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Searching Ordered Structures

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Suppose distinct real numbers are assigned to the elements of a finite partially ordered set P in an order preserving manner. The problem of determining the fewest numbers of comparisons required to locate a given number x in P is investigated. Some general bounds are provided for the problem and analyzed in detail for the case that P is a product of chains and that P is a rooted forest. © 1985 Academic Press, Inc.

I. INTRODUCTION

Suppose an $m \times n$ real matrix $A = \{a_{ij}\}$ with distinct entries is known to be increasing along rows and columns, that is, $a_{ij} \leqslant a_{kl}$ if $i \leqslant k$ and $j \leqslant l$. Given a real number x, the matrix is to be searched to determine whether x is one of its entries. What is the algorithm which minimizes the number of entries that must be searched in worst case? For $m = 1$, a binary search is optimal and requires looking at $\log_2(n + 1)$ entries in worst case. For any m, by searching each row separately, the search can be done by looking at no more than $m \log_2(n + 1)$ entries. Alternatively, by examining the upper right-hand entry a_{1n}, either the first row or the nth column of the matrix is eliminated depending on whether $x > a_{1n}$ or $x < a_{1n}$. By repeating this we obtain an algorithm requiring at most $m + n - 1$ steps. If $m = n$, this is optimal, but if $m = 1$ it is very bad. What is the optimal algorithm for arbitrary m and n?

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This is an example of the general problem of searching a partially ordered data structure. Suppose P is a finite partially ordered set. Distinct real numbers are assigned to (stored at) the elements of P in such a way that if $p <_P q$ then the number assigned to p is less than the number assigned to q. We wish to find an algorithm to locate a given number x in P or determine that it is not stored in P. The matrix problem is the case where P is the product of a chain of length m and a chain of length n.

In this paper, we investigate the (worst case) complexity of this problem. For the general problem, two lower bounds on the number of elements which need be searched are given; the main bound is an "information theoretic" type bound. We note that a recent result of Sands [9] implies that for posets whose longest chain has l elements there is a constant $K(l)$ such that the number of comparisons required for the optimal algorithm is at most $K(l)$ times the information theoretic lower bound.

For the matrix case, we discuss the equivalence of this problem with the problem of merging two sorted lists of length m and n, which has been studied extensively [4, 11, 2, 1, 3]. The known results for the merge problem provide a lower bound of $\log_2\binom{m+n}{m}$ and an algorithm requiring at most $\log_2\binom{m+n}{m} + m$ comparisons.

We also investigate the case where P is the product of three or more chains (higher dimensional arrays) and give some lower and upper bounds on the complexity. In the case of an $n \times n \times n$ array, we describe an algorithm requiring at most $\frac{3}{2}n^2 + cn\log_2 n$ comparisons and prove a lower bound of $\frac{3}{2}n^2$.

Next we consider the class of posets which are rooted forests (each element covers at most one element). We prove that a poset of this type can be searched in at most $3/\log_2 5$ times the number of steps given by the information theoretic bound and describe the class of rooted forests for which this number of steps is necessary.

Throughout this paper, we use $\lg x$ to denote $\log_2 x$. The notation $f(n) = o(g(n))$ means that $f(n)/g(n)$ tends to 0 as n approaches infinity and $f(n) = O(g(n))$ means that $f(n)/g(n)$ is bounded as n approaches infinity.

II. Preliminaries

Let P denote a finite partially ordered set with relation $<$. An *ideal* of P is a subset $I \subseteq P$ with the property that if $p \in I$ and $q < p$ then $q \in I$. A *filter* of P is the complement of an ideal. If S is any subset of P, then the ideal (resp. filter) generated by S, denoted $I(S)$ (resp. $F(S)$) is the set of all elements which are less than (resp. greater than) some element of S. A *section* of P (also called a *convex subset*) is a subset S with the property that if $p_1, p_2 \in S$ and $p_1 < q < p_2$ then $q \in S$; equivalently, S is the intersection of an ideal and a filter.

We define the function $i(P)$ to be the number of ideals of P; for $p \in P, i(p; P)$ is the number of ideals of P which contain p. The following observation will be useful.

PROPOSITION 2.1. *For any element* $p \in P$,

$$i(P) = i(P \setminus I(p)) + i(P \setminus F(p)).$$

Proof. The ideals of P which contain p (resp. do not contain p) are in natural correspondence with the ideals of $P \setminus I(p)$ (resp. $P \setminus F(p)$).

For n a positive integer, we let $[n]$ denote the totally ordered set $\{1, 2, \ldots, n\}$. We will be interested in one-to-one order preserving functions s from P to $\mathbb{R}$, i.e., functions satisfying

(i) $s(p) \neq s(q)$ if $p \neq q$,

(ii) $s(p) < s(q)$ if $p < q$,

for all $p, q \in P$. Such functions are called *storage functions* and we say that the real number $s(p)$ is "stored" at the element p. The problem we consider is: given an unknown storage function s on P and a real number x find $s^{-1}(x)$ or show it does not exist by evaluating s on as few elements of P as possible. Let $c(P)$ denote the minimum over all search algorithms for P of the maximum number of elements that the algorithm must search in order to determine $s^{-1}(x)$.

If $s(p)$ is evaluated for some $p \in P$, we obtain the information that either $x = s(p)$, $x < s(p)$, or $x > s(p)$. In the first case s need not be evaluated on any additional elements. In the second case, since s is order preserving, we know that $x < s(q)$ for all $q \in F(p)$ and the problem reduces to finding the optimal search procedure for the poset $P \setminus F(p)$. Similarly, if $x > s(p)$, the problem is reduced to searching $P \setminus I(p)$. This gives the following recursion for $c(P)$.

