Going up in dimensions: Combinatorial and probabilistic aspects of simplicial complexes

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- The asymptotic perspective.
- Extremal combinatorics and its connections to other parts of mathematics.
- The emergence of the probabilistic method.
- The computational perspective.

So, what is the next frontier?

The ubiquity of graphs

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Because they are the tool of choice in modeling pairwise interactions.
But what if we have relations involving more than two objects at a time?

A little about simplicial complexes

This is one of the major contact points between combinatorics and geometry (more specifically - with topology).
From the combinatorial point of view, this is a very simple and natural object. Namely, a down-closed family of sets.

Definition

Let V be a finite set of *vertices*. A collection of subsets $X \subseteq 2^V$ is called a *simplicial complex* if it satisfies the following condition:

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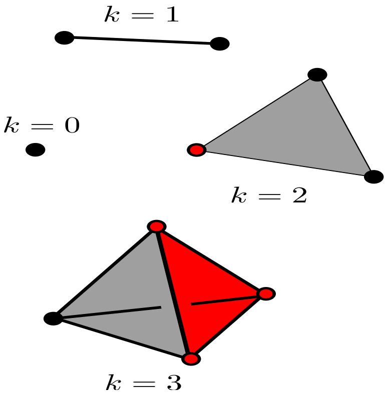
The dimension of X is the largest dimension of a face in X.

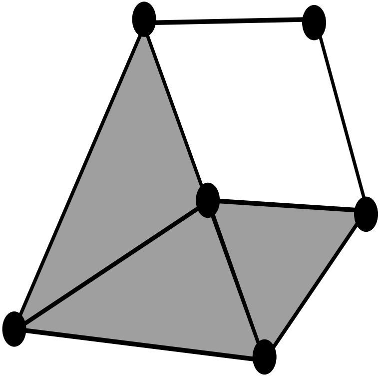
Simplicial complexes as geometric objects

We view $A \in X$ and $|A| = k + 1$ as a k-dimensional simplex.



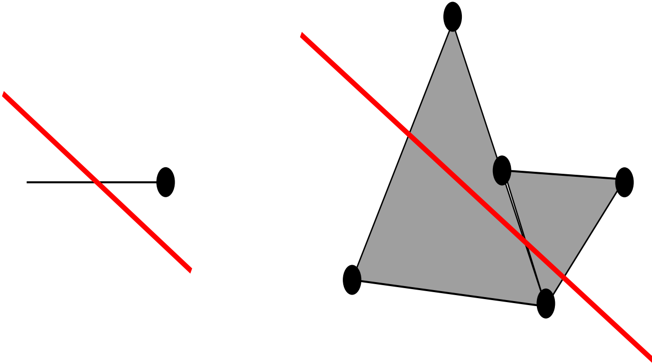
Putting simplices together properly

The intersection of every two simplices in X is a common face.



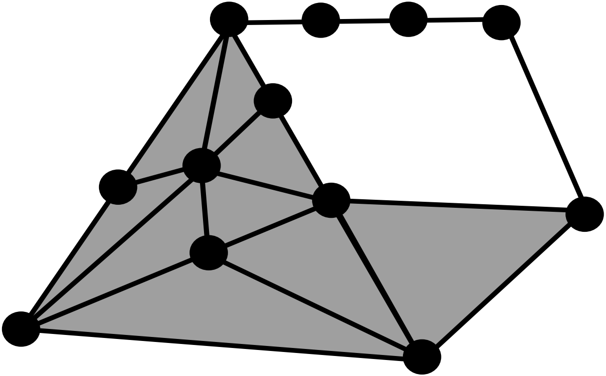
How NOT to do it

Not every collection of simplices in $\mathbb{R}^d$ is a simplicial complex



Geometric equivalence

Combinatorially different complexes may correspond to the same geometric object (e.g. via subdivision)



