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LIFTS, DISCREPANCY AND NEARLY OPTIMAL SPECTRAL GAP*

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We present a new explicit construction for expander graphs with nearly optimal spectral gap. The construction is based on a series of 2-lift operations. Let G be a graph on n vertices. A 2-lift of G is a graph H on $2n$ vertices, with a covering map $\pi : H \to G$. It is not hard to see that all eigenvalues of G are also eigenvalues of H. In addition, H has n "new" eigenvalues. We conjecture that every d-regular graph has a 2-lift such that all new eigenvalues are in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (if true, this is tight, e.g. by the Alon–Boppana bound). Here we show that every graph of maximal degree d has a 2-lift such that all "new" eigenvalues are in the range $\left[-c\sqrt{d\log^3 d}, c\sqrt{d\log^3 d}\right]$ for some constant c. This leads to a deterministic polynomial time algorithm for constructing arbitrarily large d-regular graphs, with second eigenvalue $O\left(\sqrt{d\log^3 d}\right)$.

The proof uses the following lemma (Lemma 3.3): Let A be a real symmetric matrix with zeros on the diagonal. Let d be such that the l_1 norm of each row in A is at most d. Suppose that $\frac{|x^t A y|}{\|x\|\|y\|} \le \alpha$ for every $x, y \in \{0,1\}^n$ with $\langle x, y\rangle = 0$. Then the spectral radius of A is $O(\alpha(\log(d/\alpha)+1))$. An interesting consequence of this lemma is a converse to the Expander Mixing Lemma.

1. Introduction

An d-regular graph is called a λ-*expander*, if all its eigenvalues but the first are in $[-\lambda, \lambda]$. Such graphs are interesting when d is fixed, $\lambda < d$, and the number of vertices in the graph tends to infinity. Applications of such graphs

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in computer science and discrete mathematics are many, see for example [24] for a survey.

It is known that random d-regular graphs are good expanders ([13], [18], [16], [15]), yet many applications require an explicit construction. Some known constructions of such graphs appear in [26], [20], [7], [25], [5], [27], [1] and [30]). The Alon–Boppana bound says that $\lambda \geq 2\sqrt{d-1} - o(1)$ (cf. [29]). The graphs of [25] and [27] satisfy $\lambda \leq 2\sqrt{d-1}$, for infinitely many values of d, and are constructed very efficiently. However, the analysis of the eigenvalues in these construction relies on deep mathematical results. Thus, it is interesting to look for construction whose analysis is elementary.

The first major step in this direction is a construction based on iterative use of the zig-zag product [30]. This construction is simple to analyze, and is very explicit, yet the eigenvalue bound falls somewhat short of the Alon–Boppana bound. The graphs constructed with the zig-zag product have second eigenvalue $O(d^{3/4})$, which can be improved, with some additional effort to $O(d^{2/3})$. Here we introduce an iterative construction based on 2-lifts of graphs, which is close to being optimal and gives $\lambda = O\big(\sqrt{d \log^3 d}\big)$.

A graph $\hat{G}$ is called a k-lift of a "base graph" G if there is a $k:1$ *covering map* $\pi : V(\hat{G}) \to V(G)$. Namely, if $y_1, \ldots, y_d \in G$ are the neighbors of $x \in G$, then every $x' \in \pi^{-1}(x)$ has exactly one vertex in each of the subsets $\pi^{-1}(y_i)$. See [9] for a general introduction to graph lifts.

The study of lifts of graphs has focused so far mainly on random lifts [9–11, 23, 17]. In particular, Amit and Linial show in [10] that w.h.p. a random k-lift has a strictly positive edge expansion. It is not hard to see that the eigenvalues of the base graph are also eigenvalues of the lifted graph. These are called by Joel Friedman the "old" eigenvalues of the lifted graph. In [17] he shows that w.h.p. a random k-lift of a d-regular graph on n vertices is "weakly Ramanujan". Namely, that all new eigenvalues are, in absolute value, $O(d^{3/4})$. In both cases the probability tends to 1 as k tends to infinity.

Here we study 2-lifts of graphs. We conjecture that every d regular graph has a 2-lift with all new eigenvalues at most $2\sqrt{d-1}$ in absolute value. It is not hard to show (e.g., using the Alon–Boppana bound [29]) that if this conjecture is true, it is tight. We prove (in Theorem 3.1) a slightly weaker result; every graph of maximal degree d has a 2-lift with all new eigenvalues $O\big(\sqrt{d \log^3 d}\big)$ in absolute value. Under some natural assumptions on the base graph, such a 2-lift can be found efficiently. This leads to a deterministic polynomial time algorithm for constructing families of d-regular expander graphs, with second eigenvalue $O\big(\sqrt{d \log^3 d}\big)$.

