Random Lifts of Graphs

Nati Linial

27th Brazilian Math Colloquium, July ’09
Plan of this talk

- A brief introduction to the probabilistic method.
- A quick review of expander graphs and their spectrum.
- Lifts, random lifts and their properties.
- Spectra of random lifts.
What is the probabilistic method? Introduction by example.

**Theorem**

*In every party of 6 people there are either 3 who know each other or 3 who are strangers to each other.*
What is the probabilistic method?

Introduction by example.

Theorem

In every party of 6 people there are either 3 who know each other or 3 who are strangers to each other.

In other words: If you color the edges of $K_6$ (the complete graph on 6 vertices) blue and red, you necessarily find a monochromatic (either red or blue triangle.)
The result is tight. There is a red-blue coloring of the edges of $K_5$ with no monochromatic triangle.
The result is tight. There is a red-blue coloring of the edges of $K_5$ with no monochromatic triangle.
More generally,

**Theorem (Ramsey; Erdős-Szekeres)**

Let $N = \binom{r+s-2}{r-1}$. If you color the edges of $K_N$ red and blue, then you necessarily get a red $K_r$ or a blue $K_s$. 
Diagonal Ramsey Numbers

Theorem

In every red-blue coloring of the edges of $K_N$ there is a monochromatic complete subgraph on at least

$$\frac{1}{2} \log_2 N \text{ vertices.}$$
Diagonal Ramsey Numbers

Theorem

In every red-blue coloring of the edges of $K_N$ there is a monochromatic complete subgraph on at least

$$\frac{1}{2} \log_2 N$$

vertices.

Theorem

Every $N$-vertex graph contains either a clique or an anti-clique on at least

$$\frac{1}{2} \log_2 N$$

vertices.
Theorem (Erdős ’49)

There are red-blue colorings of $K_N$ where no monochromatic subgraph has more than

$$2 \log_2 N \text{ vertices.}$$
Diagonal Ramsey Numbers (contd.)

Theorem (Erdős '49)

There are red-blue colorings of $K_N$ where no monochromatic subgraph has more than

$$2 \log_2 N \text{ vertices.}$$

Theorem

There are $N$-vertex graphs where no clique or anti-clique has more than

$$2 \log_2 N \text{ vertices.}$$
How can you prove such a statement?

One may expect that (like the coloring of $K_5$ we saw before) I would show you now a method of coloring that has no large monochromatic subgraphs.
How can you prove such a statement?

One may expect that (like the coloring of $K_5$ we saw before) I would show you now a method of coloring that has no large monochromatic subgraphs.

We do not know how to do this. In fact it is a major challenge to find such explicit colorings.
How can you prove such a statement?

One may expect that (like the coloring of $K_5$ we saw before) I would show you now a method of coloring that has no large monochromatic subgraphs.

We do not know how to do this. In fact it is a major challenge to find such explicit colorings. Instead, we use the probabilistic method.
Introducing the probabilistic method

There are $2^{\binom{N}{2}}$ ways to color the edges of $K_N$ by red and blue. We think of them as a probability space $\Omega$ with the uniform distribution.
There are $2^{\binom{N}{2}}$ ways to color the edges of $K_N$ by red and blue. We think of them as a probability space $\Omega$ with the uniform distribution.

In other words, we give the following recipe for sampling from $\Omega$: For each edge of $K_N$, flip a coin (independently from the rest). If it comes out heads color the edge red if you get tails, color it blue.
Let us consider an integer $r$ (to be determined later) and a random variable $B$ defined on $\Omega$. For a given coloring $C$ of $K_N$, we define $B(C)$ to be the number of sets of $r$ vertices in $C$ all of whose edges are blue. The expectation of $X$ is:

$$\left(\binom{N}{r}\right)\frac{1}{2^\binom{r}{2}}.$$
Let us consider an integer $r$ (to be determined later) and a random variable $B$ defined on $\Omega$. For a given coloring $C$ of $K_N$, we define $B(C)$ to be the number of sets of $r$ vertices in $C$ all of whose edges are blue. The expectation of $X$ is:

$\binom{N}{r} \frac{1}{2^\binom{r}{2}}.$

We likewise define a random variable $R$ that counts red subgraphs.
Note that if $B(C) = R(C) = 0$, there is no monochromatic set of $r$ vertices in $C$, which is just what we need.
Note that if $B(C) = R(C) = 0$, there is no monochromatic set of $r$ vertices in $C$, which is just what we need.

