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Random Lifts of Graphs: Edge Expansion

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We continue the study of random lifts of graphs initiated in [4]. Here we study the possibility of generating graphs with high edge expansion as random lifts. Along the way, we introduce the method of ϵ-nets into the study of random structures. This enables us to improve (slightly) the known bounds for the edge expansion of regular graphs.

1. Introduction

In [4] we introduced a simple model for a random finite covering $\tilde{G} \to G$ of a fixed base graph G. (Here 'covering' is in the topological sense of covering maps, as in [8] and [9].) We recall the construction. Given a connected graph G and a natural number n (the order of the covering), we orient the edges of G arbitrarily and assign a permutation $\sigma_e \in S_n$ to each edge $e \in E(G)$ using some probability distribution over S_n (usually, we use the uniform distribution). The graph $\tilde{G}$ is formed by taking $V(\tilde{G}) = V(G) \times [n]$ and connecting (u, i) to $(v, \sigma_e(i))$ whenever $e = [u, v]$ is an oriented edge of G. There is a natural covering map from $\tilde{G}$ to G defined by mapping (v, i) to v. This naturally induces a map on the edges as well.[1] To distinguish this from other notions of 'coverings' in graphs (*e.g.*, edge covers, cycle covers), we call the random graphs generated in this manner *random lifts* of the base graph G. By a common abuse of language, a claim holds *almost surely* if the probability that it holds tends to 1 as $n \to \infty$.

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[1] Technically, this gives a *labelled* covering, but we have seen in [4] that the model is essentially equivalent to a uniform random unlabelled covering.

In this paper we try to utilize the model to generate graphs with high edge expansion. We begin by showing the existence of a lower bound on the edge expansion of random lifts of a general base graph. We then calculate an explicit bound for the case of the bouquet B_l as base graph (this is the graph with 1 vertex and l loops). Finally, using the method of ϵ-nets, we show the existence of graphs whose edge expansion exceeds the known bounds.

Specifically, we are concerned with the following problem. Given a fixed connected base graph G, what can be said about the edge expansion of a random lift $\tilde{G}$ of G? This should be compared with known results concerning edge expansion of random graphs (see [5] and the discussion closing Section 3). We prove the following.

Theorem 1.1. *Let $G = (V, E)$ be a connected graph with $|E| > |V|$. Then there is a positive constant $\xi_0 = \xi_0(G)$ such that almost every lift of G has edge expansion at least ξ_0.*

For specific base graphs we can produce explicit bounds on the edge expansion of a random lift. Let $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ be the entropy function.

Theorem 1.2. *Let ξ_0 be the smaller root of $H(\xi/d) = (d-2)/d$ where $d = 2l$. Then for every $\xi < \xi_0$, almost every lift $\tilde{G}$ of $G = B_l$ has edge expansion $\xi(\tilde{G}) \geqslant \xi$.*

It is possible to apply the same technique used for proving Theorem 1.2 for other base graphs, such as the complete graphs K_r, but the technical details are more tedious. Finally, we use the method of ϵ-nets to improve the known bounds on the edge expansion of random d-regular graphs. The same method applies, *mutatis mutandis*, for random lifts.

Theorem 1.3. *For every $d \geqslant 3$ there is an $\epsilon_0 = \epsilon_0(d)$ such that if $0 \leqslant \epsilon \leqslant \epsilon_0$ then the following holds. Let μ_0 be the larger solution of*

$$\frac{2}{d}(1 - H(\epsilon)) = 1 - H(\mu_0)$$

and let ξ be such that $\xi < d(1 - \mu - 2\epsilon)$. Then the edge expansion of a d-regular graph is almost surely at least ξ.

Throughout these sections, we shall often use the approximation

$$\binom{u}{v} = 2^{uH(v/u)(1-o_u(1))}, \tag{1.1}$$

which is valid for arbitrary v and $u \to \infty$. The inequality

$$\binom{u}{v} \leqslant 2^{uH(v/u)} \tag{1.2}$$

is valid for all u, v.

2. General graphs

We define the edge expansion of a graph as usual.

Definition. Given a set $S \subset V(G)$ of vertices in a graph G, let $E(S,\bar{S})$ be the set of edges with one vertex in S and one outside S. The *edge expansion* $\xi(S)$ is defined to be $|E(S,\bar{S})|/|S|$, and the edge expansion of G is

$$\xi(G) = \min\{\xi(S) \mid S \subset V(G), |S| \leqslant |V(G)|/2\}.$$

A lift $\tilde{G}$ of G cannot have higher edge expansion than G. Given $S \subset V(G)$ with some small $\xi(S)$, take $\tilde{S}$ to be the union of the vertex fibres $\tilde{G}_u = \{u\} \times [n]$ for $u \in S$. Then $\xi(\tilde{S}) = \xi(S)$ and $|\tilde{S}| \leqslant |V(\tilde{G})|/2$ if and only if $|S| \leqslant |V(G)|/2$.

The first natural question is whether the edge expansions of G's lifts are almost surely bounded away from 0, that is, whether there is a $\xi_0 = \xi_0(G) > 0$ such that almost every lift $\tilde{G}$ of G has edge expansion at least $\xi_0(G)$. We must leave out degenerate cases where G is a tree or unicyclic, since for such base graphs a random lift is not even a.s. connected. Having ruled out these graphs, the answer to our question is positive.

Theorem 2.1. *Let $G = (V,E)$ be a connected graph with $|E| > |V|$. Then there is a positive constant $\xi_0 = \xi_0(G)$ such that almost every lift of G has edge expansion at least ξ_0.*

As in the δ-connectivity theorem in [4], the probabilistic core of the proof lies in the following simple lemma.

Definition. Given two permutations $\sigma_1,\sigma_2 \in S_n$ and an $\eta > 0$, a set $A \subset [n]$ is called *η-bad* if $0 < |A| < 2n/3$ and $|A \cup \sigma_1(A) \cup \sigma_2(A)| \leqslant (1+\eta)|A|$.

