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Combinatorial characterization of read-once formulae

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Abstract

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We give an alternative proof to a characterization theorem of Gurvich for Boolean functions whose formula size is exactly the number of variables. These functions are called read-once functions. We use methods of combinatorial optimization and give, as a corollary, an alternative proof for some results of Seymour (1976, 1977).

1. Introduction

Let X be a finite set (interpreted as Boolean variables). A monotone formula is a rooted tree whose leaves are labeled with members of X, and whose internal nodes are labeled with the Boolean operations AND, OR. The root of the tree computes a monotone Boolean function $f:\{0,1\}^X \mapsto \{0,1\}$ in the natural way.

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Let f be a monotone Boolean function. A minimal set $S \subseteq X$ is a *minterm* (*maxterm*) if setting all variables in S to 1 (0), forces the value of f to 1 (0). Let $MIN(\mathrm{f})$ ($MAX(\mathrm{f})$) denote the set of all minterms (maxterms).

The monotone formula complexity of a monotone Boolean function f, denoted by $L_{\mathrm{m}}(f)$, is the minimum number of leaves in any formula computing f. *Read-once formulae* are formulae in which every variable of X appears exactly once. These are, of course, the smallest possible for functions that depend on all their variables. A Boolean function f is *read-once* if it has a read-once formula. A monotone Boolean function $f:\{0,1\}^X \mapsto \{0,1\}$ depends on all its variables if $\bigcup_{S\in MIN(f)} S = X$. We give the following simple characterization of monotone read-once functions, originally proved by Gurvich in 1977 [2, 3].

Theorem 1.1. *A monotone Boolean function f that depends on all its variables is read-once if and only if*

$$T\in MAX(f),\ S\in MIN(f) \ \Rightarrow\ |S\cap T| = 1 \qquad (*)$$

We get Theorem 1.1 as a corollary of Theorem 2.6, in which we state some connections between certain clutters, their blockers and antiblockers and their 'graphs'.

We have recently been informed that the characterization theorem was proved by Gurvich [2] back in 1977 and more recently by Beynon and Paterson as well [1]. (Both cases use somewhat different proofs that does not go through the equivalence of (2) and (3) in Theorem 2.6). We also note here that the results on the blocker antiblocker relation (in Theorem 2.6) can be deduced from the work of Seymour [6, 7], studying binary clutters, cuts and paths of series–parallel graphs, with different methods.

We will need some definitions and notations.

2. Definitions and main theorem

Let $f:\{0,1\}^X \mapsto \{0,1\}$ be a Boolean function, $|X| = n$. We identify the arguments of f with subsets of X in the natural way ($f(S) = f(x_1, \ldots, x_n)$, where $x_i = 1$ if $x_i \in S$ and $x_i = 0$ otherwise, $1 \leqslant i \leqslant n$).

Let $\mathscr{C} \subseteq 2^X$ be an antichain of sets; $\mathscr{C}$ is called *a clutter*. Note that, by the definition of $MAX(f)$, $MIN(f)$, both these families are clutters.

Let $\mathscr{C}$ be a clutter; we will denote by $V(\mathscr{C}) = \bigcup_{S\in\mathscr{C}} S$ the set of elements of X that actually appear in $\mathscr{C}$.

Let $G = (V, E)$ be a graph, G^{C} will denote the graph which is the complement of G, i.e. $V(G^{\mathrm{C}}) = V$, $E(G^{\mathrm{C}}) = \{(u, v) \mid (u, v) \notin E\}$.

A P_4 is the simple path with 4 vertices and 3 edges. We say that a graph $G = (V, E)$ is P_4-free if it has no induced subgraph isomorphic to P_4.

Definition 2.1. For a clutter $\mathscr{C}$ on a set of points X, define:

(1) The blocker of $\mathscr{C}$, $\mathscr{C}^{\mathrm{B}}$, is a clutter defined by

$$\mathscr{C}^{\mathrm{B}}=\{S\subseteq X \mid \forall T\in\mathscr{C}\ |S\cap T|\geqslant 1, \text{ and } S \text{ is minimal}\},$$

(2) The antiblocker of $\mathscr{C}$, $\mathscr{C}^{\mathrm{A}}$, is a clutter defined by

$$\mathscr{C}^{\mathrm{A}}=\{S\subseteq V(\mathscr{C}) \mid \forall T\in\mathscr{C}\ |S\cap T|\leqslant 1, \text{ and } S \text{ is maximal}\}.$$

Definition 2.2. For a clutter $\mathscr{C}$ on a set of points X, define a Boolean function $f_{\mathscr{C}}:\{0,1\}^X\mapsto\{0,1\}$ by $f(S)=1$ if for some $T\subseteq S$, $T\in\mathscr{C}$.

