What is high-dimensional combinatorics?

Nati Linial

Random-Approx, August '08
A (very) brief and (highly) subjective history of combinatorics

Combinatorics must always have been fun. But when and how did it become a serious subject? I see several main steps in this development:

- The asymptotic perspective.
- Extremal combinatorics (in particular extremal graph theory).
- The emergence of the probabilistic method.
- The computational perspective.
So, what is the next frontier?
The ubiquity of graphs

Why do we see graphs all around us in computer science and in all other mathematical sciences, theoretical or applied?
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Why do we see graphs all around us in computer science and in all other mathematical sciences, theoretical or applied? Because they are the tool of choice in modeling pairwise interactions. But what if we have relations involving more than two objects at a time?
A little about simplicial complexes

This is one of the major contact points between combinatorics and geometry (more specifically - with topology).
From the combinatorial point of view, this is a very simple and natural object. Namely, a down-closed family of sets.
Definition
Let $V$ be a finite set of vertices. A collection of subsets $X \subseteq 2^V$ is called a simplicial complex if it satisfies the following condition:

$$A \in X \text{ and } B \subseteq A \Rightarrow B \in X.$$
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What is high-dimensional combinatorics?
We view \( A \in X \) and \(|A| = k + 1\) as a \( k \)-dimensional simplex.
Putting simplices together properly

The intersection of every two simplices in $X$ is a common face.
How NOT to do it

Not every collection of simplices in $\mathbb{R}^d$ is a simplicial complex
Geometric equivalence

Combinatorially different complexes may correspond to the same geometric object (e.g. via subdivision)
Geometric equivalence

So
Geometric equivalence

and

What is high-dimensional combinatorics?
Geometric equivalence

are two different combinatorial descriptions of the same geometric object
To make a long story short

- Graphs need no advertising for computer scientists.

What is high-dimensional combinatorics?
To make a long story short

- Graphs need no advertising for computer scientists.
- A graph may be viewed as a one-dimensional simplicial complex.

Higher dimensional complexes have a very geometric (mostly topological) aspect to them.

Can we benefit from investigating higher dimensional complexes?

How should this be attacked?

1. Using extremal combinatorics
2. With the probabilistic method
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- Can we benefit from investigating higher dimensional complexes?
- How should this be attacked?
  1. Using extremal combinatorics
  2. With the probabilistic method
My description of past relevant work is extremely incomplete.
My deep and sincere apology to all those whose work I have no time to mention.
This (academic) misdemeanor will be justified if the audience finds this interesting and starts learning the subject at depth.
Do SC’s have anything to do with TCS?

So far there are only very few (but very impressive) such examples. I believe that there are many important connections waiting to be discovered. Some of the past achievements:

▶ Applications to the evasiveness conjecture (See below).
▶ Impossibility theorems in distributed asynchronous computation (Starting with [Herlihy, Shavit '93] and [Saks, Zaharoglou '93]).
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- Applications to the evasiveness conjecture (See below).
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Here the list is a bit longer, e.g.,

- To graph connectivity (Lovász’s proof of A. Frank’s conjecture 1977).
- Lower bounds on chromatic numbers of Kneser’s graphs and hypergraphs. (Starting with [Lovász ’78]).
- To matching in hypergraphs (Starting with [Aharoni Haxell ’00]).
The evasiveness game

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This is done through a two-person game as follows:

At each round Alice points at two vertices \( x, y \in V \) and Bob answers whether they are adjacent in \( G \), i.e. whether or not \( xy \in E \).
The evasiveness game

Fix a down-monotone graph property $\mathcal{P}$ (e.g., being disconnected, being planar, being $k$-colorable, containing a large independent set...). We want to determine if a (presently unknown) $n$-vertex graph $G = (V, E)$ has property $\mathcal{P}$. This is done through a two-person game as follows: At each round Alice points at two vertices $x, y \in V$ and Bob answers whether they are adjacent in $G$, i.e. whether or not $xy \in E$. The game ends when Alice knows with certainty whether $G$ has property $\mathcal{P}$. 