PROPOSITION 2.2. $c(P) = 1 + \min_{p \in P} \max\{c(P \setminus I(p)), c(P \setminus F(p))\}$.

Of course, computing $c(P)$ directly in this way is cumbersome (to put it mildly) and we would like to find ways to simplify the computation.

Note that each time that $s(p)$ is evaluated for some $p \in P$, the problem is reduced to an equivalent problem on a smaller poset. Thus to describe an algorithm for the search problem for arbitrary posets it suffices to give a rule for selecting the first element to be searched in any poset.

III. SOME BOUNDS ON $c(P)$

Suppose the search procedure is applied in a case where x is not stored in P. In this case, the search will terminate only after each element $p \in P$ is classified according to whether $s(p) < x$ or $s(p) > x$. Thus, in worst case, the problem requires identifying the ideal of P for which the values stored are less than x. Therefore the standard "information theoretic" bound [K] for $c(P)$ is

PROPOSITION 3.1. $c(P) \geqslant \lg i(P)$.

Proposition 3.1 suggests that an optimal algorithm for the search problem would be obtained by evaluating s on an element p contained in exactly half of the ideals of P. If every poset had such an element then this algorithm would attain the lower bound. Unfortunately, it is easy to construct posets that have no such element. However, one could still employ an algorithm which evaluates s on an element p which is contained in as close to half of the ideals as possible. This idea leads to

PROPOSITION 3.2. *If $\mathcal{P}$ is a class of posets and $\frac{1}{2} \leqslant \beta < 1$ is a constant such that*

(i) *every section of a member of $\mathcal{P}$ is a member of $\mathcal{P}$*

(ii) *every poset $P \in \mathcal{P}$ possesses an element p such that*

$$1 - \beta \leqslant i(p; P)/i(P) \leqslant \beta$$

then $c(P) \leqslant -1/\lg\beta \lg i(P)$.

Proof. By induction on $|P|$. If $P \in \mathcal{P}$ and $p \in P$ satisfies condition (ii) then by Proposition 2.2, condition (i) and the induction hypothesis,

$$\begin{aligned} c(P) &\leqslant 1 + \max\{c(P \setminus I(p)), c(P \setminus F(p))\} \\ &\leqslant 1 + \frac{-1}{\lg\beta}\max\{\lg i(P \setminus I(p)), \lg i(P \setminus F(p))\} \\ &\leqslant 1 + \frac{-1}{\lg\beta}\max\{\lg i(p; P), \lg(i(P) - i(p; P))\} \\ &\leqslant 1 + \frac{-1}{\lg\beta}\lg(\beta i(P)) \\ &= \frac{-1}{\lg\beta}\lg i(P). \end{aligned}$$

□

Thus for any class of posets satisfying the conditions (i) and (ii), $c(P) = O(\lg_2 i(P))$. Let P_l denote the class of finite posets with no $l + 1$ element chain. For $p \in P$, let $a(p; P)$ denote the number of antichains (sets of incomparable elements) of P containing p. Sands [9] recently proved

THEOREM 3.3. *For each integer $l > 1$ there exists a number $\beta(l)$, $\frac{1}{2} < \beta(l) < 1$ such that every poset $P \in \mathscr{P}_l$ has an element satisfying $a(p; P)/i(P) \geqslant 1 - \beta(l)$.*

It is easy to show that $a(p; P) \leqslant i(p; P)$ and $a(p; P) \leqslant i(P) - i(p; P)$. With Theorem 3.3 this yields

PROPOSITION 3.4. *For each $l > 1$, there exists a number $\beta(l)$, $\frac{1}{2} < \beta(l) < 1$, such that every poset in $\mathscr{P}_l$ has an element p satisfying*

$$1 - \beta(l) \leqslant i(p; P)/i(P) \leqslant \beta(l).$$

Letting $K(l) = -1/\lg \beta(l)$, Propositions 3.2 and 3.4 imply

THEOREM 3.5. *For each integer $l > 1$, there exist a constant $K(l)$ such that for each $P \in \mathscr{P}_l$,*

$$c(P) \leqslant K(l) \lg i(P).$$

Sands [9] conjectured that there is a constant β independent of l that satisfies Proposition 3.4. If true, this would imply that the search problem can always be done in $O(\lg i(P))$ comparisons.

To obtain another lower bound on $c(P)$ we first observe

PROPOSITION 3.6. *If Q is a section of P then $c(Q) \leqslant c(P)$.*

Proof. By induction on $|P|$. If $p \in Q$ then $Q \setminus I(P)$ is a section of $P \setminus I(p)$ and $Q \setminus F(p)$ is a section of $P \setminus F(p)$, so by induction,

$$\max\{c(P \setminus I(p)), c(P \setminus F(p))\} \geqslant \max\{c(Q \setminus I(p)), c(Q \setminus F(p))\}.$$

If $p \in P \setminus Q$ then Q is a section of either $P \setminus I(p)$ or $P \setminus F(p)$, so by induction,

$$\max\{c(P \setminus I(p)), c(P \setminus F(p))\} \geqslant c(Q).$$

Thus by Proposition 2.2,

$$\begin{aligned} c(P) &= 1 + \min_{p \in P} \max\{c(P \setminus I(p)), c(P \setminus F(p))\} \\ &\geqslant 1 + \min_{p \in Q} \max\{c(Q \setminus I(p)), c(Q \setminus F(p))\} = c(Q). \qquad \square \end{aligned}$$

It should be noted that if Q is an arbitrary subposet of P, that is, not a section, then $c(Q)$ may be larger than $c(P)$.