In fact, $f_{\mathscr{C}}$ is a monotone Boolean function, (write $f(x_1,\ldots,x_n)=\bigvee_{S\in\mathscr{C}}\bigwedge_{x_i\in S}x_i$), and $\mathscr{C}$ is $MIN(f_{\mathscr{C}})$. (Unless $V(\mathscr{C})=\emptyset$).

Lemma 2.3 (Edmonds and Fulkerson [4]). *Let* $\mathscr{C}\subseteq 2^X$ *be a clutter; then* $(\mathscr{C}^{\mathrm{B}})^{\mathrm{B}}=\mathscr{C}$.

Proof. Easy and appears in [4]. □

Claim 2.4. *If f is a monotone Boolean function with* $MIN(f)=\mathscr{C}$ *then* $\mathscr{C}^{\mathrm{B}}=MAX(f)$.

Proof. Follows directly from the definitions.

Definition 2.5. For a clutter $\mathscr{C}\subseteq 2^X$, define the graph $G(\mathscr{C})=(V(\mathscr{C}),E)$, where $(u,v)\in E$ if there is some $T\in\mathscr{C}$, with $u,v\in T$.

We can state now our main theorem.

Theorem 2.6. *Let* $f:\{0,1\}^X\mapsto\{0,1\}$ *be a monotone Boolean function with* $\mathscr{C}=MIN(f)$ *and* $X=V(\mathscr{C})\neq\emptyset$, *then the following conditions are equivalent:*

(C1) *f is read-once.*

(C2) $\mathscr{C}^{\mathrm{B}}\subseteq\mathscr{C}^{\mathrm{A}}$.

(C3) $\mathscr{C}^{\mathrm{B}}=\mathscr{C}^{\mathrm{A}}$.

(C4) *$\mathscr{C}$ is the set of maximal cliques of $G(\mathscr{C})$ and, for any induced subgraph $G'\subseteq G(\mathscr{C})$, every maximal clique intersects every maximal independent set.*

(C5) *$\mathscr{C}$ is the set of maximal cliques of $G(\mathscr{C})$, and $G(\mathscr{C})$ is P_4-free.*

We will need the following two definitions.

Definition 2.7. For a clutter $\mathscr{C}$ on a set of points X and $W\subset X$,

(1) The deletion of W from $\mathscr{C}$ is the clutter

$$\mathscr{C}\setminus W=\{S\in\mathscr{C} \mid S\cap W=\emptyset\};$$

(2) The contraction of $\mathscr{C}$ by W is the clutter $\mathscr{C}/W$, obtained by taking all minimal sets of $\mathscr{C}(W)$, where $\mathscr{C}(W)=\{S-W \mid S\in\mathscr{C}\}$.

3. Proof of the main theorem

Lemma 3.1 (C1→C2). *Let f be a monotone read-once Boolean function on a variable set X, with $MIN(f)=\mathscr{C}$; then $\mathscr{C}^{\mathrm{B}}\subseteq\mathscr{C}^{\mathrm{A}}$.*

Proof. We prove that f satisfies condition C2 by induction on the size of the variable set $|X|$. By Claim 2.4, an equivalent reformulation of condition C2 is property $(*)$ stated in Theorem 1.1.

If $|X|=1$ then, clearly, f has property $(*)$. Assume first that f has a read-once formula in which the output gate is an OR gate, i.e. $f=g\vee h$. Clearly, g and h are both monotone read-once on some disjoint sets X_g and X_h, respectively. We have $MIN(f)=MIN(g)\cup MIN(h)$ and

$$MAX(f)=\{S \mid S=T\cup Q,\ T\in MAX(g),\ Q\in MAX(h)\}.$$

It is easily verified that f has property $(*)$ by the fact that X_g, X_h are disjoint. In the other case where the output gate of the read-once formula of f is an AND gate, a similar argument holds. □

Lemma 3.2 (C2→C3). *For a clutter $\mathscr{C}$, if $\mathscr{C}^{\mathrm{B}}\subseteq\mathscr{C}^{\mathrm{A}}$, then $\mathscr{C}^{\mathrm{B}}=\mathscr{C}^{\mathrm{A}}$.*

Proof. First we observe that if $\mathscr{C}$ meets the assumptions of the lemma, so does $\mathscr{C}\backslash W$ for any $W\subset X$, and that the lemma is trivial for $|X|=1$.