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What is high-dimensional combinatorics?
Conjecture

For every monotone graph property $\mathcal{P}$, Bob has a strategy that forces Alice to query all $\binom{n}{2}$ pairs of vertices in $V$.
Q: How is this related to simplicial complexes, topology etc.? 
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A: Fix $n$, the number of vertices in the graphs we consider. Think of an $n$-vertex graph as a subset of $W = \binom{[n]}{2}$. (Careful: $W$ is the set of vertices of the complex we consider).
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If $\mathcal{G}$ is the collection of all $n$-vertex graphs that have property $\mathcal{P}$, then $\mathcal{G}$ is a simplicial complex (since $\mathcal{P}$ is monotone).
The (simple but useful) observation with which they start is

**Lemma**

*A non-evasive complex is collapsible.*

Collapsibility is a simple combinatorial property of simplicial complexes which we do not define here. It can be thought of as a higher-dimensional analogue of being a forest.
The additional ingredient is that $\mathcal{P}$ is a graph property. Namely, it does not depend on vertex labeling. This implies that the complex $\mathcal{G}$ is highly symmetric. Using some facts from group theory they conclude:

**Theorem (KSS ’83)**

*The evasiveness conjecture holds for all graphs of order $n$ when $n$ is prime.*
How can topology help?

- Fixed-point theorems (Borsuk-Ulam, Sperner's Lemma...).
- Collapsibility, contractibility.
- The "size" of homology, Betti numbers...
- Topological connectivity.
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The major (two-way) major challenge. Or "I have a dream".

To start a systematic attack on topology from a combinatorial perspective.

Using the extremal/asymptotic paradigm.

Introduce the probabilistic method into topology.

Use ideas from topology to develop new probabilistic models (random lifts of graphs should be a small step in this direction...).

Introduce ideas from topology into computational complexity.
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We want to develop a theory of random complexes, similar to random graph theory. Specifically we seek a higher-dimensional analogue to $G(n,p)$. We consider (Though this is by no means the only sensible model).
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We denote by $X(n, p)$ this probability space of two-dimensional complexes.
The art of asking good questions

What properties of these random complexes should we investigate? Perhaps the simplest nontrivial property of a graph is being connected. Here is what Erdős-Rényi showed nearly 50 years ago:

**Theorem (ER ’60)**

The threshold for graph connectivity in $G(n, p)$ is

$$p = \frac{\ln n}{n}$$
When is a simplicial complex connected?

Unlike the situation in graphs, this question has many (in fact infinitely many) meaningful answers when it comes to complexes.

- The vanishing of the first homology (with any ring of coefficients).
- Being simply connected (vanishing of the fundamental group).

What is high-dimensional combinatorics?
A little motivation

There is a simple way to state that "$G = (V, E)$ is connected" in the language of linear algebra.

Consider $M$ the incidence $V \times E$ matrix of $G$ as a matrix over $F_2$. Clearly, $1^TM = 0$, since every column of $M$ contains exactly two 1's.

Likewise, if $S$ is the vertex set of a connected component of $G$, then $1^SM = 0$.

It is not hard to see that $G$ is connected iff the only vector $x$ that satisfies $xM = 0$ is $x = 1$.

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The simplest case: $\left(\mathbb{F}_2\right)$-homology in two dimensions

Let $A_1$ be the $n \times \binom{n}{2}$ inclusion matrix of singletons vs. pairs.

Let $A_2$ be the $\binom{n}{2} \times \binom{n}{3}$ inclusion matrix of pairs vs. triples.

The transformations associated with $A_1$ resp. $A_2$ are called the boundary operator (of the appropriate dimension) and are denoted $\partial$ (perhaps with an indication of the dimension).

It is an easy exercise to verify that $A_1A_2 = 0$ (in general there holds $\partial\partial = 0$, a key fact in homology theory).

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Vanishing of the first homology

Since $A_1A_2 = 0$, it follows that every vector in the column space of $A_2$ is in the right kernel of $A_1$. 