Lemma 2.2. *There exists a positive ϵ such that, for two uniformly chosen random permutations σ_1, $\sigma_2 \in S_n$, almost surely there are no ϵ-bad sets.*

Put differently, the probability that a bad set exists tends to 0 as $n \to \infty$. Notice that this is meaningful even for sets A that are singletons: almost surely, no element is a fixed point of both permutations.

Proof. For a fixed set A of size m, the probability that A is ϵ-bad is bounded above by

$$\binom{n-m}{\lfloor \epsilon m \rfloor} \left(\frac{\binom{\lfloor (1+\epsilon)m \rfloor}{m}}{\binom{n}{m}} \right)^2,$$

and since there are $\binom{n}{m}$ possible such sets A, it suffices to show that

$$\sum_{m=1}^{2n/3} B(m) = o(1)$$

where

$$B(m) = \binom{n}{m}\binom{n-m}{\lfloor \epsilon m \rfloor}\left(\frac{\binom{\lfloor (1+\epsilon)m \rfloor}{m}}{\binom{n}{m}}\right)^2.$$

First, consider the range $n/10 < m < 2n/3$. Setting $\mu = m/n$ and using (1.1), which is valid since $m \to \infty$, we obtain (in this range we can neglect the floor brackets):

$$\frac{1}{n}\log B(m) = \left[(1-\mu)H\left(\frac{\epsilon\mu}{1-\mu}\right) + 2(1+\epsilon)\mu H\left(\frac{1}{1+\epsilon}\right) - H(\mu)\right](1+o(1)).$$

Since $\mu/(1-\mu) \leqslant 2$ and assuming $\epsilon < 1/4$, we have $H(\epsilon\frac{\mu}{1-\mu}) \leqslant H(2\epsilon)$. Using the easy inequality $H(\frac{1}{1+\epsilon}) \leqslant H(\epsilon) \leqslant H(2\epsilon)$,

$$\begin{aligned}\frac{1}{n}\log B(m) &\leqslant \left[(1-\mu)H(2\epsilon) + 3\mu H(2\epsilon) - H(\mu)\right](1+o(1)) \\ &= \left[(1+2\mu)H(2\epsilon) - H(\mu)\right](1+o(1)),\end{aligned}$$

which can certainly be made negative in the range $1/10 \leqslant \mu \leqslant 2/3$ by choosing ϵ small enough. The contribution of $B(m)$ in this range to the sum is, therefore, negligible.

In the lower range, $m \leqslant n/10$, we can bound $B(m)$ from above by

$$B(m) \leqslant \frac{\binom{n}{\lfloor \epsilon m \rfloor}5^m}{\binom{n}{m}}.$$

If $\lfloor \epsilon m \rfloor = 0$, this is $O\big(\frac{1}{\binom{n}{m}}\big)$, which is $O(1/n)$ for $m = 1$ but $o(1/n)$ for $m > 1$. Now assume $\lfloor \epsilon m \rfloor \geqslant 1$. Using $\binom{n}{k} \leqslant \left(\frac{ne}{k}\right)^k$ we get

$$\binom{n}{\lfloor \epsilon m \rfloor} \leqslant \left(\frac{ne}{\lfloor \epsilon m \rfloor}\right)^{\lfloor \epsilon m \rfloor} \leqslant \left(\frac{2ne}{\epsilon m}\right)^{\epsilon m}.$$

For the denominator we use $\binom{n}{m} \geqslant \left(\frac{n}{m}\right)^m$, so

$$B(m) \leqslant \left(\frac{5\left(\frac{2ne}{\epsilon m}\right)^{\epsilon}}{\frac{n}{m}}\right)^m = \left(5\left(\frac{2e}{\epsilon}\right)^{\epsilon}\left(\frac{n}{m}\right)^{\epsilon-1}\right)^m \leqslant \left(10\left(\frac{n}{m}\right)^{\epsilon-1}\right)^m.$$

Finally, $(n/x)^{(\epsilon-1)x}$ is decreasing for $x \leqslant n/e$, so $B(m) = o(1/n)$ throughout the range. □

We now turn to the proof of Theorem 2.1.

Proof. Fix a vertex $z \in V(G)$ and two closed walks P_1, P_2 starting and ending at z. We assume that the P_i are *independent* in the sense that each of them passes through an edge which is not in the other, and it passes through that edge only once. This is possible since G is not unicyclic (whence the cycle space of G is at least 2-dimensional), so we can first choose some edge e_1 in a cycle in G, and another edge e_2 in a cycle in $G \setminus e_1$; choosing P_i as minimal paths from z to e_i and back ensures that they satisfy the requirement.

A random lift $\tilde{G}$ of G is determined by assigning permutations in S_n to the oriented edges of G, and we let σ_i be the product of these permutations along P_i, inverting the ones assigned to edges that are traversed against their orientation. An immediate consequence of the definition of $\tilde{G}$ is that, for every $k \in [n]$, there is a unique path in $\tilde{G}$ starting at (z,k) and ending at $(z,\sigma_i(k))$ which is a lift of P_i. Call this path $W_i(k)$. From the covering property, the paths $W_i(k)$ for fixed i are disjoint.

We let $\epsilon > 0$ be as in Lemma 2.2, and assume further that $\epsilon < 1/4$. Almost surely, the permutations σ_1 and σ_2 are such that there are no ϵ-bad sets $A \subset [n]$. This follows from Lemma 2.2 and the fact that σ_1, σ_2 are indeed uniformly distributed, independent permutations (their independence follows from the independence of the paths P_i). In the rest of the proof, we show that this property of σ_1 and σ_2 ensures that $\xi(\tilde{G}) > \xi_0$ where $\xi_0 = \frac{\epsilon(1-\epsilon)}{2|V(G)|}$.

Let $T \subset V(\tilde{G})$ be a set with $0 < |T| \leqslant \frac{1}{2}|V(\tilde{G})|$. We need to show that $|E(T,\bar{T})| \geqslant \xi_0|T|$. For a vertex $v \in V(G)$, let $T_v = T \cap \tilde{G}_v$ be the part of T that lies above v, let $t_v = |T_v|$, and set $m = \max_{v\in V(G)} t_v$. Note that $|T| \leqslant m|V(G)|$.

