, x_2 \in D$, then together with an edge in C, this is a set of four vertices and five or more edges, contrary to the assumed 4-sparsity of G.

Proposition 2.16 *C is a proper subset of A.*

Proof We show that if $A = C$, then $B \backslash x$ is b-internal, contrary to the assumption that B is $(b-1)$-degenerate. Every vertex v in $B \backslash D$, i.e., every $v \in B$ other than x satisfies $d_B(v) \geq b(v)$. Thus, if v is not a neighbor of x, then also $d_{B \backslash x}(v) \geq b(v)$, as claimed.

On the other hand, if $v \in B$ is a neighbor of x, then it has no neighbors in $A = C$ due to the assumed 4-sparsity of G. But then $d_B(v) = d(v) \geq b(v) + 1$, so that $d_{B \backslash x}(v) \geq b(v)$, as claimed. □

Let $y, z \in C$ be neighbors and let $A' := A \backslash \{y, z\}$. We claim:

Claim 2.17 $A' \in \mathcal{F}$.

Proof By Proposition 2.16, A' is not empty. Clearly A' is a-degenerate, being a subset of $A \in \mathcal{F}$. We show next that $d_{A \backslash \{y,z\}}(v) \geq a(v)$ for every $v \in A'$.

Consider first a vertex $v \in C \backslash \{y, z\}$. Since yz is an edge in the perfect matching spanned by C, there holds $d_{A'}(v) = d_A(v) \geq a(v)$, because v is adjacent to neither y nor z.

Consider next some vertex $v \in A \backslash C$. It satisfies $d_A(v) \geq a(v) + 1$ by definition of C. Also, since G is 4-sparse, v can have at most one neighbor in $\{y, z\}$. It follows that $d_{A'}(u) \geq a(u)$ for every $u \in A'$, so A' is not $(a-1)$-degenerate, as claimed. □

To conclude the proof of Lemma 2.11 and with it of Theorem 2.2, we note that applying Equation (1) sequentially to y and then to z, yields $\Delta w = 2$, i.e., $w(A') > w(A)$, contrary to the defining condition of A. □

3 Computational and Experimental Results

3.1 Algorithmic Realization

As mentioned, to the best of our knowledge, all previous existence proofs of (a, b)-internal partitions translate into polynomial-time algorithms (under specific assumptions). This applies as well to Theorem 2.2 and its proof. The observation that we need is that the argument goes through even if rather than work with the globally optimal set $A \in \mathcal{F}$ we settle for a local optimum, as follows. For two sets $A, A' \subseteq V$ we denote $A' \sim A$ if their symmetric difference is small, $|A \oplus A'| \leq 2$.

We observe that our proof of 2.2 holds even if we relax our requirements and seek an $A \in \mathcal{F}$ such that condition (2) holds for all $A \in \mathcal{F}$ with $A' \sim A$. This yields the following polynomial-time algorithm that finds an (a, b)-internal partition of G.

The validity of the output and the polynomial-time computability of each step is proven as follows:

- *Initialization* of A can be done in polynomial time, since (i) for any $\varphi : V \to \mathbb{N}$ it takes linear time to check whether a given $S \subseteq V$ is φ-degenerate, (ii) G is $(a+b)$-internal, (iii) if S is not $(\varphi + 1)$-degenerate, then $S \backslash x$ is not φ-degenerate for every $x \in S$ and (iv) if, moreover, $d(x) \leq \varphi(x)$, then $S \backslash x$ is not $(\varphi + 1)$-degenerate.
- For a given $A \in \mathcal{F}$, finding all sets $A' \sim A$ such that A' is also in $\mathcal{F}$ can also be done in polynomial time.

Algorithm 1: Internal Partition

Input: $G(V, E)$: A 4-sparse graph, such that $d(x) = a(x) + b(x)$ for every $x \in V$

Output: An (a, b)-internal partition $\langle A, V \backslash A \rangle$

1 **Initialization.** Find $A \in \mathcal{F}$. Define $B \leftarrow V \backslash A$

2 **if** B *is not* $(b - 1)$-*degenerate* **then**

3   Find b-internal inclusion-maximal subset $B' \subset B$

4   Update $A \leftarrow V \backslash B'$

5   Return $\langle A, V \backslash A \rangle$;

6 **else**

7      $C \leftarrow \{v \in A | d_A(v) = a(v)\}$ and $D \leftarrow \{v \in B | d_B(v) \leq b(v)\}$

8   **while** $D \neq \emptyset$ **do**

9   **if** *there is* $A' \sim A$ *s.t.* $A' \in \mathcal{F}$ *and* $w(A) < w(A')$ **then**