PROPOSITION 3.7. *If Q is a poset with no three element chain, then $c(Q) = |Q|$.*

Proof. Restrict attention to storage functions $s: Q \to \mathbb{R}$ for which $s(p) \leqslant x$ if p is a minimal element of Q and $s(p) \geqslant x$ otherwise. Each comparison of x with $s(p)$ for any p only eliminates the possibility $s(p) = x$ and thus in worst case all comparisons are required. □

Combining Proposition 3.6 and 3.7 gives

PROPOSITION 3.8. *If Q is any section of P containing no three element chain then $c(P) \geqslant |Q|$.*

IV. PRODUCTS OF TWO CHAINS

We consider the search problem mentioned in the introduction, where P is the product of an m element chain and an n element chain, i.e., $P = [m] \times [n] = \{(i, j) | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ with $(i, j) \leqslant (k, l)$ if and only if $i \leqslant k$ and $j \leqslant l$. A storage function s on P can be thought of as an $m \times n$ matrix $A = \{a_{ij}\}$, with $a_{ij} = s((i, j))$, which is increasing along rows and columns. Let $\Phi(m, n) = c([m] \times [n])$.

In the introduction we noted $\Phi(1, n) = [\log_2(n + 1)]$ and $\Phi(m, n) \leqslant m + n - 1$ for all m and n. When $m = n$ the set $Q = \{(i, j) | i + j = n$ or $i + j = n + 1\}$ satisfies the conditions of Proposition 3.8 and $|Q| = 2n - 1$ so we obtain

PROPOSITION 4.1 [2]. $\Phi(n, n) = 2n - 1$.

Computing $\Phi(m, n)$ in general is much more difficult. The search problem on an $m \times n$ matrix is closely related to a well-known sorting problem. Let $v_1 < v_2 < \cdots < v_m$ and $w_1 < w_2 < \cdots < w_n$ be lists of distinct real numbers. The problem is to merge these two lists into a single sorted list, that is, determine whether $v_i < w_j$ holds or $v_i > w_j$ holds for each pair (v_i, w_j). $M(m, n)$ is defined to be the minimum number of comparisons needed in worst case to merge the lists. The following result was noticed by Shearer [10].

THEOREM 4.2. $\Phi(m, n) = M(m, n)$.

Proof. Given a procedure for one problem we obtain one for the other by the following correspondence:

merge problem		**search problem**
compare v_i to w_j	$\Leftrightarrow$	compare x to a_{in+1-j}
branch on $v_i < w_j$	$\Leftrightarrow$	branch on $x > a_{in+1-j}$
branch on $v_i > w_j$	$\Leftrightarrow$	branch on $x < a_{in+1-j}$

It is easy to see that the trees representing these algorithms are the same. □

The merge problem is well studied. By applying the known results for $M(m, n)$ and Theorem 4.2, we obtain

THEOREM 4.3.

$$\lg\binom{m+n}{m} + \min(m,n) \geqslant \phi(m,n) \geqslant \lg\binom{m+n}{m} \qquad [4]$$

$$\phi(2,n) = \lceil \lg\tfrac{7}{12}(n+1) \rceil + \lceil \lg\tfrac{14}{17}(n+1) \rceil \qquad [1,5]$$

$$\phi(3,n) = \lceil \lg\tfrac{28}{43}(n+2) \rceil + \lceil \lg\tfrac{56}{107}(n+2) \rceil + \lceil \lg(7n + 13/17) \rceil \quad (n \geqslant 8). \qquad [3]$$

For more details on the merge problem see Knuth [6].

V. PRODUCTS OF THREE OR MORE CHAINS

Suppose now that P is a product of d chains, $[n_1] \times [n_2] \times \cdots \times [n_d]$. As in the 2-dimensional case, think of a storage function on P as a d-dimensional array A with distinct real entries satisfying $a_{i_1,\ldots,i_d} \leqslant a_{j_1,\ldots,j_d}$ if $i_1 \leqslant j_1, i_2 \leqslant j_2, \ldots, i_d \leqslant j_d$. We want to bound $\Phi(n_1, n_2, \ldots, n_d) = c([n_1] \times [n_2] \times \cdots \times [n_d])$.

Consider first the case where $n_1 = n_2 = \cdots = n_d = n$. Define $\tau(n,d) = \Phi([n]^d)$.

THEOREM 5.1. *For $d \geqslant 2, n \geqslant 1$,*

$$c_1(d)n^{d-1} \geqslant \tau(n,d) \geqslant c_2(d)n^{d-1} + o(n^{d-1}),$$

where (i) *$c_1(d)$ is a nonincreasing function of d and $c_1(d) \leqslant 2$ and*
(ii) $c_2(d) = \sqrt{(24/\pi)}\, d^{-1/2} + o(d^{-1/2})$.

Proof. Partition $[n]^d$ into n isomorphic copies of $[n]^{d-1}$ each consisting of all elements whose dth coordinates are the same. By searching each separately we have

$$\hat{\tau}(n,d) \leqslant n\tau(n,d-1),$$

and in the last section we saw $\tau(n,2) < 2n$ so the first inequality follows by induction.

For the second inequality consider the subset of $[n]^d$ given by $Q(n,d) = Q_1(n,d) \cup Q_2(n,d)$, where

$$Q_1(n,d) = \left\{ (i_1,\ldots,i_d) \in [n]^d \,\middle|\, \sum_{j=1}^{d} i_j = \left\lfloor \frac{(n+1)d}{2} \right\rfloor \right\}$$

$$Q_2(n,d) = \left\{ (i_1,\ldots,i_d) \in [n]^d \,\middle|\, \sum_{j=1}^{d} i_j = \left\lfloor \frac{(n+1)d}{2} \right\rfloor + 1 \right\}$$

$Q(n,d)$ is a section of $[n]^d$ with no chain of length 3, so by Proposition 3.8, $\tau(n,d) \geqslant |Q(n,d)|$. To estimate $|Q(n,d)|$ for large n, we estimate $|Q_1(n,d)|$ for n odd, enabling us to drop the greatest integer brackets; a nearly identical calculation yields the same estimate for n even and for $Q_2(n,d)$.