Assume then, for contradiction, that $\mathscr{C}$ is a counterexample with $|X|$ minimal. Let $T\in\mathscr{C}^{\mathrm{A}}-\mathscr{C}^{\mathrm{B}}$.

We may assume that $|T|\neq 1$. Otherwise, $T=\{x\}$. Take some $B\in\mathscr{C}^{\mathrm{B}}$, $x\in B$ (it is always possible to find such a B); thus, by condition C2, $B\in\mathscr{C}^{\mathrm{A}}$. But $T\subset B$, $T\neq B$, which contradicts the fact that $\mathscr{C}^{\mathrm{A}}$ is a clutter.

Define, for every $t\in T$, $\mathscr{C}_t=\{S\in\mathscr{C},\ S\cap T=\{t\}\}$, and define $\mathscr{D}=\mathscr{C}-(\bigcup_{t\in T}\mathscr{C}_t)$. By the definition of $\mathscr{C}^{\mathrm{A}}$, we get that the families $\mathscr{D}$ and $\mathscr{C}_t$, $t\in T$, are pairwise disjoint, and, by assumption, $\mathscr{D}\neq\emptyset$. Define, for $t\in T$, $W_t=(V(\mathscr{D})-\bigcup_{s\neq t}V(\mathscr{C}_s))\cap V(\mathscr{C}_t)$, that is, W_t is the set of all the points from X that appear in $\mathscr{D}$ and in $\mathscr{C}_t$ but not in any other $\mathscr{C}_s$, $s\neq t$. Note that $W_t\cap W_s=\emptyset$ for $s\neq t$.

Claim 3.3. *For any $t\in T$ and any $D\in\mathscr{D}$, $W_t\cap D\neq\emptyset$.*

Proof. Assume to the contrary that $W_t\cap D=\emptyset$ for some $D\in\mathscr{D}$; look at $\mathscr{C}'=\mathscr{C}\backslash(W_t\cup\{t\})$. Note the following:

- $D\in\mathscr{C}'$.

• $T-\{t\}\in\mathscr{C}'^{\mathrm{A}}$ since $|(T-\{t\})\cap S|\leqslant 1$ for any $S\in\mathscr{C}'$. If there exists $x\in V(\mathscr{C}')$ such that $(T-\{t\})\cup\{x\}$ has the same property then, clearly, $x\notin\bigcup_{s\neq t}V(\mathscr{C}_s)$. Thus, $x\in V(\mathscr{D})$, but $T\cup\{x\}\notin\mathscr{C}^{\mathrm{A}}$; so, $x\in V(\mathscr{C}_t)$. It follows that $x\in W_t$, which contradicts the fact that $x\in V(\mathscr{C}')$.

• $(T-\{t\})\cap D=\emptyset$; so, $T-\{t\}\notin\mathscr{C}'^{\mathrm{B}}$.

We get that $\mathscr{C}'$ is a counterexample on $X-(W_t\cup\{t\})$ (a smaller counterexample). □

Thus, by the claim we have that $W_t\cup T$ intersects every set in $\mathscr{C}$, that is, there is some $S_t\in\mathscr{C}^{\mathrm{B}}$, $S_t\subseteq W_t\cup T$. Clearly, $T-\{t\}\subseteq S_t$ since W_t does not intersect the sets of $\mathscr{C}_s$, $s\neq t$. So, we have $S_t=(T-\{t\})\cup V_t$, where $V_t\subseteq W_t$, and V_t intersects every set of $\mathscr{D}\cup\mathscr{C}_t$. Since $S_t\in\mathscr{C}^{\mathrm{B}}$ and, therefore, $S_t\in\mathscr{C}^{\mathrm{A}}$ by condition C2, it implies that V_t is a minimal set of points that intersect every set in $\mathscr{C}_t$. Thus, $V=\bigcup_{t\in T}V_t$ is a member of $\mathscr{C}^{\mathrm{B}}$, with $|V|\geqslant|T|\geqslant 2$. But, since every V_t intersects every set of $\mathscr{D}$, V intersects any set of $\mathscr{D}$ in more than one point, contradicting the fact that $V\in\mathscr{C}^{\mathrm{A}}$ (by condition C2). This completes the proof of the lemma. □