Suppose that $t_u < (1-\epsilon)m$ for some $u \in V(G)$. Let v be any vertex for which $t_v = m$, and let Q be a $v-u$ path in G. Using again the unique lifting property of paths, there are n disjoint paths in $\tilde{G}$ starting at $\tilde{G}_v$ and ending in $\tilde{G}_u$. At least ϵm of those connect a vertex in T_v to a vertex *outside* T_u. Therefore $E(T,\bar{T})$ contains at least ϵm edges, so $\xi(T) \geqslant \epsilon/|V(G)| \geqslant \xi_0$ and we are done.

We can now assume that $t_u \geqslant (1-\epsilon)m$ for every $u \in V(G)$ (this means that T is quite evenly distributed across the vertex fibres). It follows that $m \leqslant \frac{2}{3}n$ and $t_u > 0$ for every $u \in V(G)$. Let $A = \{k \in [n] \mid (z,k) \in T_z\}$ where z is the vertex chosen above. Since $|A| = |T_z|$ we have $0 < |A| < \frac{2}{3}n$. It follows that $|A \cup \sigma_1(A) \cup \sigma_2(A)| \geqslant (1+\epsilon)|A|$. Without loss of generality, there are at least $\frac{\epsilon}{2}|A|$ indices $k \in A$ for which $\sigma_1(k) \notin A$. For such indices, the path $W_1(k)$ contains an edge in $E(T,\bar{T})$, so

$$|E(T,\bar{T})| \geqslant \frac{\epsilon}{2}|A| \geqslant \frac{\epsilon(1-\epsilon)}{2}m$$

and since $|T| \leqslant m|V(G)|$ we have $\xi(T) \geqslant \frac{\epsilon(1-\epsilon)}{2|V(G)|} = \xi_0$ as required. □

In what follows we seek more precise information on the edge expansion of random lifts. For example, let G be a connected d-regular base graph. We suspect that the behaviour of edge expansion in random lifts of G resembles that of random d-regular graphs, in the following sense. Let

$$\xi^*(d) = \limsup\{\xi(G) \mid G \text{ a } d\text{-regular graph}\}.$$

We conjecture that for every connected d-regular G, almost every lift $\tilde{G}$ of G has edge expansion

$$\xi(\tilde{G}) \geqslant \min(\xi(G), \xi^*(d)).$$

(Perhaps even equality holds.) We cannot decide this question in general, but we were able to explicitly bound from below the edge expansion of almost every lift for some specific base graphs, as will be discussed in Section 3.

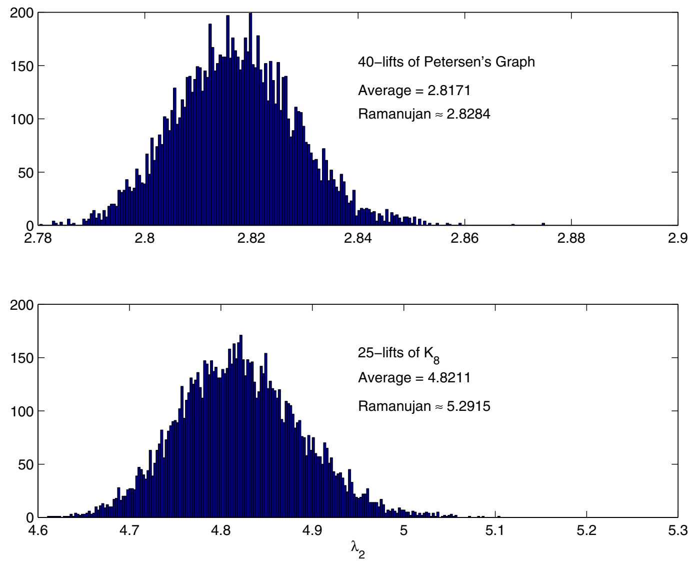

Figure 1. Histograms of λ_2 for lifts of two graphs.

2.1. The second eigenvalue

It is well known (see, *e.g.*, [2], [1]) that there are close relations between the expansion properties of a regular graph and the second-largest eigenvalue of its adjacency matrix. It is interesting to see what can be said about the spectrum of a lift, in particular random lifts. We denote the spectrum, as usual, by $d = \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_v$.

A simple observation is that if $\tilde{G}$ is a lift of G, then every eigenvalue of G is also an eigenvalue of $\tilde{G}$: any eigenvector $f(v)$ of G can be lifted to $\tilde{G}$ by setting $\tilde{f}(\tilde{v}) = f(v)$ whenever $\tilde{v} \in \tilde{G}_v$. It follows that $\lambda_2(\tilde{G}) \geqslant \lambda_2(G)$, which should be compared with the fact that a lift cannot have better edge expansion than the base graph.

In Figure 1 we present empirical results for the distribution of the second eigenvalue of lifts of Petersen's graph and the complete graph K_8. In both cases, 10^4 random lifts were generated and the second eigenvalue was numerically calculated for each one. Even though the graphs are different, there is a striking similarity between the distributions.

The average value of λ_2, compared with the so-called Ramanujan bound $2\sqrt{d-1}$, (*cf.* [10], [11], [6]), are shown for the two cases. We see that the (approximate) expectation of the second eigenvalue is below the Ramanujan bound. Recall that a d-regular graph is called *Ramanujan* if $\lambda_2 \leqslant 2\sqrt{d-1}$. A natural problem is to what extent random lifts satisfy the Ramanujan property. We hope to pursue this question, and similar ones concerning the spectra of lifts, in future papers.