If the sum of the expectations

$$\mathbb{E}(R) + \mathbb{E}(B) < 1.$$ 

Then a coloring $C$ exists with no monochromatic set of $r$ vertices. An easy calculation yields that this holds for
Note that if $B(C) = R(C) = 0$, there is no monochromatic set of $r$ vertices in $C$, which is just what we need.

If the sum of the expectations

$$E(R) + E(B) < 1.$$ 

Then a coloring $C$ exists with no monochromatic set of $r$ vertices. An easy calculation yields that this holds for

$$r = 2 \log_2 N.$$
What has happened here?

Why does the probabilistic method work so well?
What has happened here?

Why does the probabilistic method work so well?

In every mathematical field intuition is created from examples that we know.
What has happened here?

Why does the probabilistic method work so well?

In every mathematical field intuition is created from examples that we know. But it is hard to analyze large specific examples and the probabilistic method allows us to bypass this difficulty. It serves us as an observational tool, much like the astronomer’s telescope or a biologist’s microscope.
What we saw is a close relative of the most basic model of random graphs, Erdős-Rényi’s $G(n, p)$ model. In this model we start with $n$ vertices. For each pair of vertices $x, y$ we decide, independently and with probability $p$, to put an edge between $x$ and $y$. 
What we saw is a close relative of the most basic model of random graphs, Erdős-Rényi’s $G(n, p)$ model. In this model we start with $n$ vertices. For each pair of vertices $x, y$ we decide, independently and with probability $p$, to put an edge between $x$ and $y$.

There are other important and interesting models of graphs. For example, we have known for 30 years now how to sample random $d$-regular graphs.
Random graphs come up a lot in (mathematical) statistical mechanics. A major example is Percolation Theory.
Random graphs come up a lot in (mathematical) statistical mechanics. A major example is Percolation Theory.

Start e.g., from the graph of the $d$-dimensional lattice. Maintain every edge with probability $p$, and discard it with probability $1-p$. (Independently over edges).
Random graphs come up a lot in (mathematical) statistical mechanics. A major example is Percolation Theory.

Start e.g., from the graph of the $d$-dimensional lattice. \textbf{Maintain} every edge with probability $p$, and discard it with probability $1 - p$. (Independently over edges).

A typical basic question in this area: What is the probability that an infinite connected component remains.
You could also seek models to describe natural or artificial phenomena such as the Internet graph or biological control networks.
In particular, one shortcoming of the $G(n, p)$ model is the lack of control we have over the graph’s structure.
In particular, one shortcoming of the $G(n, p)$ model is the lack of control we have over the graph’s structure.

We want ”more structured” models of random graphs.
In particular, one shortcoming of the $G(n, p)$ model is the lack of control we have over the graph’s structure.

We want ”more structured” models of random graphs.

Random lifts of graphs, our subject today, are such a model.
A very quick review on expansion in graphs

There are three main perspectives of expansion:

- Combinatorial - isoperimetric inequalities
- Linear Algebraic - spectral gap
- Probabilistic - Rapid convergence of the random walk (which we do not discuss today)

For (much) more on this: see our survey article with Hoory and Wigderson.
A graph $G = (V, E)$ is said to be $\epsilon$-edge-expanding if for every partition of the vertex set $V$ into $X$ and $X^c = V \setminus X$, where $X$ contains at most a half of the vertices, the number of cross edges

$$e(X, X^c) \geq \epsilon|X|.$$
A graph $G = (V, E)$ is said to be \( \epsilon \)-edge-expanding if for every partition of the vertex set $V$ into $X$ and $X^c = V \setminus X$, where $X$ contains at most a half of the vertices, the number of cross edges

\[
e(X, X^c) \geq \epsilon |X|.
\]

In words: in every cut in $G$, the number of cut edges is at least proportionate to the size of the smaller side.
The edge expansion ratio of a graph $G = (V, E)$, is

$$h(G) = \min_{S \subseteq V, |S| \leq |V|/2} \frac{|E(S, \overline{S})|}{|S|}.$$
The linear-algebraic perspective

The **Adjacency Matrix** of an $n$-vertex graph $G$, denoted $A = A(G)$, is an $n \times n$ matrix whose $(u, v)$ entry is the number of edges in $G$ between vertex $u$ and vertex $v$. Being real and symmetric, the matrix $A$ has $n$ real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. 