A useful property of expander graphs is the so-called Expander Mixing Lemma. Roughly, this lemma states that the number of edges between two subsets of vertices in an expander graph is what is expected in a random graph, up to an additive error that depends on the second eigenvalue.

A key lemma in this paper (Lemma 3.3) shows a close connection between the combinatorial discrepancy in a symmetric martix, and its spectral radius. This key lemma implies the following converse to the Expander Mixing Lemma: Let G be a d-regular graph on n vertices, such that for every two subsets of vertices, A and B, $\left|e(A,B) - d|A||B|/n\right| \le \alpha\sqrt{|A||B|}$ for some $\alpha < d$ (where $e(A,B)$ is the number of edges between A and B). Then all eigenvalues of G but the first are, in absolute value, $O(\alpha\log(d/\alpha))$. The fact that the bound is tight up to a logarithmic factor is surprising. It is known that expansion implies a spectral gap (cf. [3]), but the actual bounds are weak, and indeed expansion alone does not imply strong bounds on the spectral gap [2].

The paper is organized as follows. After defining the basic objects – expander graphs, signed graphs and 2-lifts – in section 2, we present the main results in section 3. In sub-section 3.1 we observe that the spectrum of 2-lifts has a simple description, which suggests an iterative construction of expander graphs (described in sub-section 3.2). It reduces the problem of constructing expander graphs to finding a *signing* of the edges with a small spectral radius. In sub-section 3.3 we show that such a signing always exists. In sub-section 3.4 we show how to find such a signing efficiently. An alternative method is given in section 4, which leads to a somewhat stronger notion of explicitness. Finally, in section 5 we prove the converse to the Expander Mixing Lemma mentioned above.

2. Definitions

Let $G = (V,E)$ be a graph on n vertices, and let A be its adjacency matrix. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be the eigenvalues of A. We denote by $\lambda(G) = \max_{i=2,\ldots,n} |\lambda_i|$. We say that G is an $(n,d,\mu)-$*expander* if G is d-regular, and $\lambda(G) \le \mu$. If $\lambda(G) \le 2\sqrt{d-1}$ we say that G is *Ramanujan*.

A *signing* of the edges of G is a function $s : E(G) \to \{-1,1\}$. The *signed adjacency matrix* of a graph G with a signing s has rows and columns indexed by the vertices of G. The (x,y) entry is $s(x,y)$ if $(x,y) \in E$ and 0 otherwise.

The 2-*lift* of G associated with a signing s is a graph $\hat{G}$ defined as follows. Associated with every vertex $x \in V$ are two vertices, x_0 and x_1, called the *fiber* of x. If $(x,y) \in E$, and $s(x,y) = 1$ then the corresponding edges in $\hat{G}$ are (x_0,y_0) and (x_1,y_1). If $s(x,y) = -1$, then the corresponding edges in $\hat{G}$ are (x_0,y_1) and (x_1,y_0). The graph G is called the *base graph*, and $\hat{G}$ a 2-lift of G. By the *spectral radius of a signing* we refer to the spectral radius of the corresponding signed adjacency matrix. When the spectral radius of a signing of a d-regular graph is $\tilde{O}(\sqrt{d})$ we say that the signing (or the lift) is *Quasi-Ramanujan*.

For $v,u \in \{-1,0,1\}^n$, denote $S(u) = supp(u)$ (the set of indices i s.t. $u_i \neq 0$), and $S(u,v) = supp(u) \cup supp(v)$.

It will be convenient to assume throughout that $V(G) = \{1,\dots,n\}$.

3. Quasi-Ramanujan 2-Lifts and Quasi-Ramanujan Graphs

3.1. The eigenvalues of a 2-lift

The eigenvalues of a 2-lift of G can be easily characterized in terms of the adjacency matrix and the signed adjacency matrix:

Lemma 3.1. *Let A be the adjacency matrix of a graph G, and A_s the signed adjacency matrix associated with a 2-lift $\hat{G}$. Then every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\hat{G}$. Furthermore, the multiplicity of each eigenvalue of $\hat{G}$ is the sum of its multiplicities in A and A_s.*

Proof. It is not hard to see that the adjacency matrix of $\hat{G}$ is:

$$\hat{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix},$$

where A_1 is the adjacency matrix of $(V, s^{-1}(1))$ and A_2 the adjacency matrix of $(V, s^{-1}(-1))$. (So $A = A_1 + A_2$, $A_s = A_1 - A_2$.) Let v be an eigenvector of A with eigenvalue μ. It is easy to check that $\hat{v} = (v\,v)$ is an eigenvector of $\hat{A}$ with eigenvalue μ.