For fixed n, let $X_1, X_2, \ldots, X_d$ be identically distributed random variables with discrete distribution,

$$P(X_i = a) = \frac{1}{n} \qquad \text{if } a = \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}.$$

We associate an instantiation $(x_1, x_2, \ldots, x_d)$ of $(X_1, X_2, \ldots, X_d)$ with the element $(nx_1, \ldots, nx_d)$ of $[n]^d$. Thus

$$|Q_1(n,d)| = n^d P\left((X_1 + \cdots + X_d) = \frac{(n+1)d}{2n}\right) \tag{1}$$

and by an elementary argument,

$$\begin{aligned} &P\left(X_1 + \cdots + X_d = \frac{(n+1)d}{2n}\right) \\ &= \sum_{i=1}^{n} P\left(X_d = \frac{i}{n}\right) P\left(X_1 + \cdots + X_{d-1} = \frac{(n+1)d}{2n} - \frac{i}{n}\right) \\ &= \frac{1}{n} P\left(\frac{(n+1)d - 2n}{2n} \leqslant X_1 + \cdots + X_{d-1} \leqslant \frac{(n+1)d-2}{2n}\right). \end{aligned}$$

With (1) this gives

$$\frac{|Q_1(n,d)|}{n^{d-1}} = P\left(\frac{(n+1)d - 2n}{2n} \leqslant X_1 + \cdots + X_{d-1} \leqslant \frac{(n+1)d-2}{2n}\right).$$

If we let n approach infinity, the distribution of each X_i approaches that of a continuous random variable U_i with uniform density on the interval $[0,1]$. So

$$\lim_{n\to\infty} \frac{Q_1(n,d)}{n^{d-1}} = P\left(\frac{d}{2} - 1 \leqslant U_1 + U_2 + \cdots + U_{d-1} \leqslant \frac{d}{2}\right). \tag{2}$$

To evaluate this we use the formula

$$\begin{aligned} &P\left(\frac{d}{2} - 1 \leqslant U_1 + \cdots + U_{d-1} \leqslant \frac{d}{2}\right) \\ &= \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-it(d/2-1)} - e^{-it(d/2)}}{it} [\varphi(t)]^{d-1} dt, \end{aligned} \tag{3}$$

where $\varphi(t)$ is the Fourier transform of a uniform [0, 1] variable and is equal to $E(e^{tu}) = (e^{it} - 1)/it$ (see, e.g., [12]).

Simplifying (3) and substituting $x = t/2$ yields

$$P\left(\frac{d}{2} - 1 \leqslant U_1 + \cdots + U_{d-1} \leqslant \frac{d}{2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^d dx. \qquad (4)$$

Combining (2) and (4) and using a similar calculation to estimate $|Q_2(n, d)|$ we get

$$\lim_{n\to\infty} \frac{|Q(n,d)|}{n^{d-1}} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^d dx. \qquad (5)$$

The integrand in (5) can be expanded as Fourier series and evaluated to obtain

$$\lim_{n\to\infty} \frac{Q(n,d)}{n^{d-1}} = \frac{1}{2^{d-1}(d-1)} \sum_{0\leqslant k\leqslant d/2} (-1)^k \binom{d}{k} (d-2k)^{d-1}.$$

Finally, we can approximate (1) for large d by applying the central limit theorem (CLT): The distribution of the sum of d independent identically distributed random variables approaches that of a normally distributed variable with mean and variance equal to d times the mean and variance of X_i. A simple computation gives $E(X_i) = (n + 1)/2n$ and $\mathrm{Var}(X_i) = (n^2 - 1)/12n^2$.

Actually we need a local form of the CLT (see [7, 8, 12]), since we want to make a point estimate for $P(X_1 + \cdots + X_d = (n + 1)d/2n)$. Applying this we get that for large d,

$$Q_1(n,d) = n^d P\left(X_1^n + \cdots + X_d^n = \frac{(n+1)d}{2n}\right)$$

$$= n^d\left(1/\sqrt{(2\pi d)\left(\frac{n^2-1}{12}\right)} + \epsilon(d)\right)$$

$$\text{where } \frac{\epsilon(d)}{\sqrt{d}} \to 0 \text{ as } d \to \infty,$$

$$= \sqrt{\frac{6}{\pi d}}\, n^{d-1} + O(n^{d-2}) + \epsilon(d).$$

Since $Q(n, d) = 2Q_1(n, d)$, we get

$$c_2(d) \geqslant \frac{Q(n,d)}{n^{d-1}} = \sqrt{\frac{24}{\pi}}\, d^{-1/2} + o(d^{-1/2}). \qquad \square$$

For the case $d = 3$ we can make a much more precise statement than that given in Theorem 5.1.

THEOREM 5.2. *There exists a constant $C > 0$ so that for $n \geqslant 2$,*

$$\frac{3n^2}{2} + cn\log n \geqslant \tau(n,3) \geqslant \lfloor 3n^2/2 \rfloor .$$

Proof. For $d = 3, Q(n, d)$ can be calculated explicitly by simple combinatorial methods to be $\lfloor 3n^2/2 \rfloor$.

To prove the upper bound, we describe an algorithm which solves the problem in $3n^2/2 + cn\log n$ steps.

Step 1. Binary search for x along each of the following six subvectors of A.

$$u_1 = (a_{1,n,k} | 1 \leqslant k \leqslant n) \qquad v_1 = (a_{n,j,1} | 1 \leqslant j \leqslant n)$$
$$w_1 = (a_{i,1,n} | 1 \leqslant i \leqslant n)$$
$$u_2 = (a_{n,1,k} | 1 \leqslant k \leqslant n) \qquad v_2 = (a_{1,j,n} | 1 \leqslant j \leqslant n)$$
$$w_2 = (a_{i,n,1} | 1 \leqslant i \leqslant n)$$

(see Fig. 1).