3. Random lifts of B_l

Let $G = B_l$ be the pseudo-graph with one vertex and l loops. An n-lift $\tilde{G}$ of G has a single vertex fibre $V(\tilde{G}) = [n]$ and the edges are formed by choosing randomly l permutations

$\sigma_1, \ldots, \sigma_l$ and connecting k to $\sigma_i(k)$, for every $1 \leqslant i \leqslant l$ and $k \in [n]$. The result is, of course, a $d = 2l$-regular graph, possibly with loops and multiple edges. In the census of edges in $E(S, \bar{S})$ for $S \subset [n]$, we count edges according to their multiplicity. Loops are never in $E(S, \bar{S})$ so there is no question of how to count them.

In this section we present a lower bound for the edge expansion of almost every lift generated in this way.

Theorem 3.1. *Let ξ_0 be the smaller root of $H(\xi/d) = (d-2)/d$ where $d = 2l$. Then for every $\xi < \xi_0$, almost every lift $\tilde{G}$ of $G = B_l$ has edge expansion $\xi(\tilde{G}) \geqslant \xi$.*

Let $\xi_0(d)$ be the constant appearing in the theorem. One can calculate $\xi_0(4) \approx 0.440111$ and $\xi_0(6) \approx 1.04371$. As d grows, $\xi_0(d) \to d/2$ since $H^{-1}(1) = 1/2$. This is quite easily seen to be tight (see [5]). Exactly the same formula for a lower bound on the edge-expansion appears in [5] for random regular graphs. Recently, it was shown [7] that our model for random lifts of B_l and random regular graphs of degree $2l$ are contiguous models, which means that they satisfy the same asymptotic properties. The proof we present, however, is relatively simple, and can easily be adapted to base graphs other than B_l, which correspond to decidedly different models of random regular graphs.

The proof has two main parts. The first (Proposition 3.2) is a bound on the probability that a random lift has edge expansion below a given ξ. The second, given in Propositions 3.3 and 3.5, is an analytical calculation which proves that for $\xi < \xi_0$, this bound tends to 0 as the order of the lift $n \to \infty$. We start with the bound, which is essentially a simple union bound.

Proposition 3.2. *Let $P(a, \xi)$ be the probability that a random lift of B_l contains a set S of cardinality $|S| = a$ and $\xi(S) \leqslant \xi$. Then*

$$P(a, \xi) \leqslant \binom{n}{a}^{1-l} \sum_{\substack{0 \leqslant b_1, \ldots, b_l \leqslant a \\ \sum b_i \geqslant a(l-\xi/2)}} \prod_i \binom{a}{b_i} \binom{n-a}{a-b_i}. \tag{3.1}$$

Proof. Fix a subset $S \subset [n]$ of the vertices of $\tilde{G}$. We can calculate the distribution of $|E(S, \bar{S})|$ over the sample space of random lifts. Naturally, this random variable depends only on a, the size of S. For each permutation σ_i, let $b_i = |\{k \in S | \sigma_i(k) \in S\}|$ be the number of elements of S that σ_i maps into S. The number of elements in S mapped *outside* S is $a - b_i$, and the number of elements *outside* S that σ_i maps into S is also $a - b_i$, so σ_i contributes $2(a - b_i)$ to $E(S, \bar{S})$ and

$$|E(S, \bar{S})| = 2la - 2\sum_{i=1}^{l} b_i$$

whence

$$\xi(S) = 2l - \frac{2}{a}\sum_{i=1}^{l} b_i,$$

so $\xi(S) \leqslant \xi$ if and only if $\sum b_i \geqslant a(l - \xi/2)$.

The distribution of b_i is easy to calculate explicitly: the probability that $b_i = b$ is

$$\frac{\binom{a}{b}\binom{n-a}{a-b}}{\binom{n}{a}}.$$

The bound in (3.1) is simply a union bound over all possible choices of such b_i and choices of S of size a. □

The probability $P(\xi)$ that the lift has edge expansion smaller than ξ is therefore bounded by

$$P(\xi) \leqslant \sum_{a=1}^{n/2} B(a),$$

where $B(a)$ is the right-hand side of (3.1). Let us set

$$\lambda = 1 - \frac{\xi}{2l} = 1 - \frac{\xi}{d} \tag{3.2}$$

$$Q(a, b_1, \ldots, b_l) = \binom{n}{a}^{1-l} \prod_i \binom{a}{b_i}\binom{n-a}{a-b_i}, \tag{3.3}$$

so

$$B(a) = \sum_{\substack{0 \leqslant b_1, \ldots, b_l \leqslant a \\ \frac{1}{l}\sum b_i \geqslant \lambda a}} Q(a, b_1, \ldots, b_l). \tag{3.4}$$

We seek the maximal ξ_0 that guarantees $P(\xi) \to 0$ as $n \to \infty$ whenever $\xi < \xi_0$. As in the proof of Lemma 2.2, we need to treat separately the small sets $a < \delta n$ and the large ones $\delta n \leqslant a \leqslant n/2$. Here δ is some small constant that will be determined later (in fact, $\delta = 1/10^5$ will suffice). For convenience, set

$$\begin{aligned} \sum_{a=1}^{n/2} B(a) &= \sum_{a=1}^{\delta n} B(a) + \sum_{a=\delta n}^{\frac{1}{2}n} B(a) \\ &= B_1 + B_2. \end{aligned}$$

In the following two propositions we prove $B_2 \to 0$ and $B_1 \to 0$ for the required ξ.

Proposition 3.3. *If $\xi < \xi_0$ then $B_2 \to 0$.*

Proof. Since the number of summands in B_2 is $O(n)$, and the number of summands in $B(a)$ is also polynomial in n, it suffices to prove an exponentially small upper bound on $Q(a, b_1, \ldots, b_l)$ in the range

(1) $\delta n < a \leqslant \frac{1}{2}n$,
(2) $0 \leqslant b_i \leqslant a$,
(3) $\frac{1}{l}\sum_{i=1}^{l} b_i \geqslant \lambda a$.