Similarly, if u is an eigenvector of A_s with eigenvalue λ, then $\hat{u} = (u\,{-u})$ is an eigenvector of $\hat{A}$ with eigenvalue λ.

As the $\hat{v}$'s and $\hat{u}$'s are perpendicular and $2n$ in number, they span all the eigenvectors of $\hat{A}$. ∎

3.2. The construction scheme

We follow Friedman's ([17]) nomenclature, and call the eigenvalues of A the *old* eigenvalues of $\hat{G}$, and those of A_s the *new* ones.

Consider the following scheme for constructing (n,d,λ)-expanders. Start with $G_0 = K_{d+1}$, the complete graph on $d+1$ vertices[1]. Its eigenvalues are d,

[1] We could start with any small d-regular graph with a large spectral gap. Such graphs are easy to find.

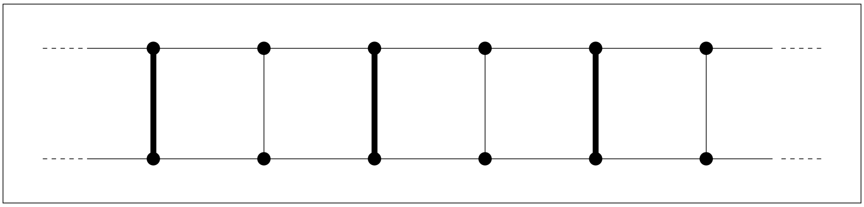

Figure 1. The Railway Graph. Edges where the signing is -1 are bold.

with multiplicity 1, and -1, with multiplicity d. We want to define G_i as a 2-lift of G_{i-1}, such that all new eigenvalues are in the range $[-\lambda,\lambda]$. Assuming such a 2-lifts always exist, the G_i constitute an infinite family of (n,d,λ)-expanders.

It is therefore natural to look for the smallest $\lambda=\lambda(d)$ such that every graph of degree at most d has a 2-lift with new eigenvalues in the range $[-\lambda,\lambda]$. In other words, a signing with spectral radius $\le\lambda$.

We note that $\lambda(d)\ge 2\sqrt{d-1}$. Otherwise, using the scheme above we could get graphs that violate the Alon–Boppana bound. We next observe:

Proposition 3.1. *Let G be a d-regular graph which contains a vertex that does not belong to any cycle of bounded length, then no signing of G has spectral radius below $2\sqrt{d-1}-o(1)$.*

To see this, note first that all signings of a tree have the same spectral radius. This follows e.g., from the easy fact that any 2-lift of a tree is a union of two disjoint trees, isomorphic to the base graph. The assumption implies that G contains an induced subgraph that is a full d-ary tree T of unbounded radius. The spectral radius of T is $2\sqrt{d-1}-o(1)$. The conclusion follows now from the interlacing principle of eigenvalues (cf. [21]).

There are several interesting examples of arbitrarily large d-regular graphs for which there is a signing with spectral radius bounded away from $2\sqrt{d-1}$. One such example is the 3-regular graph R defined as follows. $V(R)=\{0,\dots,2k-1\}\times\{0,1\}$. For $i\in\{0,\dots,2k-1\}$, $j\in\{0,1\}$, the neighbors of $(i,j)\in R$ are $((i-1)\bmod 2k,j)$, $((i+1)\bmod 2k,j)$ and $(i,1-j)$. Define s, a signing of R, to be -1 on the edges $((2i,0),(2i,1))$, for $i\in\{0,\dots,k-1\}$, and 1 elsewhere (see Figure 1). Let A_s be the signed adjacency matrix. It is easy to see that A_s^2 is a matrix with 3 on the diagonal, and two 1's in each row and column. Thus, its spectral radius is 5, and that of A_s is $\sqrt{5}<2\sqrt{2}$.

3.3. Quasi-Ramanujan 2-lifts for every graph

We conjecture that every graph has a signing with small spectral radius:

Conjecture 3.1. Every d-regular graph has a signing with spectral radius at most $2\sqrt{d-1}$.