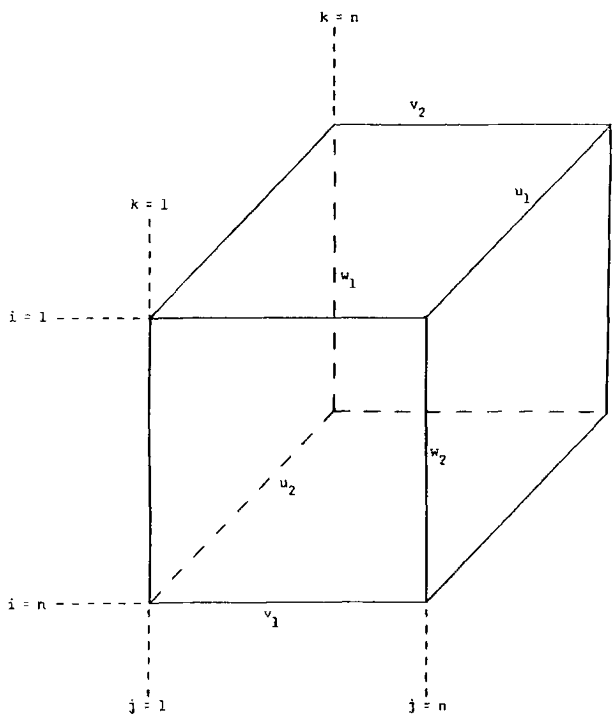

FIGURE 1

These searches find integers $k_1, k_2, j_1, j_2, i_1, i_2$ such that

$$a_{1,n,k_1} < x < a_{1,n,k_1+1}$$
$$a_{n,1,k_2} < x < a_{n,1,k_2+1}$$
$$a_{n,j_1,1} < x < a_{n,j_1+1,1}$$
$$a_{1,j_2,n} < x < a_{1,j_2+1,n}$$
$$a_{i_1,1,n} < x < a_{i_1+1,1,n}$$
$$a_{i_2,n,1} < x < a_{i_2+1,n,1}.$$

Consider now the face $i = 1$ of A (the array of entries $a_{1,j,k}$): $\{a_{1,j,k} | 1 \leqslant j, k \leqslant n\}$. The searches of Step 1 have left only $\{a_{1,j,k} | n \geqslant j > j_2, n \geqslant k > k_1\}$ as entries which possibly equal x, so out of this face of A there remains an $(n - j_2) \times (n - k_1)$ submatrix to be checked. On the $i = n$ face we only have to check the $j_1 \times k_2$ matrix $\{a_{n,j,k} | 1 \leqslant j \leqslant j_1, 1 \leqslant k \leqslant k_2\}$. Similarly for the $j = 1$ and $j = n$ faces we have $(n - i_1) \times (n - k_2)$ and

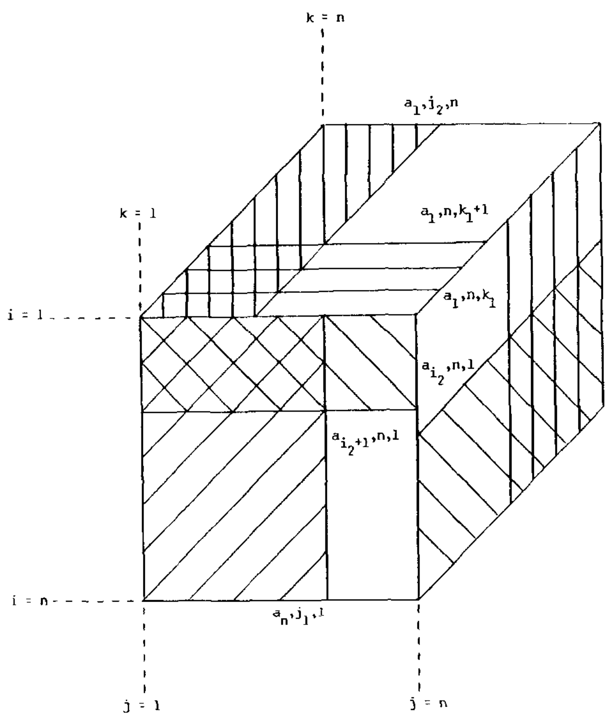

FIG. 2. After searching u_1, u_2, v_1, v_2, w_1, and w_2 only the unshaded portions of each face need be searched. (Back faces are not shown, but the situation is similar.)

$(i_2 \times k_1)$ matrices to be checked, and for the $k = 1$ and $k = n$ faces we have $(n - j_1) \times (n - i_2)$ and $j_2 \times i_1$ matrices to be checked (see Fig. 2.)

Step 2. Apply the "$m + n - 1$" algorithm described in the introduction to each of these six matrices.