Using (1.1) and (1.2) we obtain

$$\frac{1}{n}\log_2 Q(a, b_1, \ldots, b_l) \leqslant q(\alpha, \beta_1, \ldots, \beta_l) + o(1) \tag{3.5}$$

where $\alpha = a/n, \beta_i = b_i/n$ and

$$q(\alpha, \beta_1, \dots, \beta_l) = (1-l)H(\alpha) + \sum_{i=1}^{l} \left(\alpha H\left(\frac{\beta_i}{\alpha}\right) + (1-\alpha) H\left(\frac{\alpha - \beta_i}{1-\alpha}\right)\right). \tag{3.6}$$

If q is negative then $Q \to 0$ exponentially as required. The following lemma determines the points where q is maximized, which enables us to prove that $q < 0$ in the required range, for the specified ξ.

Lemma 3.4. *Let $\Delta = \Delta_\lambda \subset \mathbb{R}^{l+1}$ be the region determined by*

(1) $0 \leqslant \alpha \leqslant 1/2$,

(2) $0 \leqslant \beta_i \leqslant \alpha$,

(3) $\frac{1}{l}\sum_{i=1}^{l} \beta_i \geqslant \lambda\alpha$,

with $l \geqslant 2$ and $1/2 < \lambda$. The function q given by (3.6) attains its global maximum in Δ_λ either when $\alpha = 1/2$ and $\beta_i = \lambda/2$ for all i, or when $\alpha = 0$ and $\beta_i = 0$ for all i.

Proof. The concavity of H implies immediately that if $\bar{\beta} = (\sum \beta_i)/l$ then

$$q(\alpha, \bar{\beta}, \dots, \bar{\beta}) \geqslant q(\alpha, \beta_1, \dots, \beta_l),$$

so we may assume that all β_i are equal, $\beta_i = \beta$ where $\beta \geqslant \lambda\alpha$. Note that $(\alpha, \beta_1, \dots, \beta_l) \in \Delta$ implies $(\alpha, \bar{\beta}, \dots, \bar{\beta}) \in \Delta$.

We fix α and optimize over $\alpha \geqslant \beta \geqslant \lambda\alpha$. Now

$$\frac{\partial}{\partial \beta} q(\alpha, \beta, \dots, \beta) = l \log\left(\frac{\alpha - \beta}{\beta}\right) - l \log\left(\frac{1 - 2\alpha + \beta}{\alpha - \beta}\right) = l \log\left(\frac{(\alpha - \beta)^2}{\beta(1 - 2\alpha + \beta)}\right).$$

This expression is always negative, because $\beta \geqslant \lambda\alpha \geqslant \frac{1}{2}\alpha \geqslant \alpha^2$ since $\lambda \geqslant 1/2 \geqslant \alpha$. Therefore, to maximize q, β needs to be as small as possible, namely $\beta = \lambda\alpha$. We are left with the univariate function $f(\alpha) = q(\alpha, \lambda\alpha, \dots, \lambda\alpha)$. Explicitly, f is given by

$$f(\alpha) = (1-l)H(\alpha) + l\alpha H(\lambda) + l(1-\alpha) H\left((1-\lambda)\frac{\alpha}{1-\alpha}\right),$$

and its derivatives are

$$f'(\alpha) = (1-l)\log\frac{1-\alpha}{\alpha} + lH(\lambda) - lH\left((1-\lambda)\frac{\alpha}{1-\alpha}\right) + \frac{l(1-\lambda)}{1-\alpha}\log\frac{1 - 2\alpha + \lambda\alpha}{(1-\lambda)\alpha}$$

and

$$f''(\alpha) = \log(e)\frac{\lambda l - 1 - \alpha(l-1)(2-\lambda)}{\alpha(1-\alpha)(1-2\alpha+\lambda\alpha)}.$$

Notice that $f''(\alpha) = 0$ has at most one root, so f' has at most a single extremum. Further,

$$\lim_{\alpha \to 0^+} f'(\alpha) = -\infty$$

since $l\lambda > 1$, while

$$f'\left(\frac{1}{2}\right) = 2l(1-\lambda)\log\frac{\lambda}{1-\lambda} > 0.$$

It follows that $f'(\alpha) = 0$ has exactly one root in the range $0 \leqslant \alpha \leqslant 1/2$, so $f(\alpha)$ has a unique minimum in that range, and the maximal value of f is achieved either at $\alpha = 0$ or $\alpha = 1/2$ as claimed. □

Returning to the proof of Proposition 3.3, we set $\lambda_0 = 1 - \frac{\xi_0}{2l}$. Then $\lambda_0 > 1/2$, the lemma applies and at the nonzero critical point we have

$$q(1/2, \lambda_0/2, \ldots, \lambda_0/2) = (1-l) + lH(\lambda_0) = 0.$$

Since $\lambda > \lambda_0$, $q(1/2, \lambda/2, \ldots, \lambda/2) < 0$. We are assuming here $a \geqslant \delta n$, namely $\alpha \geqslant \delta$, so q is bounded away from 0 throughout the whole range, and Q is exponentially small as required. □

This proof cannot be applied to $a = o(n)$ because $q(0, 0, \ldots, 0) = 0$. For this reason we have to deal separately with B_1. In fact, small sets a.s. have larger edge expansion than that given by Theorem 1.2.

Proposition 3.5.

(1) *If $l = 2$ and $\xi < 1$ then $B_1 \to 0$.*

(2) *If $l \geqslant 3$ and $\xi < l$ then $B_1 \to 0$.*

Proof. Fix a in the range $1 \leqslant a < \delta n$. Call $\vec{b} = (b_1, \ldots, b_l)$ *admissible* if $0 \leqslant b_i \leqslant a$ and $\frac{1}{l}\sum b_i \geqslant \lambda a$. Suppose $\vec{b} = (b_1 + 1, b_2, \ldots, b_l)$ is admissible and $b_1 \geqslant b_2 + 1$. Then $\vec{b}' = (b_1, b_2 + 1, \ldots, b_l)$ is also admissible, and since

$$\frac{Q(a, b_1, b_2 + 1, \ldots, b_l)}{Q(a, b_1 + 1, b_2, \ldots, b_l)} = \frac{b_1 + 1}{b_2 + 1} \cdot \frac{n - 2a + b_1 + 1}{n - 2a + b_2 + 1} \cdot \frac{(a - b_2)^2}{(a - b_1)^2}$$

we have $Q(a, \vec{b}') > Q(a, \vec{b})$. Starting from any $\vec{b}$ and iterating this procedure we can find a $\vec{b}'$ with $Q(a, \vec{b}') > Q(a, \vec{b})$ and (up to rearrangements) $\vec{b}' = (b_0, b_0, \ldots, b_0 + 1, \ldots, b_0 + 1)$ where $b_0 + j/l \geqslant \lambda a$ for some $0 \leqslant j < l$.