Nati Linial
Random Lifts of Graphs
Simple things that the spectrum of $A(G)$ tells about $G$

- If $G$ is $d$-regular, then $\lambda_1 = d$. In the corresponding eigenvector all coordinates are equal.
Simple things that the spectrum of $A(G)$ tells about $G$

- If $G$ is $d$-regular, then $\lambda_1 = d$. In the corresponding eigenvector all coordinates are equal.

- The graph is connected iff $\lambda_1 > \lambda_2$. We call $\lambda_1 - \lambda_2$ the spectral gap.
Simple things that the spectrum of $A(G)$ tells about $G$

- If $G$ is $d$-regular, then $\lambda_1 = d$. In the corresponding eigenvector all coordinates are equal.
- The graph is connected iff $\lambda_1 > \lambda_2$. We call $\lambda_1 - \lambda_2$ the spectral gap.
- The graph is bipartite iff $\lambda_1 = -\lambda_n$. 

Nati Linial
Random Lifts of Graphs
Simple things that the spectrum of $A(G)$ tells about $G$

- If $G$ is $d$-regular, then $\lambda_1 = d$. In the corresponding eigenvector all coordinates are equal.
- The graph is connected iff $\lambda_1 > \lambda_2$. We call $\lambda_1 - \lambda_2$ the spectral gap.
- The graph is bipartite iff $\lambda_1 = -\lambda_n$.
- $\chi(G) \geq -\frac{\lambda_1}{\lambda_n} + 1$. 

A substantial spectral gap implies logarithmic diameter.
Simple things that the spectrum of $A(G)$ tells about $G$

- If $G$ is $d$-regular, then $\lambda_1 = d$. In the corresponding eigenvector all coordinates are equal.
- The graph is connected iff $\lambda_1 > \lambda_2$. We call $\lambda_1 - \lambda_2$ the spectral gap.
- The graph is bipartite iff $\lambda_1 = -\lambda_n$.
- $\chi(G) \geq -\frac{\lambda_1}{\lambda_n} + 1$.
- A substantial spectral gap implies logarithmic diameter.
Theorem

Let $G$ be a $d$-regular graph with spectrum $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{(d + \lambda_2)(d - \lambda_2)}.$$  

The bounds are tight.
What’s a ”large” spectral gap?

If expansion is “good” and if a large spectral gap yields large expansion, then it’s natural to ask:

Question

How small can $\lambda_2$ be in a $d$-regular graph? (i.e., how large can the spectral gap get?)

Theorem (Alon, Boppana)

$\lambda_2 \geq 2\sqrt{d - 1} - o(1)$
What’s a ”large” spectral gap?

If expansion is “good” and if a large spectral gap yields large expansion, then it’s natural to ask:

**Question**

*How small can $\lambda_2$ be in a $d$-regular graph? (i.e., how large can the spectral gap get)?*
What’s a ”large” spectral gap?

If expansion is “good” and if a large spectral gap yields large expansion, then it’s natural to ask:

**Question**

*How small can $\lambda_2$ be in a $d$-regular graph? (i.e., how large can the spectral gap get)?*

**Theorem (Alon, Boppana)**

$$\lambda_2 \geq 2\sqrt{d - 1} - o(1)$$
The meaning of the number $2\sqrt{d-1}$

A good approach to extremal problems is to come up with a candidate for an ideal example, and show that there are no better instances.
A good approach to extremal problems is to come up with a candidate for an ideal example, and show that there are no better instances.

What, then, is the ideal expander? A good candidate is the infinite $d$-regular tree. Using (a little) spectral theory it is possible to define a spectrum for infinite graphs.
A good approach to extremal problems is to come up with a candidate for an ideal example, and show that there are no better instances.