Let us now count the number of comparisons done in Steps 1 and 2. In Step 1, at most 6 lg$(n + 1)$ comparisons were made. In Step 2, we need at most

$$(n - j_2 + n - k_1 - 1) + (j_2 - k_2 - 1) + (n - i_1 + n - k_2 - 1) + (i_2 + k_1 - 1) + (n - i_2 + n - j_1 - 1) + (i_1 + j_2 - 1) = 6(n - 1)$$

comparisons. Steps 1 and 2 leave an $(n - 2) \times (n - 2) \times (n - 2)$ array so

$$\tau(n,3) \leqslant \tau(n - 2,3) + 6\lg(n + 1) + 6(n - 1).$$

Solving this recursion gives

$$\tau(n,3) \leqslant \frac{3n^2}{2} + 3n\lg(n + 2). \qquad \square$$

We now return to the general case of a d-dimensional array in which the n_i's are not necessarily equal. The best result we have is

THEOREM 5.3. *For $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_d \geqslant 1$, there exists a nonincreasing function $k_1(d)$ and a function $k_2(d)$ such that*

$$k_1(d)n_d n_{d-1} \cdots n_2 \lg\left(\frac{n_1}{n_2} + 1\right) \geqslant \Phi(n_1,\dots,n_d) \geqslant k_2(d)n_d n_{d-1} \cdots n_2 \lg\left(\frac{n_1}{n_2} + 1\right).$$

Proof. The first inequality is obtained inductively. For $d = 2$, Theorem 4.3 shows that $k_1(2) = 2$ works, since

$$\lg\binom{n_1 + n_2}{n_2} \leqslant \lg\binom{n_1 + n_2}{n_1}^{n_2}.$$

For $d > 2$, by searching separately the n_d copies of $[n_1] \times \cdots \times [n_{d-1}]$ obtained by fixing the dth coordinate, we get

$$\Phi(n_1,\dots,n_d) \leqslant n_d \Phi(n_1,\dots,n_{d-1}),$$

which gives the desired bound.

To obtain the second inequality, we again use induction. Note for $d = 2$, Theorem 4.3 shows that $k_1(2) = 1$ works. For $d > 2$, think of A as an

$n_d \times n_d \times \cdots \times n_d$ array A' each of whose entries is a $l_1 \times l_2 \times \cdots \times l_{d-1}$ array, where l_i is equal to either $\lfloor n_i/n_d \rfloor$ or $\lceil n_i/n_d \rceil$. Consider the set $Q_1(n_d, d)$ defined in the proof of Theorem 5.1, which is a set of elements in A' and corresponds to a section of $[n_1] \times \cdots \times [n_d]$ which we call Q' and so by Proposition 3.6,

$$\Phi(n_1, \ldots, n_d) \geq c(Q').$$

Now Q' consists of $Q_1(n_d, d)$ arrays whose dimensions are at least $\lfloor n_1/n_d \rfloor \times \lfloor n_2/n_d \rfloor \times \cdots \times \lfloor n_{d-1}/n_d \rfloor$ with the elements of different arrays unrelated, so

$$c(Q') \geq Q_1(n_d, d)\Phi\left(\left\lfloor \frac{n_1}{n_d} \right\rfloor, \left\lfloor \frac{n_2}{n_d} \right\rfloor, \ldots, \left\lfloor \frac{n_{d-1}}{n_d} \right\rfloor\right)$$

$$\geq c_2(d) n^d k_2(d-1) \left\lfloor \frac{n_{d-1}}{n_d} \right\rfloor \left\lfloor \frac{n_{d-2}}{n_d} \right\rfloor \cdots \left\lfloor \frac{n_2}{n_d} \right\rfloor \lg\left(\frac{\left\lfloor \frac{n_1}{n_d} \right\rfloor}{\left\lfloor \frac{n_2}{n_d} \right\rfloor} + 1 \right),$$

by Theorem 5.1 and the induction hypothesis. Note that

$$n_d \left\lfloor \frac{n_i}{n_d} \right\rfloor \geq \frac{n_i}{2} \qquad \text{and} \qquad \lg\left(\left\lfloor \frac{n_1}{n_d} \right\rfloor \Big/ \left\lfloor \frac{n_2}{n_d} \right\rfloor + 1 \right) \geq \lg\left(\frac{n_1}{n_2} + 1 \right) \Big/ 2$$

so

$$c(Q') \geq \frac{c_2(d) k_2(d-1)}{2^d} n_d n_{d-1} \cdots n_2 \lg\left(\frac{n_1}{n_2} + 1 \right),$$

and by defining $k_2(d) = c_2(d)k_2(d-1)/2^d$ we prove the theorem. □

VI. Rooted Forests

A partially ordered set P is a *rooted forest* if every element covers at most one element (p *covers* q in P if $p > q$ and there does not exist r with $p > r > q$). We adopt standard graph theoretic terminology: If q covers p in a rooted forest P then p is the *father* of q and q is a *son* of p. The minimal elements of P are *roots*, the maximal elements are *leaves*. A maximal connected component of P is a *rooted tree*. If $p \in P$, the subtree rooted at p is the filter generated by p; if q is a son of a root the subtree is a *principal subtree*.

The ideal generated by an element p of a rooted forest P is a chain C. Thus in any subposet P' containing p, p covers at most one element, namely the maximum element of $C \cap P'$, so we have

PROPOSITION 6.1. *If P is a rooted forest, all subposets of P are rooted forests.*

Throughout this section α is the constant $5^{-1/3}$. For a poset P let $b(P) = 3/\lg 5 \lg i(P)$. The main result of the section is

THEOREM 6.2. *For any rooted forest P, $c(P) \leqslant b(P)$ with equality if and only if each connected component of P is either a chain of four elements or one element covered by two elements ($K_{1,2}$).*

Proof. If P is a chain of four elements or a $K_{1,2}$ then $i(P) = 5$ and $c(P) = 3$ so the equality holds. If P is a disjoint union of $P_1, \dots, P_k$ then $c(P)$ and $b(P)$ are the sums, respectively, of $c(P_j)$ and $b(P_j)$ so the theorem holds if it holds for each P_j. Hence it suffices to consider the case where P has a single component.

We prove that $c(P) \leqslant b(P)$ for any connected poset by induction on the number of elements of P. Fix P such that $c(P) \geqslant b(P)$ and assume that the theorem holds for any P' with fewer elements. We will show that P is a chain of four elements or $K_{1,2}$.