Geometric equivalence

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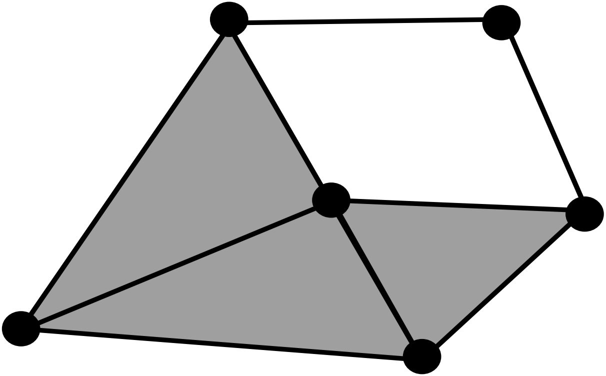

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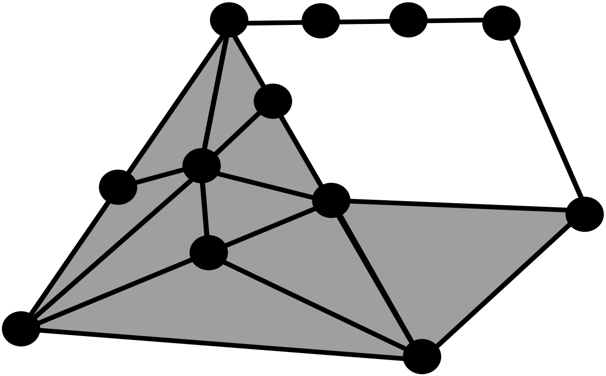
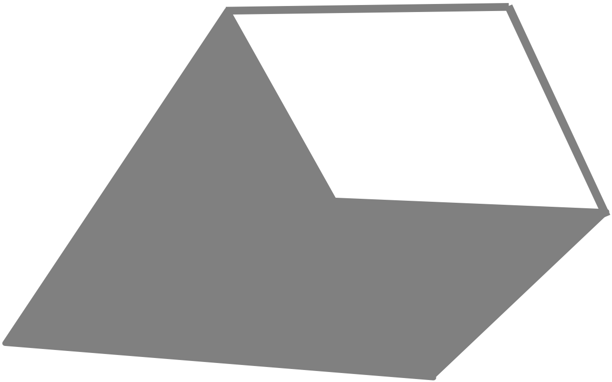


Geometric equivalence

are two different combinatorial descriptions of the same geometric object



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- Graphs need no advertising in this forum.
- A graph may be viewed as a one-dimensional simplicial complex.
- Higher dimensional complexes have a very geometric (mostly topological) aspect to them.
- Can we benefit from investigating higher dimensional complexes?
- How should this be attacked?
  1. Using extremal combinatorics
  2. With the probabilistic method

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- Impossibility theorems in distributed asynchronous computation (Starting with [Herlihy, Shavit '93] and [Saks, Zaharoglou '93]).

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- Characterization of graph connectivity (Lovász's proof of A. Frank's conjecture 1977).

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The game ends when Alice knows with certainty whether G has property $\mathcal{P}$.

The evasiveness conjecture

Conjecture

For every monotone graph property $\mathcal{P}$, Bob has a strategy that forces Alice to query all $\binom{n}{2}$ pairs of vertices in V.

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Q: How is this related to simplicial complexes, topology etc.?
A: Fix n, the number of vertices in the graphs we consider. Think of an n-vertex graph as a subset of $W = \binom{[n]}{2}$. (Careful: W is the set of vertices of the complex we consider).
If $\mathcal{G}$ is the collection of all n-vertex graphs that have property $\mathcal{P}$, then $\mathcal{G}$ is a simplicial complex (since $\mathcal{P}$ is monotone).

Kahn Saks and Sturtevant (contd.)

The (simple but useful) observation with which they start is

Lemma

A non-evasive complex is collapsible.

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Collapsibility is a simple combinatorial property of simplicial complexes which can be thought of as a higher-dimensional analogue of being a forest.

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Collapsibility is a simple combinatorial property of simplicial complexes which can be thought of as a higher-dimensional analogue of being a forest.

We will later return to this notion.