We have numerically tested this conjecture quite extensively.
In this subsection we show a close upper bound:

Theorem 3.1. *Every graph of maximal degree d has a signing with spectral radius $O\left(\sqrt{d\cdot\log^3 d}\right)$.*

The theorem is an easy consequence of the following two lemmata. The first one uses a probabilistic argument to show the existence of a signing for which the Rayleigh quotient is small for vectors in $v,u\in\{-1,0,1\}^n$. The second shows how to conclude from this that the Rayleigh quotient for *all* vectors is small – and therefore all new eigenvalues are small as well.

Lemma 3.2. *For every graph of maximal degree d, there exists a signing s such that for all $v,u\in\{-1,0,1\}^n$ the following holds:*

$$\frac{|v^t A_s u|}{\|v\|\|u\|} \le 10\sqrt{d\log_2 d}, \tag{1}$$

where A_s is the signed adjacency matrix.

Lemma 3.3. *Let A be an $n\times n$ real symmetric matrix such that the l_1 norm of each row in A is at most d, and all diagonal entries of A are, in absolute value, $O(\alpha(\log(d/\alpha)+1))$. Assume that for any two vectors, $u,v\in\{0,1\}^n$, with $supp(u)\cap supp(v)=\emptyset$:*

$$\frac{|u^t A v|}{\|u\|\|v\|} \le \alpha.$$

Then the spectral radius of A is $O(\alpha(\log(d/\alpha)+1))$.

Proof of Lemma 3.2. First note that it's enough to prove this for u's and v's such that the set $S(u,v)$ spans a connected subgraph. Indeed, suppose that $S(u,v)$ is not connected. Let $S_1,\dots,S_k$ be the connected subgraphs of $S(u,v)$. Split $u=\sum_{i=1}^k u^i$, so that $supp(u^i)\subset S_i$. Define $v^1,\dots,v^k$ similarly. Observe that for $i\neq j$, $(u^i)^t A v^j = 0$, since there are no edges between S_i and S_j. Assume that the lemma holds for connected components, that is, for $i=1,\dots,k$, $|(u^i)^t A v^i|\le 10\sqrt{d\log_2 d}\|u^i\|\|v^i\|$. We have that:

$$\begin{aligned}|u^t A v| \le \sum_{i,j=1}^k \left|(u^i)^t A v^j\right| = \sum_{i=1}^k \left|(u^i)^t A v^i\right| &\le \sum_{i=1}^k 10\sqrt{d\log_2 d}\left\|u^i\right\|\left\|v^i\right\| \\ &\le 10\sqrt{d\log_2 d}\|u\|\|v\|,\end{aligned}$$

where the last inequality follows from the Cauchy–Schwartz inequality, and the fact that $\|u\|^2=\sum_{i=1}^{k}\|u^i\|^2$, $\|v\|^2=\sum_{i=1}^{k}\|v^i\|^2$.

Thus, henceforth we assume that $S(u,v)$ is connected. We will also assume that d is somewhat large. This is justified by that fact that for any signing, $\frac{|v^tA_su|}{\|v\|\|u\|}\le d$. Thus, the claim holds trivially for $d<997$, since $d\le 10\sqrt{d\log_2 d}$ in this range.

Consider some $u,v\in\{-1,0,1\}^n$. Suppose we choose the sign of each edge uniformly at random. Denote the resulting signed adjacency matrix by A_s, and let $E_{u,v}$ be the "bad" event that $\frac{|v^tA_su|}{\|v\|\|u\|}>10\sqrt{d\log_2 d}$. Assume w.l.o.g. that $|S(u)|\ge\frac{1}{2}|S(u,v)|$. Observe that $v^tA_su=\sum_{i\in S(v),j\in S(u)}(A_s)_{i,j}$ is the sum of independent variables, attaining values of either ± 2, when both $(A_s)_{i,j}$, $(A_s)_{j,i}$ appear in the sum, or ± 1 otherwise. By the Chernoff inequality:

$$\begin{aligned}Pr[E_{u,v}] &\le 2\exp\left(-\frac{100d\log_2 d|S(u)||S(v)|}{8e(S(u),S(v))}\right)\\ &\le 2\exp\left(-\frac{100d\log_2 d|S(u)||S(v)|}{8d|S(v)|}\right)\\ &< d^{(-6|S(u,v)|)},\end{aligned}$$

where $e(S(u),S(v))$ is the number of edges between $S(u)$ and $S(v)$.

We want to use the Lovász Local Lemma [14], with the following dependency graph on the $E_{u,v}$: There is an edge between $E_{u,v}$ and $E_{u',v'}$ iff $S(u,v)\cap S(u',v')\neq\emptyset$. Denote $k=|S(u,v)|$. How many neighbors, $E_{u',v'}$, does $E_{u,v}$ have, with $|S(u',v')|=l$?