What, then, is the ideal expander? A good candidate is the infinite $d$-regular tree. Using (a little) spectral theory it is possible to define a spectrum for infinite graphs. It turns out that the spectrum of the $d$-regular infinite tree spans the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. 

Nati Linial
Random Lifts of Graphs
The meaning of the number $2\sqrt{d - 1}$

A good approach to extremal problems is to come up with a candidate for an ideal example, and show that there are no better instances.

What, then, is the ideal expander? A good candidate is the infinite $d$-regular tree. Using (a little) spectral theory it is possible to define a spectrum for infinite graphs. It turns out that the spectrum of the $d$-regular infinite tree spans the interval

$$(-2\sqrt{d - 1}, 2\sqrt{d - 1})$$
Some questions

How tight is this bound?

Problem
Are there $d$-regular graphs with second eigenvalue

$$\lambda_2 \leq 2\sqrt{d - 1} \quad ?$$

When such graphs exist, they are called Ramanujan Graphs.
What is the typical behavior?

Problem

How likely is a (large) random $d$-regular graph to be Ramanujan?
What is currently known about Ramanujan Graphs?

Margulis; Lubotzky-Phillips-Sarnak; Morgenstern: 
$d$-regular Ramanujan Graphs exist when $d - 1$ is a prime power. The construction is easy, but the proof uses a lot of heavy mathematical machinery.
What is currently known about Ramanujan Graphs?

Margulis; Lubotzky-Phillips-Sarnak; Morgenstern: $d$-regular Ramanujan Graphs exist when $d - 1$ is a prime power. The construction is easy, but the proof uses a lot of heavy mathematical machinery.

Friedman: If you are willing to settle for $\lambda_2 \leq 2\sqrt{d - 1} + \epsilon$, they exist. Moreover, almost every $d$-regular graph satisfies this condition.
The distribution of the second eigenvalue

\[ n = 400000 \]
\[ n = 100000 \]
\[ n = 40000 \]
\[ n = 10000 \]

\[ 2\sqrt{3} \]
Some open problems on Ramanujan Graphs

- Are there arbitrarily large $d$-regular Ramanujan Graphs (i.e. $\lambda_2 \leq 2\sqrt{d - 1}$) for every $d \geq 3$? The first unknown case is $d = 7$. 

Can we find combinatorial/probabilistic methods to construct graphs with large spectral gap (or even Ramanujan)? As we'll see random lifts of graphs (Bilu-L.) yield graphs with $\lambda_2 \leq O(\sqrt{d \log \frac{3}{d}})$.
Some open problems on Ramanujan Graphs

- Are there arbitrarily large $d$-regular Ramanujan Graphs (i.e. $\lambda_2 \leq 2\sqrt{d - 1}$) for every $d \geq 3$? The first unknown case is $d = 7$.

- Can we find combinatorial/probabilistic methods to construct graphs with large spectral gap (or even Ramanujan)? As we’ll see random lifts of graphs (Bilu-L.) yield graphs with

$$\lambda_2 \leq O(\sqrt{d} \log^{3/2} d).$$
Covers and lifts - the abstract approach

**Definition**

A map $\varphi : V(H) \to V(G)$ where $G, H$ are graphs is a **covering map** if for every $x \in V(H)$, the neighbor set $\Gamma_H(x)$ is mapped $1:1$ onto $\Gamma_G(\varphi(x))$. This is a special case of a fundamental concept from topology. Recall that a graph is a one-dimensional simplicial complex, so covering maps can be defined and studied for graphs.
Covers and lifts - the abstract approach

Definition
A map \( \varphi : V(H) \to V(G) \) where \( G, H \) are graphs is a covering map if for every \( x \in V(H) \), the neighbor set \( \Gamma_H(x) \) is mapped 1 : 1 onto \( \Gamma_G(\varphi(x)) \).

This is a special case of a fundamental concept from topology. Recall that a graph is a one-dimensional simplicial complex, so covering maps can be defined and studied for graphs.
A little terminology

When there is a covering map from $H$ to $G$, we say that $H$ is a lift of $G$. 