Remark 3.4. If $\mathscr{C}$ satisfies condition C2, so does $\mathscr{C}'=\mathscr{C}\backslash\{x\}$, ($\mathscr{C}'=\mathscr{C}/\{x\}$) for any $x\in X$ such that $V(\mathscr{C}')\neq\emptyset$. Thus, by the lemma, these clutters satisfy conditions C3 *too.*

Lemma 3.5 (C3→C4). *Let $\mathscr{C}$ be a clutter for which $\mathscr{C}^{\mathrm{B}}=C^{\mathrm{A}}$; then $\mathscr{C}$ is the set of maximal cliques of $G(\mathscr{C})$ and, for any induced subgraph $G'\subseteq G(\mathscr{C})$, every maximal clique intersects every maximal independent set.*

Proof. We begin by making the following claims.

Claim 3.6. *If $\mathscr{C}^{\mathrm{B}}=\mathscr{C}^{\mathrm{A}}$ then $\mathscr{C}=(\mathscr{C}^{\mathrm{A}})^{\mathrm{A}}$.*

Proof of the Claim 3.6. We have $\mathscr{C}^{\mathrm{B}}=\mathscr{C}^{\mathrm{A}}$. But, since $\mathscr{C}=(\mathscr{C}^{\mathrm{B}})^{\mathrm{B}}$, $\mathscr{C}^{\mathrm{B}}$, as a clutter, also satisfies condition C2; thus, $(\mathscr{C}^{\mathrm{B}})^{\mathrm{B}}=(\mathscr{C}^{\mathrm{B}})^{\mathrm{A}}$ (Lemma 3.2). Substituting $(\mathscr{C}^{\mathrm{B}})^{\mathrm{B}}$ with $\mathscr{C}$, and $\mathscr{C}^{\mathrm{B}}$ with $\mathscr{C}^{\mathrm{A}}$, gives the required claim. □

Claim 3.7 (Fulkerson [5]). *Let $\mathscr{C}$ be a clutter. If $(\mathscr{C}^{\mathrm{A}})^{\mathrm{A}}=\mathscr{C}$ then $\mathscr{C}$ is the maximal cliques of $G(\mathscr{C})$, and, in that case, $\mathscr{C}^{\mathrm{A}}$ is the set of maximal independent sets of G.*

Proof of Claim 3.7. By the definition of $G(\mathscr{C})$, every $T\in\mathscr{C}$ induces a clique of $G(\mathscr{C})$ and every $S\in\mathscr{C}^{\mathrm{A}}$ is a maximal independent set. Moreover, every maximal independent set is in $\mathscr{C}^{\mathrm{A}}$; thus, $\mathscr{C}^{\mathrm{A}}$ is the set of maximal independent set of $G(\mathscr{C})$. It follows that every maximal clique of $G(\mathscr{C})$ is in $(\mathscr{C}^{\mathrm{A}})^{\mathrm{A}}$, but $(\mathscr{C}^{\mathrm{A}})^{\mathrm{A}}=\mathscr{C}$, which completes the proof of the claim.

Claim 3.8. *Let $\mathscr{C}$ be a clutter satisfying condition C3, let $G=G(\mathscr{C})=(X,E)$; then $G-\{x\}$ is the graph of $\mathscr{C}/\{x\}$ or the graph of $\mathscr{C}\backslash\{x\}$.*

Proof of Claim 3.8. Observe that the following properties are true for any clutter [4]:

(1) $(\mathscr{C}/\{x\})^{\mathrm{B}}=\mathscr{C}^{\mathrm{B}}\backslash\{x\}$,

(2) $(\mathscr{C}\backslash\{x\})^{\mathrm{B}}=\mathscr{C}^{\mathrm{B}}/\{x\}$.

Assume $V(\mathscr{C}\backslash\{x\})=X-\{x\}$. Then $G-\{x\}$ is the graph of $\mathscr{C}\backslash\{x\}$ since, for every $(u,v)\in E(G-\{x\})$, if $(u,v)\notin G(\mathscr{C}\backslash\{x\})$ then $(u,v)\in G^{\mathrm{C}}(\mathscr{C}\backslash\{x\})=G((\mathscr{C}\backslash\{x\})^{\mathrm{B}})=G(\mathscr{C}^{\mathrm{B}}/\{x\})$, that is, u,v are in some $T\in\mathscr{C}^{\mathrm{B}}$. But $(u,v)\in E(G-\{x\})$ implies that there is a set $S\in\mathscr{C}$ such that $u,v\in S$; thus, $\{u,v\}\subseteq T\cap S$, contradicting condition C3.