Let $\vec{b}_0 = (b_0, b_0, \ldots, b_0)$. A similar calculation shows that (for large n) $Q(a, \vec{b}_0) > Q(a, \vec{b}')$. It is possible that $\vec{b}_0$ is not admissible, but it is not far – in fact,

$$b_0 \geqslant \lambda a - \frac{l-1}{l} \geqslant \lambda a - 1. \tag{3.7}$$

We have shown that, given a, there exists an integer b_0 satisfying (3.7) so that

$$Q(a, b_1, \ldots, b_l) < Q(a, b_0, \ldots, b_0)$$

for every admissible $(b_1, \ldots, b_l)$. It follows that

$$B(a) \leqslant (a+1)^l Q(a, b_0, \ldots, b_0) = (a+1)^l \binom{n}{a}^{1-l} \left(\binom{a}{b_0}\binom{n-a}{a-b_0}\right)^l = C(a, b_0).$$

Assume first that $l \geqslant 3$. Here $\xi < l$ so $\lambda > 1/2$. Let us consider even values of a; the odd case is similarly handled. Since b_0 is an integer greater than $\lambda a - 1$ we have $b_0 \geqslant a/2$. Decreasing b_0 causes $C(a, b_0)$ to increase, so we may take $b_0 = a/2$.

Set $f(a) = C(a, a/2)$. This is decreasing as a function of l, so it is enough to consider the case $l = 3$. We have

$$f(a) = \frac{(a+1)^3 \binom{a}{a/2}^3 \binom{n-a}{a/2}^3}{\binom{n}{a}^2}.$$

If δ is small enough, then $f(a) \geqslant f(a+2)$ for $1 \leqslant a \leqslant \delta n$. This follows from a straightforward calculation:

$$\begin{aligned}
\frac{f(a+2)}{f(a)} &= \frac{(a+3)^3(a+1)^5(a+2)^5(n-\frac{3}{2}a)^3(n-\frac{3}{2}a-1)^3(n-\frac{3}{2}a-2)^3}{(a+1)^3(a/2+1)^9(n-a)^5(n-a-1)^5} \\
&\leqslant 2^9(1+\frac{1}{a})^4(a+3)\frac{(n-\frac{3}{2}a)^9}{(n-a-1)^{10}} \\
&\leqslant 2^{13}\frac{a+3}{n-a-1} \\
&\leqslant 2^{13}\frac{\delta+3/n}{1-\delta-1/n} \leqslant 1.
\end{aligned}$$

Finally, $f(2) = O(1/n)$ and $f(4) = O(1/n^2)$ so $\sum_1^{\delta n} B(a) = O(1/n)$.

The case $l = 2$ is done in the same manner. The difference is that $C(a, a/2)$ is not decreasing in this case, but $C(a, 3a/4)$ is. Since we just assumed $\xi < 1$ for $l = 2$, then indeed $\lambda > 3/4$ and $b_0 \geqslant 3a/4 - 1/2$ and we proceed as before. □

We close this section with some remarks concerning other base graphs. Let $G = (V, E)$ be a connected graph. Trying to obtain tighter lower bounds on the edge expansion of lifts of G (tighter than the one obtained from the proof of Theorem 2.1), we can use a similar method as the one we used for B_l.

To do this, we need a bound on the probability that a set in the lift has many internal edges (*i.e.*, small expansion). It is easy to write such a bound, similar to equation (3.1), using variables a_v to parametrize the size of the set in each fibre $\tilde{G}_v$ (instead of the single a in (3.1)) and b_{uv} for counting the edges connecting vertices in the fibre $\tilde{G}_u$ to vertices in $\tilde{G}_v$. It is then necessary to find the critical value ξ_0 of ξ for which the bound tends to 0.

Natural candidates for ξ_0 are obtained by setting $a_v = n/2$ and $b_{uv} = b_0$ on the one hand, or else taking $a_v = 0$ for all $v \in V_0$ and $a_v = 1$ for $v \in V \setminus V_0$.

We have been able to carry out this program for the complete graphs K_r and the results are, once again, the same as those obtained for random regular graphs. It should be noted that a random lift of K_r is different from a random $(r-1)$-regular graph – for example, a random regular graph is not expected to cover a complete graph.

4. An improved bound through ϵ-nets

We now digress from the subject of random lifts to show how the method of ϵ-nets can be applied to improve the lower bound on the edge expansion of regular graphs. We demonstrate the method in the standard model for random regular graphs, although the same can be done for random lifts. To prepare the ground, we recall briefly the proof of a lower bound on the edge expansion of random regular graphs from [5].

4.1. Random regular graphs

The standard probability space used to randomly generate regular graphs is the collection of perfect matchings among nd vertices, grouped into n clusters of size d. By shrinking each cluster into a single vertex, we get (with probability bounded away from zero for fixed d and growing n) a random d-regular graph G on n vertices.