Convention: We will always assume that the base graph is connected. This creates no loss in generality.
A little terminology

When there is a covering map from $H$ to $G$, we say that $H$ is a lift of $G$.

We also call $G$ the base graph.
A little terminology

When there is a covering map from $H$ to $G$, we say that $H$ is a lift of $G$.

We also call $G$ the base graph.

Convention: We will always assume that the base graph is connected. This creates no loss in generality.
An example - The 3-cube is a 2-lift of $K_4$
The icosahedron is a 2-lift of $K_6$
We see in the previous examples that the covering map $\varphi$ is $2 : 1$.

- The 3-cube is a 2-lift of $K_4$.
- The graph of the icosahedron is a 2-lift of $K_6$.

In general, if $G$ is a connected graph, then every covering map $\varphi : V(H) \rightarrow V(G)$ is $n : 1$ for some integer $n$ (easy).
We call $n$ the fold number of $\varphi$. 

$H$ is an $n$-lift of $G$, or an $n$-cover of $G$. 

The set of those graphs that are $n$-lifts of $G$ is denoted by $L_n(G)$. 

Nati Linial

Random Lifts of Graphs
Fold numbers etc.

- We call $n$ the fold number of $\varphi$.
- We say that $H$ is an $n$-lift of $G$, or an $n$-cover of $G$. 
Fold numbers etc.

- We call $n$ the fold number of $\varphi$.
- We say that $H$ is an $n$-lift of $G$, or an $n$-cover of $G$.
- The set of those graphs that are $n$-lifts of $G$ is denoted by $L_n(G)$.
Every $H \in L_n(G)$, has vertex set $V(H) = V(G) \times [n]$. 
A direct, constructive perspective

- Every $H \in L_n(G)$, has vertex set $V(H) = V(G) \times [n]$.
- We call the set $F_x = \{x\} \times [n]$ the fiber over $x$. 

Nati Linial
Random Lifts of Graphs
Every $H \in L_n(G)$, has vertex set $V(H) = V(G) \times [n]$. We call the set $F_x = \{x\} \times [n]$ the fiber over $x$. For every edge $e = xy \in E(G)$ we have to select some perfect matching between the fibers $F_x$ and $F_y$, i.e., a permutation $\pi = \pi_e \in S_n$ and connect $(x, i)$ with $(y, \pi(i))$ for $i = 1, \ldots, n$. 
A direct, constructive perspective

- Every $H \in L_n(G)$, has vertex set $V(H) = V(G) \times [n]$.
- We call the set $F_x = \{x\} \times [n]$ the fiber over $x$.
- For every edge $e = xy \in E(G)$ we have to select some perfect matching between the fibers $F_x$ and $F_y$, i.e., a permutation $\pi = \pi_e \in S_n$ and connect $(x, i)$ with $(y, \pi(i))$ for $i = 1, \ldots, n$.
- This set of edges is denoted by $F_e$, the fiber over $e$. 
Figure 1: Lifting an edge

$F_u$ \quad \quad \quad F_v

$u$ \quad \quad \quad v
Random lifts of graphs

When the permutations $\pi_e$ are selected at random, we call the resulting graph a random $n$-lift of $G$. They can be used in essentially every way that traditional random graphs are employed:

- To construct graphs with certain desirable properties. In our case, to achieve large spectral gaps.
- To model various phenomena.
- To study their typical properties.
Random lifts of graphs

- When the permutations $\pi_e$ are selected at random, we call the resulting graph a random $n$-lift of $G$.
- They can be used in essentially every way that traditional random graphs are employed:
When the permutations $\pi_e$ are selected at random, we call the resulting graph a random $n$-lift of $G$.

They can be used in essentially every way that traditional random graphs are employed:

- To construct graphs with certain desirable properties. In our case, to achieve large spectral gaps.
When the permutations $\pi_e$ are selected at random, we call the resulting graph a random $n$-lift of $G$.

They can be used in essentially every way that traditional random graphs are employed:

- To construct graphs with certain desirable properties. In our case, to achieve large spectral gaps.
- To model various phenomena.
Random lifts of graphs

When the permutations $\pi_e$ are selected at random, we call the resulting graph a random $n$-lift of $G$.