Since we are interested only in connected subsets, this is clearly bounded by the number of rooted directed subtrees on l vertices, with a root in $S(u,v)$. It is known (cf. [22]) that there are at most $k\binom{d(l-1)}{l-1}\approx kd^{l-1}$ such trees (a similar argument appears in [19]. The bound on the number of trees is essentially tight by [4]).

In order to apply the Local Lemma, we need to define for such u and v a number $0\le X_{u,v}<1$. It is required that:

$$X_{u,v}\prod_{(u',v'):E_{u,v}\sim E_{u',v'}}(1-X_{u',v'})\ge d^{-6|S(u,v)|}. \tag{2}$$

Observe that for $S\subseteq[n]$ there are at most $2^{4|S|}$ distinct pairs $v,u\in\{-1,0,1\}^n$ such that $S(u,v)=S$.

For all u,v set $X_{u,v}=d^{-3k}$, where $k=|S(u,v)|$. Then in (2) we get:

$$X_{u,v}\cdot\prod_{(u',v'):E_{u,v}\sim E_{u',v'}}(1-X_{u',v'})=d^{-3k}\prod_{l=1}^{n}\left(1-d^{-3l}\right)^{kd^l2^{4l}}$$
$$\geq d^{-3k}\exp\left(-2k\sum_{l=1}^{n}d^{-3l}d^l2^{4l}\right)\geq d^{-3k}e^{-3k}>d^{-6k}$$

as required (the last two inequalities rely on the fact that d is large). ∎

Proof of Lemma 3.3. For simplicity, assume first that all diagonal entries of A are zeros. We explain at the end of the proof how to deal with a general matrix.

We start be showing that our assumptions imply that for any $u\in\{0,1\}^n$,

$$\frac{|u^tAu|}{\|u\|^2}\leq 2\alpha. \tag{3}$$

For any $u_1,u_2\in\{0,1\}^n$ such that $S(u_1)\cap S(u_2)=\emptyset$ we have that

$$|u_1Au_2|\leq\alpha\|u_1\|\|u_2\|. \tag{4}$$

Let $u\in\{0,1\}^n$, and denote $k=|S(u)|$. For simplicity we assume that k is even, a similar argument works for the odd case. Set $K=\binom{k}{k/2}$. Summing up inequality (4) over all subsets of $S(u)$ of size $k/2$, we have that:

$$\sum_{u_1:S(u_1)\subset S(u),|S(u_1)|=k/2}|u_1A(u-u_1)|\leq K\alpha k/2.$$

For each $i\neq j\in S(u)$, $a_{i,j}$ is added up $\binom{k-2}{k/2-1}$ times in the sum on the LHS, hence (since diagonal entries are by assumption zero):

$$\binom{k-2}{k/2-1}|u^tAu|\leq K\alpha k/2,$$

or:

$$|u^tAu|\leq\alpha 2k.$$

Thus, inequality 3 indeed holds.

Next, it follows that for any $u,v\in\{-1,0,1\}^n$, such that $S(u)=S(v)$, or $S(u)\cap S(v)=\emptyset$:

$$\frac{|u^tAv|}{\|u\|\|v\|}\leq 4\alpha.$$

Fix $u,v \in \{-1,0,1\}^n$. Denote $u = u^+ - u^-$ and $v = v^+ - v^-$, where $u^+,u^-,v^+,v^- \in \{0,1\}^n$, and $S(u^+)\cap S(u^-) = S(v^+)\cap(v^-) = \emptyset$.

$$\begin{aligned}
\left|u^tAv\right| &= \left|(u^+ - u^-)^t A(v^+ - v^-)\right| \\
&\le 2\alpha\left(\|u^+\|\|v^+\| + \|u^+\|\|v^-\| + \|u^-\|\|v^+\| + \|u^-\|\|v^-\|\right) \\
&\le 4\alpha\sqrt{\|u^+\|^2\|v^+\|^2 + \|u^+\|^2\|v^-\|^2 + \|u^-\|^2\|v^+\|^2 + \|u^-\|^2\|v^-\|^2} \\
&= 4\alpha\sqrt{\left(\|u^+\|^2 + \|u^-\|^2\right)\left(\|v^+\|^2 + \|v^-\|^2\right)} = 4\alpha\|u\|\|v\|\,.
\end{aligned}$$

The first inequality follows from the hypothesis on vectors in $\{0,1\}^n$, and the second from the l_2 to l_1 norm ratio.