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Theorem (KSS '83)

The evasiveness conjecture holds for all graphs of order n when n is prime.

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- The "size" of homology, Betti numbers...
- Topological connectivity.

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- Introduce ideas from topology into computational complexity

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We want to develop a theory of random complexes, similar to random graph theory. Specifically we seek a higher-dimensional analogue to $G(n, p)$. For the purpose of illustration let us mostly consider:

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We denote by $X(n, p)$ this probability space of two-dimensional complexes.

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Theorem (ER '60)

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- The vanishing of the $(d-1)$-st homology (with any ring of coefficients).
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- Likewise, if S is the vertex set of a connected component of G, then $\mathbf{1}_S M = 0$.
- It is not hard to see that G is connected iff the only vector x that satisfies $xM = 0$ is $x = \mathbf{1}$.

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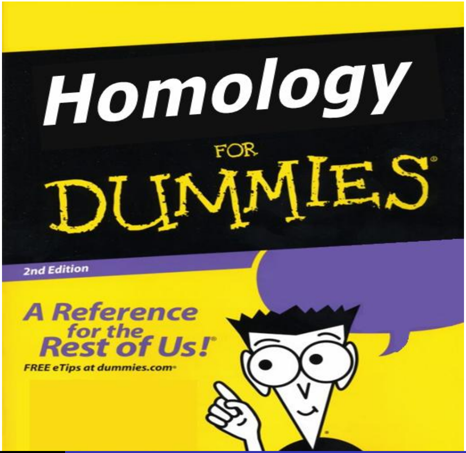

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- The transformations associated with A_1 resp. A_2 are called *the boundary operator* (of the appropriate dimension) and are denoted ∂ (perhaps with an indication of the dimension).

It is an easy exercise to verify that $A_1 A_2 = 0$ (in general there holds $\partial\partial = 0$, a key fact in homology theory).

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Is this a proper inclusion or an equality?

This is quantified by considering the quotient space

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In our situation where X and Y are inclusion matrices of k vs. $(k+1)$-dimensional faces of a simplicial complex, these quotient spaces are the relevant homology and cohomology groups.

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Several things are clear: We now start from the $\binom{n}{2} \times \binom{n}{3}$ inclusion matrix and select a random subset of the columns where every column is selected independently and with probability p.

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That should be clear now: The row space of the $n \times \binom{n}{2}$ matrix.

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The process of selecting the columns yields a random two-dimensional complex with a full one-dimensional skeleton. We call this model of random complexes $X_2(n, p)$. (So, e.g. $X_1(n, p)$ is nothing but good old $G(n, p)$).

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We have asked for the critical p where there a non-trivial left kernel exists.
In topological language: What is the critical p at which the first homology with $\mathbb{F}_2$ coefficients of a random $X \in X_2(n,p)$ vanishes?

...and the answer is...

Theorem (L. + Meshulam '06)

The threshold for the vanishing of the first homology of $X_2(n, p)$ with $\mathbb{F}_2$ coefficients is

$$p = \frac{2 \ln n}{n}$$

More generally

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We still do not know, however:

Question

What is the threshold for the vanishing of the $\mathbb{Z}$*-homology?*

The vanishing of the fundamental group

Theorem (Babson, Hoffman, Kahle '09 ?)

The threshold for the vanishing of the fundamental group in $X(n, p)$ is near

$$p = n^{-1/2}.$$

Comment: When the field is not $\mathbb{F}_2$

We have to select an (arbitrary but fixed) orientation to the triples and pairs. The entries of the inclusion matrix are ± 1 depending on whether the orientation of the edge and the 2-face containing it are consistent or not.

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The d-dimensional case is similar (with an appropriate adaptation).

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Again let's start with the graphical case. The right kernel of the $V \times E$ inclusion matrix of a graph $G = (V, E)$ is G's cycle space. So the relevant 1-dimensional theorem is:

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Theorem

The critical probability for almost sure existence of a cycle in $G(n, p)$ is

$$p = \frac{1}{n}.$$

And the higher-dimensional analogue?