They can be used in essentially every way that traditional random graphs are employed:

- To construct graphs with certain desirable properties. In our case, to achieve large spectral gaps.
- To model various phenomena.
- To study their typical properties.
A few more general properties of lifts

- Vertex degrees are maintained. If $x$ has $d$ neighbors, then so do all the vertices in the fiber of $x$. In particular, a lift of a $d$-regular graph is $d$-regular.
A few more general properties of lifts

- Vertex degrees are maintained. If \( x \) has \( d \) neighbors, then so do all the vertices in the fiber of \( x \). In particular, a lift of a \( d \)-regular graph is \( d \)-regular.
- The cycle \( C_n \) is a lift of \( C_m \) iff \( m \mid n \).
A few more general properties of lifts

▶ Vertex degrees are maintained. If $x$ has $d$ neighbors, then so do all the vertices in the fiber of $x$. In particular, a lift of a $d$-regular graph is $d$-regular.

▶ The cycle $C_n$ is a lift of $C_m$ iff $m|n$.

▶ The $d$-regular tree covers every $d$-regular graph. This is the universal cover of a $d$-regular graph. Every connected base graph has a universal cover which is an infinite tree.
Here is an easy observation:

The lifted graph inherits every eigenvalue of the base graph.

Namely, if $H$ is a lift of $G$, then every eigenvalue of $G$ is also an eigenvalue of $H$. 
Here is an easy observation:

The lifted graph inherits every eigenvalue of the base graph.

Namely, if $H$ is a lift of $G$, then every eigenvalue of $G$ is also an eigenvalue of $H$ (Pf: Pullback, i.e., take any eigenfunction $f$ of $G$, and assign the value $f(x)$ to every vertex in the fiber of $x$. It is easily verified that this is an eigenfunction of $H$ with the same eigenvalue as $f$ in $G$).
Old vs. New Eigenvalues (contd.)

These are called the old eigenvalues of $H$. If $G$ is given, the old eigenvalues appear in every lift, and we can only hope to control the values of the new eigenvalues.
These are called the old eigenvalues of $H$. If $G$ is given, the old eigenvalues appear in every lift, and we can only hope to control the values of the new eigenvalues.

This suggests the following approach to the construction of $d$-regular Ramanujan Graphs by repeated lifts:
Old vs. New Eigenvalues (contd.)

These are called the old eigenvalues of $H$. If $G$ is given, the old eigenvalues appear in every lift, and we can only hope to control the values of the new eigenvalues.

This suggests the following approach to the construction of $d$-regular Ramanujan Graphs by repeated lifts:

- Start from a small $d$-regular Ramanujan Graph (e.g. $K_{d+1}$).
- In every step apply a lift to the previous graph while keeping all new eigenvalues in the interval $[-2\sqrt{d - 1}, 2\sqrt{d - 1}]$. 
How to think about 2-lifts

In an $n$-lift of a graph $G$ we associate with every edge $e = xy$ of $G$ a permutation $\pi_e \in S_n$ which tells us how to connect the $n$ vertices in the fiber $F_x$ with the $n$ vertices of $F_y$. 

But in $S_2$ the only permutations are the identity $\text{id}$ and the switch $\sigma = (12)$. So a 2-lift of $G$ is specified by deciding, for every edge $e$ whether $\pi_e$ equals $\text{id}$ or $\sigma$. Alternatively, we sign the edges of $G$ where $+1$ stands for $\text{id}$ and $-1$ for $\sigma$. 

Nati Linial
Random Lifts of Graphs
How to think about 2-lifts

In an $n$-lift of a graph $G$ we associate with every edge $e = xy$ of $G$ a permutation $\pi_e \in S_n$ which tells us how to connect the $n$ vertices in the fiber $F_x$ with the $n$ vertices of $F_y$.

But in $S_2$ the only permutations are the identity $\text{id}$ and the switch $\sigma = (12)$. So a 2-lift of $G$ is specified by deciding, for every edge $e$ whether $\pi_e$ equals $\text{id}$ or $\sigma$. 