Fix $x \in \mathbb{R}^n$. We need to show that $\frac{|x^tAx|}{\|x\|^2} = O(\alpha\log(d/\alpha))$. By losing only a multiplicative factor of 2, we may assume that the absolute value of every non-zero entry in x is a negative power of 2: Clearly we may assume that $\|x\|_\infty < \frac{1}{2}$. To bound the effect of rounding the coordinates, denote $x_i = \pm(1+\delta_i)2^{t_i}$, with $0 \le \delta_i \le 1$ and $t_i < -1$, an integer. Now round x to a vector x' by choosing the value of x'_i to be $\mathrm{sign}(x_i)\cdot 2^{t_i+1}$ with probability δ_i and $\mathrm{sign}(x_i)\cdot 2^{t_i}$ with probability $1-\delta_i$. The expectation of x'_i is x_i. As the coordinates of x' are chosen independently, and the diagonal entries of A are 0's, the expectation of x'^tAx' is x^tAx. Thus, there is a rounding, x', of x, such that $|x^tAx| \le |x'^tAx'|$. Clearly $\|x'\|^2 \le 2\|x\|^2$, so $\frac{|x^tAx|}{\|x\|^2} \le 2\frac{|x'^tAx'|}{\|x'\|^2}$.

Denote $S_i = \{j : x_j = \pm 2^{-i}\}$, $s_i = |S_i|$. Denote by k the maximal index i such that $s_i > 0$. Denote by x^i the sign vector of x restricted to S_i, that is, the vector whose j'th coordinate is the sign of x_j if $j \in S_i$, and zero otherwise. By our assumptions, for all $1 \le i \le j \le k$:

$$\left|(x^i)^tAx^j\right| \le \alpha\sqrt{s_is_j}. \tag{5}$$

Also, since the l_1 norm of each row is at most d, for all $1 \le i \le k$:

$$\sum_j \left|(x^i)^tAx^j\right| \le ds_i. \tag{6}$$

We wish to bound:

$$\frac{|x^tAx|}{\|x\|^2} \le \frac{\sum_{i,j=1}^k \left|(x^i)^tAx^j\right|2^{-(i+j)}}{\sum_i 2^{-2i}s_i}. \tag{7}$$

Denote $\gamma = \log_2(d/\alpha)$, $q_i = s_i2^{-2i}$ and $Q = \sum_i q_i$. Add up inequalities (5) and (6) as follows. For $i = j$ multiply inequality (5) by 2^{-2i}. When $i < j \le i+\gamma$

multiply it by $2^{-(i+j)+1}$. Multiply inequality (6) by $2^{-(2i+\gamma)}$. (We ignore inequalities (5) when $j>i+\gamma$.)

We get that:

$$\sum_i 2^{-2i}\big|(x^i)^t Ax^i\big| + \sum_i \sum_{i<j\le i+\gamma} 2^{-(i+j)+1}\big|(x^i)^t Ax^j\big| + \\ + \sum_i 2^{-(2i+\gamma)} \sum_j |(x^i)^t Ax^j|$$

$$\le \sum_i \alpha q_i + \sum_i \sum_{i<j<i+\gamma} 2\alpha\sqrt{q_i q_j} + \sum_i 2^{-\gamma} d\cdot q_i$$

$$\le \alpha\sum_i q_i + \alpha\sum_i\sum_{i<j<i+\gamma}(q_i+q_j) + 2^{-\gamma}d\sum_i q_i$$

$$< (2^{-\gamma}d + 2\gamma\alpha)\sum_i q_i = \big(\alpha + \alpha\log_2(d/\alpha)\big)Q.$$

Note that the denominator in (7) is Q, so to prove the lemma it's enough to show that the numerator,

$$\sum_{i<j} 2^{-(i+j+1)}\big|(x^i)^t Ax^j\big| + \sum_i 2^{-2i}\big|(x^i)^t Ax^i\big|, \tag{8}$$

is bounded by

$$\sum_i 2^{-2i}\big|(x^i)^t Ax^i\big| + \sum_i \sum_{i\le j\le i+\gamma} 2^{-(i+j)+1}\big|(x^i)^t Ax^j\big| + \\ + \sum_i 2^{-(2i+\gamma)} \sum_j \big|(x^i)^t Ax^j\big|. \tag{9}$$