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Theorem (Aroshtam, L., Meshulam; work in progress)

The critical probability where a random complex in $X_2(n, p)$ has almost surely a nontrivial second homology satisfies

$$\frac{1.34...}{n} \leq p \leq \frac{2.74...}{n}.$$

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As mentioned, this is still work in progress and we hope to soon know more.

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Let X be a simplicial complex with vertex set V, and let $x \in V$ be a vertex. The link of x, denoted $\text{link}_X(x)$, is a simplicial complex Y on vertex set $V \setminus \{x\}$. A subset $A \subseteq V \setminus \{x\}$ is a face in Y iff $A \cup \{x\}$ is a face in X.

Links

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In simple words: Your link is everything that together with you forms a face.

Regular complexes?

So, in a graph $G = (V, E)$, link(x) is the neighbor set of the vertex x. We say that G is regular if all vertex links are "the same", i.e., all these sets have the same cardinality.

Regular complexes?

So, in a graph $G = (V, E)$, link(x) is the neighbor set of the vertex x. We say that G is regular if all vertex links are "the same", i.e., all these sets have the same cardinality.

But in a two-dimensional complex X the link of a vertex $\text{link}_X(x)$ is a graph H. (Recall: yz is an edge of H iff xyz is a face in X). This leads to the following:

Regular complexes?

Open Problem

For which graphs H does there exist a two-dimensional complex X, such that $link_X(x)$ is isomorphic to H for every vertex x?

Alternatively...

We could try and restore the simplicity of the notion of regular graphs by considering links of pairs (since link(x, y) is just a set and we only care about its cardinality).

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Namely, let X be a two-dimensional simplicial complex with a full one-dimensional skeleton. Say that X is $(2, d)$-regular if for any two vertices, the cardinality of the set $\text{link}(x, y)$ is d. This, however, means that X is a Steiner Triple System = STS and leads to another open question.

Learning (a bit more) from history

The study of random regular graphs is, of course, a major part of the field. To develop a higher-dimensional analog to this, we would have to resolve:

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Open Problem

Give an efficient algorithm to uniformly generate STS's.

A high-dimensional Cayley formula?

The $n \times \binom{n}{2}$ inclusion matrix has rank $n-1$ as we saw. A column basis is a set of $n-1$ columns that is a basis for the column space.

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But a set of columns in this matrix is just a graph. Which graphs are bases?

High-dimensional Cayley (contd.)

This is not hard to see: Spanning trees of K_n.

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No.

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Definition

Let M be a matrix. In an elementary collapse we erase row i and column j of M provided that M_{ij} is the only nonzero entry in the i-th row.

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M is called collapsible if it is possible to eliminate all its columns by a series of elementary collapses.

If M is the vertex-edge incidence matrix of a graph, an elementary collapse is a step where we remove a vertex of degree 1 and the edge incident with it.

High-dimensional Cayley (contd.)

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This is a combinatorial condition and so it holds over any base field. (The most interesting cases for us are $\mathbb{F}_2$ and $\mathbb{Q}$).
As mentioned, over $\mathbb{Q}$ we work with a signed matrix, that corresponds to an (arbitrary, but fixed) orientation of the graph.

High-dimensional Cayley (contd.)

There is, of course, a lot that we know about trees - How to generate them, what they look like etc. Can this be moved up in dimension?

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We turn to the $\binom{n}{2} \times \binom{n}{3}$ inclusion matrix and consider column bases. The rank now is $\binom{n-1}{2}$. We call a column basis over $\mathbb{Q}$ a hypertree and we now know what to ask

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2. In particular, is it still equivalent to collapsibility? (It's easy to see that collapsibility is still a *sufficient* condition).
3. At any event: How many column bases does the $\binom{n}{2} \times \binom{n}{3}$ inclusion matrix have over our favorite fields?