We will show that if $V(\mathscr{C}\backslash\{x\})\neq X-\{x\}$ then $V(\mathscr{C}^{\mathrm{B}}\backslash\{x\})=X-\{x\}$; thus, by the above argument $G^{\mathrm{C}}-\{x\}$ is the graph of $\mathscr{C}^{\mathrm{B}}\backslash\{x\}$ and, thus, $G-\{x\}$ is the graph of $(\mathscr{C}^{\mathrm{B}}\backslash\{x\})^{\mathrm{B}}=\mathscr{C}/\{x\}$.

Note. If for some $z\in X-\{x\}$, $z\notin V(\mathscr{C}\backslash\{x\})$, then $\forall S\in\mathscr{C}\ z\in S\Rightarrow x\in S$. Similarly, if for some $w\in X-\{x\}$, $w\notin V(\mathscr{C}^{\mathrm{B}}\backslash\{x\})$, then $\forall T\in\mathscr{C}^{\mathrm{B}}\ w\in T\Rightarrow x\in T$.

Assume then that $z\in X-\{x\}$, $z\notin V(\mathscr{C}\backslash\{x\})$. Clearly, $z\in V(\mathscr{C}^{\mathrm{B}}\backslash\{x\})$ since $z\notin V(\mathscr{C}\backslash\{x\})\Rightarrow\exists S\in\mathscr{C}\ (x,z\in S)\Rightarrow\neg(\exists T\in\mathscr{C}^{\mathrm{B}}\ (x,z\in T))\Rightarrow z\in V(\mathscr{C}^{\mathrm{B}}\backslash\{x\})$.

Now, for every $w\in X-\{x\}$, if $(w,z)\in E(G)$, it implies that $\exists S\in\mathscr{C}$ such that $x,w\in S$ (by the assumption on z and the note above). Thus, we must have $w\in V(\mathscr{C}^{\mathrm{B}}\backslash\{x\})$; otherwise, for any T of the remark, $w,x\in S\cap T$, which contradicts condition C3. The argument for the case $(w,z)\notin E(G)$ is similar. □

Proof of Lemma 3.5 (*conclusion*). The lemma is easily verified for $|X|=2$. We proceed by induction on $|X|$. We have, by assumption, $\mathscr{C}^{\mathrm{A}}=\mathscr{C}^{\mathrm{B}}$; thus, by Claims 3.6 and 3.7, the lemma is true for $G'=G(\mathscr{C})$. Clearly, every proper induced subgraph of $G(\mathscr{C})$ is an induced subgraph of $G(\mathscr{C})-\{x\}$, for some $x\in X$. By Claim 3.8, $G'=G(\mathscr{C})-\{x\}$ is $G(\mathscr{C}')$ for $\mathscr{C}'=\mathscr{C}\backslash\{x\}$ or $\mathscr{C}'=\mathscr{C}/\{x\}$. By Remark 3.4, $\mathscr{C}'$ satisfies condition C3. Thus, by induction, we are done. □

Lemma 3.9 (C4→C5). *Let $\mathscr{C}$ be a clutter on $X=V(\mathscr{C})$ such that $G(\mathscr{C})$ satisfies condition* C4; *then $G(\mathscr{C})$ is P_4-free.*

Proof. Clearly, the graph P_4 does not satisfy condition C4; thus, $G(\mathscr{C})$ cannot have a P_4 as an induced subgraph.

Lemma 3.10 (C5→C1). *Let $\mathscr{C}$ be a clutter on $X=V(\mathscr{C})$ such that $\mathscr{C}$ is the set of maximal cliques of $G(\mathscr{C})$ and $G(\mathscr{C})$ is P_4-free; then $f_{\mathscr{C}}$ is read-once.*

Claim 3.11. *If G is a P_4-free graph on more than one vertex, then one of G, G^{C} is disconnected.*

Proof. Clearly, the claim is true for a graph on two vertices. Let G be the minimal counter example, that is, G is P_4-free and both G, G^{C} are connected. Consider any vertex x; x cannot be connected (in G) to all the vertices because, in that case, G^{C} is not connected. By the minimality assumption, one of $G-\{x\}$, $(G-\{x\}^{\mathrm{C}}$ is not connected.

Assume that $G-\{x\}$ is not connected, that is, $G-\{x\}$ has at least two components. Therefore, there is a vertex u and a vertex t in one of the components of $G-\{x\}$ such that (x, t), (t, u) are edges of $E(G)$ but (x, u) is not an edge. Take some other vertex y such that y is not in the same component with u, t, and (y, x) is an edge of G. The induced graph on the four vertices u, t, x, y is a P_4.