Let X be the set of nd vertices. For $0 \leqslant \alpha \leqslant 1/2$ let $\mathcal{S}_\alpha$ be the collection of subsets $A \subset X$ of size αnd that are unions of clusters. These are the ones that yield sets of size αn in G. The probability that $A \in \mathcal{S}_\alpha$ contains exactly βnd vertices that are paired to each other in the perfect matching is

$$p(\alpha,\beta) = \binom{\alpha nd}{\beta nd}\binom{(1-\alpha)nd}{(\alpha-\beta)nd}((\alpha-\beta)nd)!\frac{N(\beta nd)N((1-2\alpha+\beta)nd)}{N(nd)},$$

where $N(2v) = \frac{(2v)!}{2^v v!}$ is the number of perfect matchings of $2v$ vertices. If β is so large so that the union bound $|\mathcal{S}_\alpha| p(\alpha,\beta) = o(1)$, then sets of size αn with small expansion will not emerge in G. The relation between β and the edge expansion ξ is

$$\xi = d\left(1 - \frac{\beta}{\alpha}\right). \tag{4.1}$$

To calculate the union bound $U = |\mathcal{S}_\alpha| p(\alpha,\beta)$ we note that $|\mathcal{S}_\alpha| = \binom{n}{\alpha n}$, and rewriting $p(\alpha,\beta)$ we obtain[2]

$$U = \frac{\binom{n}{\alpha n}}{\binom{nd}{\alpha nd}} 2^{(\alpha-\beta)nd}\binom{nd/2}{\beta nd/2,\, 2(\alpha-\beta)nd/2}$$

and

$$\frac{1}{n}\log U = \left(H(\alpha)(1-d) + (\alpha-\beta)d + \frac{d}{2}H(\beta, 2(\alpha-\beta))\right)(1-o(1)).$$

It turns out that $\alpha = 1/2$ is the extremal case (we will prove this in the next section), and so the critical β is determined by

$$H(2\beta) = \frac{d-2}{d},$$

which yields $H(\xi/d) = \frac{d-2}{d}$ since $\xi/d = 1 - 2\beta$ by (4.1). This is the same expression we obtained in Theorem 1.2.

4.2. ϵ-nets

At the heart of the above proof lies the union bound $|\mathcal{S}_\alpha| p(\alpha,\beta)$. This argument fails, however, to take into account that if a certain set A has high expansion (low β), then

[2] We denote $\frac{x!}{y!z!(x-y-z)!}$ as $\binom{x}{y,z}$. Correspondingly, we define $H(x,y) = -x\log x - y\log y - (1-x-y)\log(1-x-y)$.

other sets that have large intersection with A tend to expand, too. We wish somehow to exploit this fact. Let us first outline the method, in a general probability space.

The general idea. In the usual setting of the union bound approach, we are given a family $\{A_i\}_{i\in I}$ of events and we wish to estimate $\Pr(\cup A_i)$. Suppose that ρ is some metric on I such that it is possible to estimate $\Pr(A_i \triangle A_j)$ in terms of $\rho(i,j)$. It is then natural to define an ϵ-net in I, namely a set $J \subset I$ with the property that, for every $i \in I$, there is some $j \in J$ such that $\rho(i,j) \leqslant \epsilon$.

Applying a union bound to the events $\{A_j\}_{j\in J}$ may yield better results, since there are fewer events in the ϵ-net J than in the whole of I. It is then possible to apply these results to all events in I using the information on $\Pr(A_i \triangle A_j)$.

In our case, let us fix an $\epsilon > 0$ and say that two sets $A_1, A_2 \in \mathscr{S}_\alpha$ are *ϵ-close* if

$$|A_1 \cap A_2| \geqslant (1-\epsilon)\alpha n d = (1-\epsilon)|A_1|. \tag{4.2}$$

Now let $\mathscr{N}_\alpha = \mathscr{N}_\alpha(\epsilon)$ be a subcollection of $\mathscr{S}_\alpha$ with the property that, for every $A \in \mathscr{S}_\alpha$, there is some $N \in \mathscr{N}_\alpha$ such that N is close to A. This is the ϵ-net. Since $\mathscr{N}_\alpha$ is smaller than $\mathscr{S}_\alpha$, the union bound implies that the condition $|\mathscr{N}_\alpha| p(\alpha, \beta') = o(1)$ already holds for some $\beta' < \beta$. That is, every $N \in \mathscr{N}_\alpha$ contains at most $\beta' nd$ vertices that are paired among themselves. This means that sets in the net have edge expansion $\xi = d(1-\beta'/\alpha)$.

By the large intersection property (4.2), every $A \in \mathscr{S}_\alpha$ has fewer than $\beta'' nd$ self-paired vertices, for some β'' which is slightly larger than β' but still smaller than β. We are left with calculating β' and β'', and then optimizing ϵ to get the best lower bound possible in this context.

We estimate β' as before, using the same formula for $p(\alpha,\beta)$. We only need to estimate the size of $\mathscr{N}_\alpha(\epsilon)$. For this we note that for every $A \in \mathscr{S}_\alpha$, the number of sets B that are ϵ-close to A is at least

$$D = \binom{\alpha n}{\epsilon \alpha n}\binom{n(1-\alpha)}{\epsilon \alpha n}.$$

By a standard probabilistic argument (see, for example, [3, Theorem 2.2, Chapter 1]), we can find an ϵ-net with size

$$|\mathscr{N}_\alpha| = |\mathscr{S}_\alpha| \frac{1+\log(D+1)}{D+1}$$

so

$$\frac{1}{n}\log|\mathscr{N}_\alpha| \leqslant \left(H(\alpha) - \left(\alpha H(\epsilon) + (1-\alpha) H\left(\epsilon \frac{\alpha}{1-\alpha}\right)\right)\right)(1-o(1)).$$

A noticeable feature of this formula is that at $\epsilon = 0$, the derivative of $\log|\mathscr{N}_\alpha|$ with respect to ϵ is infinitely negative, so that the 'immediate gain' from introducing the ϵ-net is infinite.