Nati Linial

Random Lifts of Graphs
How to think about 2-lifts

In an $n$-lift of a graph $G$ we associate with every edge $e = xy$ of $G$ a permutation $\pi_e \in S_n$ which tells us how to connect the $n$ vertices in the fiber $F_x$ with the $n$ vertices of $F_y$.

But in $S_2$ the only permutations are the identity $\text{id}$ and the switch $\sigma = (12)$. So a 2-lift of $G$ is specified by deciding, for every edge $e$ whether $\pi_e$ equals $\text{id}$ or $\sigma$.

Alternatively, we sign the edges of $G$ where $+1$ stands for $\text{id}$ and $-1$ for $\sigma$. 
A **signing** is a symmetric matrix in which some of the entries in the adjacency matrix of $G$ are changed from $+1$ to $-1$. 

We think of a signing in two equivalent ways: A way of specifying a 2-lift of $G$, and a real symmetric matrix with entries 0, 1, $-1$.

An easy but useful observation:

**Proposition**

The new eigenvalues of a 2-lift of $G$ are the eigenvalues of the corresponding signing matrix.
A signing is a symmetric matrix in which some of the entries in the adjacency matrix of $G$ are changed from $+1$ to $-1$.

We think of a signing in two equivalent ways: A way of specifying a 2-lift of $G$, 

Proposition

The new eigenvalues of a 2-lift of $G$ are the eigenvalues of the corresponding signing matrix.
A **signing** is a symmetric matrix in which some of the entries in the adjacency matrix of $G$ are changed from $+1$ to $-1$.

We think of a signing in two equivalent ways: A way of specifying a 2-lift of $G$, and a real symmetric matrix with entries $0, 1, -1$. 
A **signing** is a symmetric matrix in which some of the entries in the adjacency matrix of \( G \) are changed from \(+1\) to \(-1\).

We think of a signing in two equivalent ways: A way of specifying a 2-lift of \( G \), and a real symmetric matrix with entries 0, 1, \(-1\).

An easy but useful observation:
A signing is a symmetric matrix in which some of the entries in the adjacency matrix of $G$ are changed from $+1$ to $-1$.

We think of a signing in two equivalent ways: A way of specifying a 2-lift of $G$, and a real symmetric matrix with entries $0, 1, -1$.

An easy but useful observation:

Proposition

The new eigenvalues of a 2-lift of $G$ are the eigenvalues of the corresponding signing matrix.
Recall: The spectral radius of a matrix is the largest absolute value of an eigenvalue.
Recall: The spectral radius of a matrix is the largest absolute value of an eigenvalue.

The above approach to the construction of Ramanujan Graphs can be stated as follows:
Recall: The **spectral radius** of a matrix is the largest absolute value of an eigenvalue.

The above approach to the construction of Ramanujan Graphs can be stated as follows:

**Conjecture**

*Every d-regular Ramanujan Graph has a signing with spectral radius $\leq 2\sqrt{d} - 1$.***
The signing conjecture

But it seems that something much stronger is true
The signing conjecture

But it seems that something much stronger is true

Conjecture

Every $d$-regular graph $G$ has a signing with spectral radius $\leq 2\sqrt{d} - 1$. 
The signing conjecture

But it seems that something much stronger is true

Conjecture

Every $d$-regular graph $G$ has a signing with spectral radius $\leq 2\sqrt{d} - 1$.

This conjecture, if true, is tight.
What is known

Theorem (Yonatan Bilu + L.)

By repeated application of 2-lifts it is possible to explicitly construct $d$-regular graphs ($d \geq 3$) whose second eigenvalue

$$\lambda_2 \leq O(\sqrt{d} \log^{3/2} d)$$
A highlight of the proof

The most unexpected part of the proof is a converse of the so-called Expander Mixing Lemma.
A highlight of the proof

The most unexpected part of the proof is a converse of the so-called Expander Mixing Lemma.

Our new lemma says that $\lambda_2$ is controlled by the extent to which $G$ is pseudo-random.
A highlight of the proof

The most unexpected part of the proof is a converse of the so-called Expander Mixing Lemma.