For $q \in Q$ let $i(q; Q)$ be the number of ideals of Q containing q, $f(q; Q)$ denote the fraction $i(q; Q)/i(Q)$ of ideals of Q containing q, and $N(q) = i(F(q))$, the number of ideals in the subtree rooted at q.

LEMMA 6.3. *If q is the father of p in the rooted tree Q then*

$$\frac{N(p)-1}{N(p)} f(q; Q) = f(p; Q).$$

Proof. Each ideal containing q but not p can be extended in $N(p) - 1$ ways to an ideal containing p, by adding any nonempty ideal of $F(p)$. All ideals containing p are obtained in this way. □

COROLLARY 6.4. *If p is a leaf of Q then $f(p; Q) < \frac{1}{2}$.*

COROLLARY 6.5. *If r is the root of the tree Q then $f(p; Q) \geqslant \frac{1}{2}$.*

LEMMA 6.6. *Let P' be a subposet of P such that $k + c(P') \geqslant c(P)$. Then $i(P')/i(P) \geqslant \alpha^k$.*

Proof. By the induction hypothesis $c(P) \geqslant b(P)$ and $c(P') \leqslant b(P')$ so

$$\frac{3}{\lg 5} \lg i(P') + k \geqslant \frac{3}{\lg 5} \lg i(P).$$

Simple calculation yields $i(P')/i(P) \geqslant \alpha^k$. □

LEMMA 6.7. *Let $p \in P$.*

(i) *If $f(p; P) < \alpha$ then $c(P) \leqslant 1 + c(P \setminus F(p))$.*

(ii) *If $f(p; P) > 1 - \alpha$ then $c(P) \leqslant 1 + c(P \setminus I(p))$.*

Proof. By Proposition 2.2, $c(P) \leqslant 1 + \max\{c(P \setminus I(p)), c(P \setminus F(p))\}$.

(i) If $f(p; P) < \alpha$ then $i(P \setminus I(p))/i(P) < \alpha$, and by Lemma 6.6, $1 + c(P \setminus I(p)) < c(P)$. Hence $c(P) \leqslant 1 + c(P \setminus F(p))$.

The proof of (ii) is similar. □

COROLLARY 6.8. *For any element $p \in P$ either $f(p; P) \geqslant \alpha$ or $f(p; P) \leqslant 1 - \alpha$.*

Proof. If $1 - \alpha < f(p; P) < \alpha$ then by Lemma 6.7, $c(P) \leqslant 1 + \min\{c(P \setminus I(p)), c(P \setminus F(p))\}$ so by Lemma 6.6, $\min\{i(P \setminus I(p), i(P \setminus F(p)\}/i(P) \geqslant \alpha$, which is impossible. □

An element q of P is *critical* if $f(q; P) \geqslant \frac{1}{2}$ and $f(p; P) < \frac{1}{2}$ for any son p of q. We will determine the structure of P by studying its critical elements.

The following observations are trivial.

PROPOSITION 6.9. *If $f(p; P) \geqslant \frac{1}{2}$ then some element in $F(p)$ is critical.*

PROPOSITION 6.10. *No leaf of P is critical.*

Henceforth, we let q denote an arbitrary critical element of P.

LEMMA 6.11. *If p is a son of q then either p is a leaf or p has exactly one son and it is a leaf (such an element will be called a* preleaf).

Proof. Applying Corollary 6.8 and Lemma 6.3 we have

$$\alpha \leqslant f(q; P) = f(p; P)\frac{N(p; P)}{N(p; P) - 1} \leqslant (1 - \alpha)\frac{N(p; P)}{N(p; P) - 1}$$

which implies that $N(p; P) \leqslant \alpha/(2\alpha - 1)$, i.e., $N(p; P)$ equals two or three. $N(p; P) = 2$ only if p is a leaf and $N(p; P) = 3$ only if p is a preleaf. □

LEMMA 6.12. (i) *If q has a son that is a leaf then $f(q; P) \leqslant 2(1 - \alpha)$.*

(ii) *If q has a son that is a pre-leaf then $f(q; P) \leqslant 3/2(1 - \alpha)$.*

Proof. (i) By Lemma 6.3 and Corollary 6.8, $f(q; P) = 2f(p; P) \leqslant 2(1 - \alpha)$.

(ii) $f(q; P) = 3/2 f(p; P) \leqslant 3/2(1 - \alpha)$. □

LEMMA 6.13. (i) *If q has n or more sons that are leaves then* $f(q; P) \geqslant \alpha^{n+1}2^n$.

(ii) *If q has n or more sons that are preleaves then* $f(q; P) \geqslant \alpha^{2n+1}3^n$.

Proof. (i) By Lemma 6.7(i), $c(P) \leqslant 1 + c(P \setminus I(q))$. $P \setminus I(q)$ is the union of the n leaf sons of q (which are disconnected elements) and a poset P', so $c(P) \leqslant n + 1 + c(P')$. By Lemma 6.6, $i(P')/i(P) \geqslant \alpha^{n+1}$. Now $i(P') = i(P \setminus I(q))/2^n = i(P)f(q; P)/2^n$ so $f(q; P) \geqslant \alpha^{n+1}2^n$.
The proof of (ii) is analogous. □

LEMMA 6.14. *All of the sons of q are leaves or all are preleaves.*

Proof. The bounds of Lemma 6.12(ii) and Lemma 6.13(i), with $n = 1$, are contradictory so q cannot have both a leaf son and a preleaf son. □

LEMMA 6.15. *q has at most 2 sons.*

Proof. By Lemma 6.14 all the sons are leaves or all are preleaves. In each case, the bounds of Lemmas 6.12 and 6.13 are contradictory if $n \geqslant 3$. □

LEMMA 6.16. *Either q has exactly one son that is a preleaf or q has exactly two sons that are both leaves.*

Proof. By Lemma 6.15, q has either one or two sons. Suppose q has exactly one son and it is a leaf. Let t be a father of q. We have $f(t; P) = N(q)/(N(q) - 1)f(q; P) = \frac{3}{2}f(q; P) \geqslant 3\alpha^2 > 1$ by Lemmas 6.3 and 6.13(i), which is impossible. Thus q is the root, so P is a 2 element chain for which $c(P) < b(P)$ contradicting our initial assumption.