10   Update $A \leftarrow A'$ and $B \leftarrow V \backslash A$

11   **else**

12   Find $A' \sim A$ s.t. $A' \in \mathcal{F}$, $w(A) = w(A')$ and $|A'| < |A|$.

13   Update $A \leftarrow A'$ and $B \leftarrow V \backslash A$

14   Update $C \leftarrow \{v \in A | d_A(v) = a(v)\}$ and $D \leftarrow \{v \in B | d_B(v) \leq b(v)\}$

15   Return $\langle A, V \backslash A \rangle$;

- The condition in line (2) is checked in polynomial time, and the existence in line (3) and the correctness of output is proven in Proposition 2.10.
- The if-else dichotomy in lines (9) and (11) and the existence of A' in line (12) is proven in Claim 2.11 and in the propositions following it.
- The algorithm *terminates*, since in initialization, $\Delta w > 0$, and at each iteration line (10) increases w or line (13) decreases $|A|$ while keeping $\Delta w \geq 0$. Termination is proved by induction on lexicographically-ordered pairs $(w, |A|)$.
- $A \neq \emptyset$ remains a-internal and $B \neq \emptyset$ throughout, and upon termination $D = \emptyset$ which means B is b-internal.

3.2 Improving Previous Experimental Results

In [13] some experimental results are presented. They apply a heuristic algorithm in an attempt to find a $\left(\lceil \frac{d(x)}{2} \rceil, \lceil \frac{d(x)}{2} \rceil \right)$-internal partition in random graphs. Their algorithm starts from a random partition and at each iteration minimizes $f(A, B) = \sum_{v \in A} (d_A(v) - d_B(v))^+ + \sum_{v \in B} (d_B(v) - d_A(v))^+$ where the minimum is taken over all partitions which were achieved by switching an unsatisfied vertex. The process can terminate with either an internal partition or a trivial partition. It can also loop indefinitely. In the latter two cases, they restart the process.

We have experimented with a similar algorithm. The main change is that we consider only near-bisections $\langle A, V \backslash A \rangle$, and insist that $\left| |A| - \frac{|V|}{2} \right| \leq c(n)$ for $c(n) = \log_d(n)$. When this condition is violated, we move a random vertex from the big part to the small. This algorithm may either output an internal partition or loop forever. However, in extensive simulations with random d-regular graphs ($30 \leq n \leq 10,000$ and $4 \leq d \leq \min(50, \frac{n}{2})$) the algorithm has always found an internal partition in fewer than $5n$ iterations. It may be that this phenomenon is barely affected by the choice of c.

4 Discussion and Some Open Problems

4.1 Even vs. Odd Degrees

The problem of internal partition for d-regular graphs seems harder for *odd* degree d; for an even d, a random vertex in a random partition is biased towards being satisfied, while for an odd d, the probability is exactly one half. Thus, perhaps, it is no coincidence that we were able to settle the case of $2r$-regular graphs, but are still unable to prove the analogous statement for $2r + 1$ regular graphs. E.g., while Conjecture 2.1 is already known for 4- and 6-regular graphs, it is still open in the 5-regular case.

4.2 Further Open Problems

We have mentioned above the analogy between the existence of internal partitions and upper bounds on Cheeger constants. As shown by Alon [1], the Cheeger constant of every large d-regular graph is at most $\frac{d}{2} - c\sqrt{d}$ for some absolute $c > 0$. In this view we raise:

Problem 4.1 *Is it true that for every $\delta \geq 1$ there are integers d and n_0 such that for every $n > n_0$ almost every d-regular graph on n vertices has a $(\frac{d}{2} + \delta, \frac{d}{2} + \delta)$-internal partition?*

On the other hand, D. Cizma and the first named author [9] have constructed, for all $r \geq 2$ infinitely many $2r$-regular graphs such that in every proper partition of the vertices there is a vertex whose internal degree is at most r.

Also, the upper bound on Cheeger's constant in Alon's paper is actually attained by a bisection (the two parts differ in size by at most one). This suggests:

Problem 4.2 *Does Conjecture 2.1 hold also with "near" bisections? E.g., where the cardinalities of the two parts differ by $O_d(1)$.*

How does the computational complexity of the internal partition vary as n grows? So far, existence theorems have gone hand-in-hand with efficient search algorithms. Is this a coincidence or is there a real phenomenon?

Problem 4.3 *How hard is it to decide whether a given d-regular n-vertex graph has an internal partition? Conjecture 2.1 implies that this decision problem is trivial, since the answer is always positive provided that $n > n_0(d)$. Even if this conjecture holds, we may still wonder how hard it is to* find *an internal partition when one exists.*

Note Added in Proof

After the submission of this paper another proof of the theorem was published by Liu and Xu, see Theorem 1.3 in [17].

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