If $(G-\{x\})^C$ is not connected, observe that $(G-\{x\})^C = G^C - \{x\}$ and apply the above argument to get a P_4 as an induced subgraph of G^C. Since P_4 is self-complementary, it is an induced subgraph of G too. □

Let $\mathscr{A}$ be the clutter of maximal independent sets of $G=G(\mathscr{C})$.

Claim 3.12. $\mathscr{C}^B = \mathscr{A}$, *and* $f_{\mathscr{C}}$ *is read-once.*

Proof. We prove the claim by induction on the size of $n=|X|$. The claim can be easily checked for $n \leqslant 3$.

By Claim 3.11, one of G, G^C is disconnected. If G is disconnected then $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$, where $\mathscr{C}_1$ is the set of maximal cliques of one of the components of G, and $\mathscr{C}_2$ is the set of all other maximal cliques. Clearly, $\mathscr{C}_1, \mathscr{C}_2$ are clutters on disjoint sets of points, say X_1, X_2 respectively. Define g to be the monotone Boolean function defined by $\mathscr{C}_1$ as its set of minterms, and define h in similar way with respect to $\mathscr{C}_2$. Clearly, $f_{\mathscr{C}} = g \vee h$. Moreover, $G(\mathscr{C}_1), G(\mathscr{C}_2)$ are P_4-free (as induced subgraphs). Thus, by induction, g, h are read-once and then so is $f_{\mathscr{C}}$ (by the fact that X_1 and X_2 are disjoint). Moreover, $\mathscr{C}^B = MAX(f_{\mathscr{C}}) = \{I_1 \cup I_2 \mid I_1 \in MAX(g), I_2 \in MAX(h)\}$. But $MAX(g) = \mathscr{C}_1^B$ and, by induction on $\mathscr{C}_1$, that is the set of maximal independent sets of $G(X_1)$. Similarly, $MAX(h)$ is the set of maximal independent sets of $G(X_2)$ and then $MAX(f)$ is indeed $\mathscr{A}$.

In the case where G^C is disconnected, the above, argument applied to the components of G^C, shows that $\mathscr{A}^B = \mathscr{C}$ (note that G^C is P_4-free too). By Lemma 2.3, this gives $\mathscr{C}^B = \mathscr{A}$. Define g to be the function whose maxterms are the maximal cliques of a component of G^C and h is the function whose maxterms are all other maximal cliques of G^C. It is easy to see that $f = g \wedge h$. h, g are read-once (by induction) on disjoint variable sets; so, f is read-once too. □

4. The nonmonotone case

Theorem 1.1 may be generalized for the nonmonotone case using the following definition: Let f be a Boolean function on a set of variables X. A 1-witness of f is a pair (S, T), $S, T \subseteq X$, $S \cap T = \emptyset$, such that setting the variables in S equal to 1, and the variables in T equal to 0, forces the value of f to 1.

We say that a 1-witness (S, T) is a minterm of f if it is 'minimal', that is, there is no other 1-witness (S', T') for which $S' \subseteq S$ and $T' \subseteq T$.

Maxterms of f are defined similarly as the 'minimal' pairs that force the value of f to 0. Denote by $MIN(f)$ $(MAX(f))$ the set of all minterms (maxterms) of f. We get the following theorem.

Theorem 4.1. *A Boolean function f that depends on all its variables is read-once if and only if*

$$(S, T)\in MAX(f),\ (P, Q)\in MIN(f)\Rightarrow|(S\cup T)\cap(P\cup Q)|=1 \qquad (**)$$

Proof. The 'only if' part follows directly along the lines of the proof of Lemma 3.1. To prove the 'if' part, define $X_1=\bigcup_{(S,T)\in MIN(f)}S$, $X_2=\bigcup_{(S,T)\in MIN(f)}T$, $Y_1=\bigcup_{(S,T)\in \mathrm{MAX}(f)}S$, $Y_2=\bigcup_{(S,T)\in \mathrm{MAX}(f)}T$. Condition $(**)$ implies that $X_1\cap X_2=\emptyset$ and $X_1=Y_1$, $X_2=Y_2$. Complement the variables in X_2; $x_i\mapsto\neg x_i$. This gives a new monotone function f' that has property $(*)$; therefore, f' and, hence, f are read-once. □

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