If q has two sons p_1 and p_2, one of which is not a leaf, then by Lemma 6.14 both are preleaves with leaves p_1' and p_2'. If q is the root then P is specified and is easily seen to violate $c(P) \geqslant b(P)$. So let t be the father of q, then $f(t; P) = N(q)/(N(q) - 1) = \frac{10}{9}f(q; P)$ by Lemma 6.3. By Lemma 6.7(i), $c(P) \leqslant 1 + c(P')$, where $P' = P \setminus F(p_1)$. Also, by Proposition 2.1, $c(P') \leqslant 1 + \max\{c(P' \setminus F(t)), c(P' \setminus I(t))\}$, so $c(P) \leqslant 2 + \max\{c(P' \setminus F(t)), c(P' \setminus I(t))\}$. Now $i(P' \setminus F(t))/i(P) = 1 - f(t; P) = 1 - \frac{10}{9}f(q; P) > 1 - 10\alpha^5$ by Lemma 6.13(ii), which is less than α^2 so by Lemma 6.6, $c(P) > 2 + c(P' \setminus F(t))$. Thus $c(P) \leqslant 2 + c(P' \setminus I(t))$. In $P' \setminus I(t)$, the chain $y < p_2 < p_2'$ is a detached component requiring two steps to search. Let $P'' = P' \setminus (I(t) \cup \{y, p_2, p_2'\})$ and we have: $c(P) \leqslant 4 + c(P'')$. By Lemma 6.6, $i(P'')/i(P) \geqslant \alpha^4$. The ideals of P'' are in one to one correspondence with the ideals of P which contain t but not y, so $i(P'')/i(P) = f(t; P)/10$ so $f(t; P) \geqslant 10\alpha^4 > 1$ which is impossible. So p_1 and p_2 are leaves. □

A critical element is called type 1 or type 2 depending on the number of sons it has.

LEMMA 6.17. *If q is a type 2 element then it is the root, and P is a $K_{1,2}$.*

Proof. Let t be the father of q, then $f(t; P) = \frac{5}{4}f(q; P)$ by Lemma 6.3 and $f(q; P) \geqslant \frac{4}{5}$ by Lemma 6.13(i) so $f(t; P) \geqslant 1$ which is impossible. □

LEMMA 6.18. *If q is a type 1 element, then q is the unique son of its father t and t is the root, and thus P is a chain of four elements.*

Let q^* be a type 1 element of maximum distance from the root, and let t^* be its father. Suppose that t^* has a son $v \neq q^*$.

Case i. v is a leaf. By Lemma 6.7(ii), $c(P) \leqslant 1 + c(P \setminus I(q^*))$. v is a disconnected element in $c(P \setminus I(q^*))$, so $c(P) \leqslant 2 + c(P \setminus I(q^*) \cup \{v\})$) so by Lemma 6.6, $i(P \setminus I(q^*) \cup \{v\})/I(P) > \alpha^2$, contradicting $i(P \setminus I(q^*) \cup \{v\})/i(P) = f(q^*; P)/2 < \frac{3}{4}(1 - \alpha)$, which is obtained from Lemma 6.12(ii).

Case ii. v is not a leaf. Then $N(v) \geqslant 2$ so $f(v; P) \geqslant \frac{2}{3}f(t^*; P) = \frac{2}{3}(\frac{4}{3})f(q^*; P) \geqslant \frac{8}{9}\alpha > \frac{1}{2}$ by Lemma 6.3 and Corollary 6.8. By Proposition 6.9, some element in $F(p)$ is critical. By Lemma 6.17 this element must be type 1 and since q^* has maximum distance from the root of all type 1 elements, v must be that element. Now $c(P) \leqslant 1 + c(P \setminus I(q^*))$ by Lemma 6.7(ii), and $P \setminus I(q^*)$ is the disjoint union of a poset P' and a chain of three elements with minimum element v. Thus $c(P) \leqslant 3 + c(P')$ so by Lemma 6.6, $i(P')/i(P) \geqslant \alpha^3 = \frac{1}{5}$. This contradicts $i(P')/i(P) = f(q^*; P)/4 < \frac{3}{8}(1 - \alpha)$ obtained from Lemma 6.12(ii).

Thus q^* is the only son of t^*. If u^* is the father of t^* then by Lemmas 6.3 and 6.13(ii), $f(u^*; P) = \frac{5}{4} \cdot \frac{4}{3}f(q^*; P) \geqslant 1$ which is impossible so t^* is the root.

By Lemma 6.16, P has a type 1 or type 2 element. By Lemmas 6.17 and 6.18, this implies that P is a $K_{1,2}$ or a chain of four elements which completes the proof.

VII. OPEN PROBLEMS

In this paper we investigated the complexity of the search problem for partially ordered data structures. Several questions remain. The key question is whether there exists a constant b independent of l in Proposition 3.4, which would imply that the search problem can always be done in $-1/\lg b$ times the information theoretic bound.[1]

The bounds on the multidimensional array problem in Section V can almost certainly be sharpened. It would be interesting to try to generalize the algorithm for the 3-dimensional cube to higher dimensions.

[1] *Note added in proof.* The authors have proved this result, which will appear in *J. Combin. Theory Ser. A.* (1985).

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