Our new lemma says that $\lambda_2$ is controlled by the extent to which $G$ is pseudo-random.

What’s involved is the graph’s discrepancy, i.e. the maximum of

$$e(A, B) - \frac{d}{n} |A||B|$$

$$\frac{1}{\sqrt{|A||B|}}$$
A matching $M$ in a graph $G$ is a collection of disjoint edges. If the edges in $M$ meet every vertex in $G$, we say that $M$ is a perfect matching (PM). The defect of $G$ is the number of vertices missed by the largest matching in $G$. (So the existence of a PM is the same as zero defect).
A matching $M$ in a graph $G$ is a collection of disjoint edges. If the edges in $M$ meet every vertex in $G$, we say that $M$ is a perfect matching=PM. The defect of $G$ is the number of vertices missed by the largest matching in $G$. (So the existence of a PM is the same as zero defect).

**Question:** Given a base graph $G$ and a large even integer $n$, how likely is an $n$-lift of $G$ to contain a perfect matching?
Theorem (L. + Rozenman)

Let $G$ be a base graph and let $H$ be a random $2n$ lift of $G$. For every base graph exactly one of the following four situations occurs:

1. Every $2n$-lift $H$ of $G$ contains a PM.
2. Every $H$ must have defect $\geq \alpha n$ for some constant $\alpha > 0$.
3. The probability that $H$ has a PM is $1 - o(1)$ (but not $1$).
4. Almost surely $H$ has defect $\Theta(\log n)$.
Theorem (L. + Rozenman)

Let $G$ be a base graph and let $H$ be a random $2n$ lift of $G$. For every base graph exactly one of the following four situations occurs:

- Every $2n$-lift $H$ of $G$ contains a PM.
Theorem (L. + Rozenman)

Let $G$ be a base graph and let $H$ be a random $2n$ lift of $G$. For every base graph exactly one of the following four situations occurs:

- Every $2n$-lift $H$ of $G$ contains a PM.
- Every $H$ must have defect $\geq \alpha n$ for some constant $\alpha > 0$.
- The probability that $H$ has a PM is $1 - o(1)$ (but not $1$).
- Almost surely $H$ has defect $\Theta(\log n)$.
Theorem (L. + Rozenman)

Let $G$ be a base graph and let $H$ be a random $2n$ lift of $G$. For every base graph exactly one of the following four situations occurs:

- Every $2n$-lift $H$ of $G$ contains a PM.
- Every $H$ must have defect $\geq \alpha n$ for some constant $\alpha > 0$.
- The probability that $H$ has a PM is $1 - o(1)$ (but not 1).
- Almost surely $H$ has defect $\Theta(\log n)$. 

Nati Linial

Random Lifts of Graphs
Theorem (L. + Rozenman)

Let $G$ be a base graph and let $H$ be a random $2n$ lift of $G$. For every base graph exactly one of the following four situations occurs:

- Every $2n$-lift $H$ of $G$ contains a PM.
- Every $H$ must have defect $\geq \alpha n$ for some constant $\alpha > 0$.
- The probability that $H$ has a PM is $1 - o(1)$ (but not 1).
- Almost surely $H$ has defect $\Theta(\log n)$. 
What else?

There are theorems about the typical behavior of:

▶ The typical degree of connectivity of a lift of $G$.
▶ The chromatic numbers of typical lifts.
▶ The typical distribution of new eigenvalues.

... and more ....
What else?

There are theorems about the typical behavior

- The typical degree of connectivity of a lift of $G$. 
What else?

There are theorems about the typical behavior

- The typical degree of connectivity of a lift of $G$.
- The chromatic numbers of typical lifts.
What else?

There are theorems about the typical behavior

- The typical degree of connectivity of a lift of $G$.
- The chromatic numbers of typical lifts.
- The typical distribution of new eigenvalues.
- ... and more ....
... and much more that we do not know...

Open Problem

- *Is there a zero-one law for Hamiltonian cycles?*
Open Problem

- *Is there a zero-one law for Hamiltonian cycles?*
- *What is the typical chromatic number of an n-lift of $K_5$? Is it 3 or 4, perhaps each with positive probability?*