A little surprise

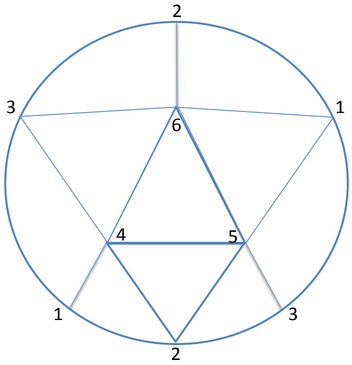

Figure: A triangulation of the projective plane

A little surprise

The example we just saw is a column basis for $\mathbb{Q}$, but not for $\mathbb{F}_2$ (in fact, it's a 2-STS). A partial remedy is given by

Theorem (Kalai '83)

$$\sum |H_{d-1}|^2 = n^{\binom{n-2}{d}}$$

where the sum is over all d-dimensional $\mathbb{Q}$-hypertrees T.

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1. *How many column basis does the* $\binom{n}{2} \times \binom{n}{3}$ *inclusion matrix have over* $\mathbb{F}_2$*?*
   Over $\mathbb{Q}$*?*
2. *How likely is such a basis to be collapsible? (Perhaps it's* $o(1)$*?).*

Extremal combinatorics of simplicial complexes

Theorem (Brown, Erdős, Sós '73)

Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a two-sphere. The bound is tight.

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- Since X contains $\Omega(n^{5/2})$ two-dimensional simplices, the average link size (number of edges in the graph) is $\Omega(n^{3/2})$.

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- Consequently, there are two vertices x, y whose links have $\Omega(n)$ edges in common.
- In particular, there is a cycle C that is contained in the link of x as well as in link(y).
- We just found a double pyramid with base C and x and y as apexes. This is homeomorphic to a two-sphere.

But many extremal questions on simplicial complexes remain widely open

Conjecture

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Conjecture

Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a torus.

- We can show that if true this bound is tight.
- This may be substantially harder than the BES theorem, since a "local" torus need not exist.
- (With Friedgut:) $\Omega(n^{8/3})$ simplices suffice.

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- How many they are: $n!$
- How to sample a random permutation.
- Numerous *typical* properties of random permutations e.g.,:
  - Number of fixed points.
  - Number of cycles.

High-dimensional permutations?

The definition naturally suggests itself: It's an $n \times n \times n$ array of zeros and ones A where every line (now with three types of lines) contains exactly a single 1.

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An alternative description: An $n \times n$ array M where m_{ij} gives the unique k for which $a_{ijk} = 1$. It is easy to verify that M is defined by the condition that every row and column in M is a permutation of $[n]$.
Such a matrix is called a Latin square.

Some challenges

So this raises

Question

Determine or estimate $\mathcal{L}_n$, the number of $n \times n$ Latin squares.

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Theorem (van Lint and Wilson)

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The (fairly easy) proof uses two substantial facts about permanents: The proof of the van der Waerden conjecture and Brégman's Theorem. This raises:

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- In dimension 1,

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- In dimension 2,

$$(\mathcal{L}_n)^{1/n^2} = (1 + o(1))\frac{n}{e^2}.$$

- In general dimension?

...and a few words on tensors...

Let us quickly recall the notion of tensor rank. But first a brief reminder of matrix rank. A matrix A has rank one iff there exist vectors x and y such that $a_{ij} = x_i y_j$.

Proposition

The rank of a matrix M is the least number of rank-one matrices whose sum is M.

More on tensors...

All of this extends to tensors almost verbatim:
A three-dimensional tensor A has rank one iff there exist vectors x, y and z such that $a_{ijk} = x_i y_j z_k$.

Definition

The rank of a three-dimensional tensor Z is the least number of rank-one tensors whose sum is Z.

Can you believe that this question is open?

Open Problem

What is the largest rank of an $n \times n \times n$ real tensor.

It is only known (and easy) that the answer is between $\frac{n^2}{3}$ and $\frac{n^2}{2}$. With A. Shraibman we have constructed a family of examples which suggests

Conjecture (L. and Shraibman)

The answer is $(1 + o(1))\frac{n^2}{2}$

THAT'S ALL, FOLKS....