Indeed, let us compare the coefficients of the terms $|(x^i)^t Ax^j|$ in both expressions (since $|(x^i)^t Ax^j| = |(x^j)^t Ax^i|$, it's enough to consider $i\le j$). For $i=j$ this coefficient is 2^{-2i} in (8), and $2^{-2i}+2^{-(2i+\gamma)}$ in (9). For $i<j\le i+\gamma$, it is $2^{-(i+j)+1}$ in (8), and in (9) it is $2^{-(i+j)+1}+2^{-(2i+\gamma)}+2^{-(2j+\gamma)}$. For $j>i+\gamma$, in (8) the coefficient is again $2^{-(i+j)+1}$. In (9) it is:

$$2^{-(2i+\gamma)} + 2^{-(2j+\gamma)} > 2^{-(2i+\gamma)} \ge 2^{-(i+j)+1}.$$

It remains to show that the lemma holds when the diagonal entries of A are not necessarily zero, but $O(\alpha(\log(d/\alpha)+1))$ in absolute value. Denote $B=A-D$, with D the matrix having the entries of A on the diagonal, and zero elsewhere. We have that for any two vectors, $u,v\in\{0,1\}^n$, with $supp(u)\cap supp(v)=\emptyset$:

$$\frac{|u^t Av|}{\|u\|\|v\|} \le \alpha.$$

For such vectors, $u^t Bv = u^t Av$, so by applying the lemma to B, we get that its spectral radius is $O(\alpha(\log(d/\alpha)+1))$. By the assumption on the diagonal entries of A, the spectral radius of D is $O(\alpha(\log(d/\alpha)+1))$ as well. The spectral radius of A is at most the sum of these bounds – also $O(\alpha(\log(d/\alpha)+1))$. ∎

An example of Bollobás and Nikiforov [12] shows that, as stated, Lemma 3.3 is tight up to constant factors (there no bound is assumed on the l_1 norm of the rows, and so the bound is $O(\log n)$). For the purpose of this paper, Lemma 3.3 is interesting mainly for matrices where the l_1 norm of all rows is the same, and independent of n. Does a tighter bound hold under these assumptions? In section 5 we construct an example showing that this is not the case.

3.4. An explicit construction of Quasi-Ramanujan graphs

For the purpose of constructing expanders, it is enough to prove a weaker version of Theorem 3.1 – that every *expander* graph has a 2-lift with small spectral radius (see subsection 3.2). In this sub-section we show that when the base graph is a good expander (in the sense of the definition below), then w.h.p. a random 2-lift, where the sign of each edge is chosen uniformly and independently, has a small spectral radius. We then derandomize the construction to get a deterministic polynomial time algorithm for constructing arbitrarily large expander graphs.

Definition 3.1. We say that a graph G on n vertices is (β,t)-sparse if for every $u,v\in\{0,1\}^n$, with $|S(u,v)|\le t$,

$$u^t Av \le \beta\|u\|\|v\|.$$

Lemma 3.4. *Let A be the adjacency matrix of a d-regular $(\gamma(d),\log_2 n)$-sparse G graph on n vertices, where $\gamma(d)=10\sqrt{d\log_2 d}$. Then for a random signing of G (where the sign of each edge is chosen uniformly at random) the following hold w.h.p.:*

1. $\forall u,v\in\{-1,0,1\}^n : |u^t A_s v|\le\gamma(d)\|u\|\|v\|.$
2. *$\hat{G}$ is $(\gamma(d),1+\log_2 n)$-sparse*

where A_s is the random signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

Proof. Assume that (2) holds. Following the same arguments and notations as in the proof of Lemma 3.2, we have that there are at most $n\cdot d^k$ connected subsets of size k. With probability at most d^{-6k} requirement (1) is violated for a given pair u,v such that $|S(u,v)|=k$. For each S there are at most $2^{4|S|}$ pairs u,v such that $S(u,v)=S$. The probability that there is some pair u,v such that $|S(u,v)|>\log_2 n$ and for which (1) does not hold, is thus (by the union bound) at most $\sum_{k>\log_2 n} d^{-6k}2^{4k} \le n^{-6\log_2 d+4}$. If $|S(u,v)|\le\log_2 n$ then by (2) there are simply not enough edges between $S(u)$ and $S(v)$ for (1) to be violated. Thus, it suffices to show that w.h.p. (2) holds.