Actually, this is the right kernel of $A_1$.

To a two-dimensional complex in $\mathbb{X}(n,p)$ corresponds a random matrix $B$ that we obtain by selecting a random subset of the $\binom{n}{3}$ columns in $A_2$ where each column is selected independently with probability $p$.

We can now consider our high-dimensional notion of being connected: The first homology with $F_2$ coefficients vanishes.
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To a two-dimensional complex in $X(n, p)$ corresponds a random matrix $B$ that we obtain by selecting a random subset of the $(\begin{bmatrix} n \\ 3 \end{bmatrix})$ columns in $A_2$ where each column is selected independently with probability $p$.

We can now consider our high-dimensional notion of being connected: The first homology with $\mathbb{F}_2$ coefficients vanishes.
Concretely, we ask

**Question**

*What is the critical \( p \) at which*

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\text{Im}(B) = \ker(A_1)
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Theorem (L. + Meshulam ’06)
The threshold for the vanishing of the first homology in $X(n, p)$ over $\mathbb{F}_2$ is

$$p = \frac{2 \ln n}{n}$$
A bit more accurately...

If $M_1$ (resp. $M_2$) are the inclusion matrices of the $(d - 1)$-dimensional vs. $d$-faces (resp. $d$-faces vs. $(d + 1)$-faces). Again the relation $M_1 M_2 = 0$ holds.
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$$\text{Im}(M_2) = \ker(M_1).$$
The quotient \( \ker(M_1)/\text{Im}(M_2) \) is the homology group (of the appropriate dimension).
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Here you can also see why our "dummies" version does not apply when the coefficients come from a ring (such as $\mathbb{Z}$) and not from a field.
The vanishing of the \((d - 1)\)-st homology

The above result extends to \(d\)-dimensional simplicial complexes with a full \((d - 1)\)-st dimensional skeleton. Also, for other coefficient groups. (Most of this was done by Meshulam and Wallach).

We still do not know, however:

**Question**

*What is the threshold for the vanishing of the \(\mathbb{Z}\)-homology?*
The vanishing of the fundamental group

Theorem (Babson, Hoffman, Kahle ’09 ?)

The threshold for the vanishing of the fundamental group in $X(n,p)$ is near

$$p = n^{-1/2}.$$
Let’s move on to some extremal problems

The grandfather of extremal graph theory is:

Theorem (Turán '41)

Of all $n$-vertex graphs that contain no $K_r + 1$ (complete graph on $r + 1$ vertices), the one with the largest number of edges is the complete $r$-partite graphs with (nearly) equal parts.

In other words, an $n$-vertex graph with more than \( \frac{(r-1)r}{2} n^2 \) edges must contain a $K_r + 1$, and the bound is tight (up to the $o(1)$ term).

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Extremal combinatorics of simplicial complexes

Theorem (Brown, Erdős, Sós '73)

Every $n$-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a two-sphere. The bound is tight.
A word on the upper bound

Definition
If $X$ is a two-dimensional complex, then the link of a vertex $u$ is a graph whose edge set contains all pairs $vw$ such that the triple $uvw$ is a simplex in $X$.

The analogous definition applies in all dimensions, of course. You can also consider links of larger sets (not just singleton), the definition is essentially the same.
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You can also consider links of larger sets (not just singleton), the definition is essentially the same.
Since $X$ contains $\Omega(n^{5/2})$ two-dimensional simplices, the average link size (number of edges in the graph) is $\Omega(n^{3/2})$. Consequently, there are two vertices say $x$ and $y$ such that their links have $\Omega(n)$ edges in common. In particular, there is a cycle that is included in the link of $x$ as well as in the link of $y$. This gives a double pyramid which is homeomorphic to a two-sphere.
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- (With Friedgut:) $\Omega(n^{8/3})$ simplices suffice.
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**Theorem**

*For every two integers $g$ and $k$ there exist graphs with girth $\geq g$ and chromatic number $\geq k$.***
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With L. Aronshtam (work in progress) we can show:

**Theorem**

*For every two integers $g$ and $k$ there exist two-dimensional complexes with a full one-dimensional skeleton, such that for every vertex $x$, the link of $x$ is a graph with girth $\geq g$ and chromatic number $\geq k$.***

Much more remains to be done here.
Even very elementary subjects in combinatorics take on a new life when you think high-dimensionally.
Even very elementary subjects in combinatorics take on a new life when you think high-dimensionally. What is a permutation? It’s an $n \times n$ array of zeros and ones where every line (i.e., a row or a column) contains exactly a single 1. We know of course:

- How many they are: $n!$
- How to sample a random permutation.
- Numerous typical properties of random permutations e.g.,:
  - Number of fixed points.
  - Number of cycles.
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Nati Linial

What is high-dimensional combinatorics?
... and when you have a hammer...

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An alternative description: An $n \times n$ array $M$ where $m_{ij}$ gives the unique $k$ for which $a_{ijk} = 1$. It is easy to verify that $M$ is defined by the condition that every row and column in $M$ is a permutation of $[n]$. 

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An alternative description: An $n \times n$ array $M$ where $m_{ij}$ gives the unique $k$ for which $a_{ijk} = 1$. It is easy to verify that $M$ is defined by the condition that every row and column in $M$ is a permutation of $[n]$. Such a matrix is called a Latin square.
Some challenges

So this raises

**Question**

*Determine or estimate $L_n$, the number of $n \times n$ Latin squares.*
Some challenges

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Currently the best known bound is:

**Theorem (van Lint and Wilson)**

$$(\mathcal{L}_n)^{1/n^2} = (1 + o(1)) \frac{n}{e^2}.$$
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The (fairly easy) proof uses two substantial facts about permanents: The proof of the van der Waerden conjecture and Brégman’s Theorem. This raises:
Some challenges

▶ Improve this bound (which only determines $\mathcal{L}_n$ up to $e^{o(n^2)}$).

▶ Solve the even higher dimensional cases.

▶ Factorials are, of course, closely related to the Gamma function. Are there higher dimensional analogues of $\Gamma$?
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- Solve the even higher dimensional cases.
- Factorials are, of course, closely related to the Gamma function. Are there higher dimensional analogues of $\Gamma$?
Let us quickly recall the notion of tensor rank. But first a brief reminder of matrix rank. A matrix $A$ has rank one iff there exist vectors $x$ and $y$ such that $a_{ij} = x_i y_j$.

**Proposition**

*The rank of a matrix $M$ is the least number of rank-one matrices whose sum is $M$.***
More on tensors...

All of this extends to tensors almost verbatim: A three-dimensional tensor $A$ has rank one iff there exist vectors $x, y$ and $z$ such that $a_{ijk} = x_i y_j z_k$.

**Definition**
The rank of a three-dimensional tensor $Z$ is the least number of rank-one tensors whose sum is $Z$. 

Nati Linial

What is high-dimensional combinatorics?
Open Problem

*What is the largest rank of an \( n \times n \times n \) real tensor.*

It is only known (and easy) that the answer is between \( \frac{n^2}{3} \) and \( \frac{n^2}{2} \). With A. Shraibman we have constructed a family of examples which suggests

Conjecture

*The answer is \((1 + o(1)) \frac{n^2}{2}\)*
Can you believe that this question is open?

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Our ignorance may be somewhat justified since tensor rank is NP-hard to determine (Hastad ’90).
... and why should you care?

The complexity of matrix multiplication is given by the rank of certain tensors (well, \textit{border rank}, but we do not go into this).
... and why should you care?

The complexity of matrix multiplication is given by the rank of certain tensors (well, border rank, but we do not go into this).
Singular Value Decomposition (SVD) of matrices is one of the most practically important algorithms in computational linear algebra. It is a fascinating challenge to develop an analogous theory for higher-dimensional tensors.
THAT’S ALL, FOLKS....