Given the bound β' on self-paired vertices in members of $\mathscr{N}_\alpha$, we need to bound from above the number of such vertices in a set $A \in \mathscr{S}_\alpha$. Given A, let $N \in \mathscr{N}_\alpha$ be a set ϵ-close to A. Obviously, the worst case is when A contains all the self-paired vertices of N, and in addition all the vertices in $A \setminus N$ are paired to ones in N (see Figure 2). This gives

$$\beta'' \leqslant \beta' + 2\alpha\epsilon.$$

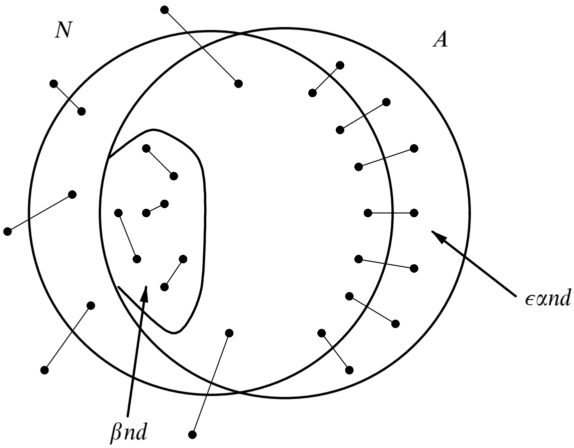

Figure 2. The worst case when moving from N to A.

It is convenient to change variables and replace β by $\mu = \beta/\alpha$. To summarize, we have the following result.

Proposition 4.1. *Let $d \geqslant 3$ and $\epsilon > 0$. Define, for $0 < \alpha \leqslant 1/2$ and $0 < \mu \leqslant 1$,*

$$\begin{aligned} g_1(\alpha) &= H(\alpha) - \alpha H(\epsilon) - (1-\alpha)H\left(\frac{\epsilon\alpha}{1-\alpha}\right), \\ g_2(\alpha,\mu) &= -dH(\alpha) + \alpha(1-\mu)d + \frac{d}{2}H(\alpha\mu, 2\alpha(1-\mu)), \\ g(\alpha,\mu) &= g_1(\alpha) + g_2(\alpha,\mu), \end{aligned}$$

and let μ_0 be such that $g(\alpha,\mu) < 0$ for every $\mu_0 < \mu \leqslant 1$ and $0 < \alpha \leqslant 1/2$. Then the edge expansion of a random d-regular graph is, almost surely, at least ξ for any $\xi < d(1-\mu_0-2\epsilon)$.

Proof. For a given ϵ, g_1 bounds the (logarithm of the) size of the ϵ-net, and g_2 bounds the probability that poorly expanding sets exist in the net. If their sum is negative, a random regular graph contains a net in which each set has edge expansion at least $d(1-\mu)$ by (4.1), and so every set in the graph has edge expansion at least $d(1-\mu-2\epsilon)$. □

We now determine μ_0 explicitly.

Theorem 4.2. *For every $d \geqslant 3$ there is an $\epsilon_0 = \epsilon_0(d)$ such that if $0 \leqslant \epsilon \leqslant \epsilon_0$ then the following holds. Let μ_0 be the larger solution of*

$$\frac{2}{d}(1 - H(\epsilon)) = 1 - H(\mu_0) \tag{4.3}$$

then the edge expansion of a d-regular graph is almost surely at least ξ whenever $\xi < d(1-\mu_0-2\epsilon)$.

Notice that for $\epsilon = 0$, this gives the same edge expansion as Theorem 1.2. Also note that taking the larger solution implies $\mu_0 \geqslant 1/2$.

Proof. Define g_1, g_2 and g as in Proposition 4.1. Equation (4.3) is just the requirement that $g(\frac{1}{2}, \mu_0) = 0$. The point of the theorem is that, indeed, $\alpha = \frac{1}{2}$ is the critical case. We need to show that if μ_0 is defined by (4.3) then for any $0 < \alpha \leqslant 1/2$ and $\mu_0 < \mu \leqslant 1$ we have $g(\alpha, \mu) < 0$. First, for any fixed α, the function $g(\alpha, \mu)$ has a unique maximum at $\mu = \alpha$. This follows from

$$\frac{\partial g}{\partial \mu} = \frac{\partial g_2}{\partial \mu} = -\alpha d + \frac{1}{2}\alpha d \log \frac{(2\alpha(1-\mu))^2}{\alpha\mu(1-2\alpha+\alpha\mu)},$$

which is easily seen to be strictly negative for $\mu > \alpha$ and positive otherwise. It therefore suffices to show that $f(\alpha) = g(\alpha, \mu_0) \leqslant 0$. We have $f(0^+) = 0$ and $f(1/2) = 0$ by definition. Also, $f'(0^+) = -\infty$ and

$$f'(1/2) = -2\epsilon \log \frac{1-\epsilon}{\epsilon} + d(1-\mu_0)\log \frac{\mu_0}{1-\mu_0},$$

which is positive for ϵ small enough. Therefore, if f is not negative throughout the interval $0 \leqslant \alpha \leqslant 1/2$, it must have at least three extremal points, so f'' must have at least two zeros. We have

$$f''(\alpha) = \log_2(e)\frac{A\alpha^2 + B\alpha + C}{2\alpha(1-\alpha)(1-\alpha-\alpha\epsilon)(1-\alpha(2-\mu))},$$

where

$$\begin{aligned} A &= (d-2)(2-\mu_0)(1+\epsilon), \\ B &= 6 - 2(d+\mu_0) - \epsilon((d-2)\mu_0 + 2), \\ C &= d\mu_0 - 2 + 2\epsilon. \end{aligned}$$

The key point is that for $\epsilon = 0$, the numerator vanishes at $\alpha = 1$ which is also a root of the denominator. Therefore the numerator becomes a linear function of α for $\epsilon = 0$, and in particular can only have a single zero. It follows that f'' has a single root for ϵ small enough. We note that, when applying this result, one needs to check that the negativity of f holds for the desired ϵ. □

To find the best ϵ we optimize $2\epsilon + \mu_0$ under the constraint (4.3). Using Mathematica, we find that for $d = 3$ the optimum is $\epsilon \approx 0.00029082$, which gives $\xi \approx 0.184682$, compared with 0.184471 obtained without ϵ-nets (as in [5]), an improvement of about 0.1%. It is not hard to verify numerically that this ϵ is small enough to satisfy the negativity requirement for f in the above proof.

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