Let s be a signing, and define A_1, A_2 and $\hat{A}$ as in Lemma 3.1. Given $u=(u_1\ u_2), v=(v_1\ v_2)\in\{0,1\}^n\times\{0,1\}^n$, we wish to prove that $u^t\hat{A}v\le\gamma(d)\|u\|\|v\|$. As in the proof of Lemma 3.2 we may assume that $S(u,v)$ is connected – in fact, that it is connected via the edges between $S(u)$ and $S(v)$. Hence, we may assume that the ratio of the sizes of these subsets is at most d. Define $x=u_1\vee u_2$, $y=v_1\vee v_2$, $x'=u_1\wedge u_2$, and $y'=v_1\wedge v_2$ (the characteristic vectors of $S(u_1,u_2)$, $S(v_1,v_2)$, $S(u_1)\cap S(u_2)$ and $S(v_1)\cap S(v_2)$). It is not hard to verify that:

$$u^t\hat{A}v = u_1A_1v_1+u_1A_2v_2+u_2A_2v_1+u_2A_1v_2\le x^tAy+x'^tAy'. \tag{10}$$

If $|S(x,y)|\le\log_2 n$, then clearly $|S(x',y')|\le\log_2 n$ and from the assumption that G is $(\gamma(d),\log_2 n)$-sparse

$$x^tAy+x'^tAy'\le\gamma(d)\big(\sqrt{|S(x)||S(y)|}+\sqrt{|S(x')||S(y')|}\big).$$

Observe that $|S(u)|=|S(x)|+|S(x')|$ and $|S(v)|=|S(y)|+|S(y')|$, so in particular $u^t\hat{A}v\le\gamma(d)\sqrt{|S(u)||S(v)|}$, and requirement (2) holds.

So assume $|S(x,y)|=|S(u,v)|=\log_2 n+1$. It is not hard to see that this entails $S(u_1,v_1)\cap S(u_2,v_2)=\emptyset$. In other words, $S(u,v)$ contains at most one vertex from each fiber. Hence, $x'=y'=\vec{0}$ and $|S(u)|=|S(x)|$, $|S(v)|=|S(y)|$.

Denote $S=S(x,y)$, and assume w.l.o.g. that $|S(y)|>\frac{1}{2}\log_2 n$. From (10) $u^t\hat{A}v\le x^tAy$, so it's enough to show that $x^tAy\le\gamma(d)\sqrt{|S(x)||S(y)|}$. If this is not the case, we can bound the ratio between $|S(x)|$ and $|S(y)|$: Since the graph is of maximal degree d we have $x^tAy\le d|S(X)|$. Hence, $\frac{|S(x)|}{|S(y)|}>\frac{\gamma(d)^2}{d^2}=\frac{100\log_2 d}{d}$.

Observe that the edges between $S(u)$ and $S(v)$ in $\hat{G}$ originate from edges between $S(x)$ and $S(y)$ in G in the following way – for each edge $e\in S(x)\times S(y)$ in G there is, with probability $\frac{1}{2}$, an edge between $S(u)$ and $S(v)$ in $\hat{G}$.

Next we bound x^tAy. Averaging over all $S\setminus\{i\}$, for $i\in S(y)$ we have that:

$$(|S(y)|-2)x^tAy\le|S(y)|\gamma(d)\sqrt{(|S(y)|-1)|S(x)|}.$$

Hence the expectation of $u^t\hat{A}v$ is at most $\frac{1}{2}c\gamma(d)\|u\|\|v\|$, where

$$c = \frac{\sqrt{|S(y)|(|S(y)|-1)}}{|S(y)|-2} \leq 1.1,$$

assuming n is not very small (which we may, since $n>d$, and the lemma holds trivially for small d's). By the Chernoff bound, the probability that $u^t\hat{A}v>\gamma(d)\|u\|\|v\|$ is at most:

$$\begin{aligned} 2\exp\left(-\frac{0.9}{2.2}\gamma(d)\|u\|\|v\|\right) &\leq \exp\left(-0.2\gamma(d)\frac{(\log_2 n+1)(10\sqrt{\log_2 d})}{\sqrt{d}}\right) \\ &= \exp\bigl(-20\log_2 d(\log_2 n+1)\bigr), \end{aligned}$$

Since $\frac{|S(x)|}{|S(y)|} > \frac{100\log_2 d}{d}$, and $|S(v)|>\frac{1}{2}\log_2 n$.

There are at most $d^{\log_2 n+1}4^{\log_2 n+1}$ pairs u,v with $S(u,v)$ connected and of size $\log_2 n+1$, so by the union bound, w.h.p., requirement (2) holds. ∎

Corollary 3.1. *Let A be the adjacency matrix of a d-regular $(\gamma(d),\log_2 n)$-sparse G graph on n vertices, where $\gamma(d)=10\sqrt{d\log_2 d}$. Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:*

1. *The spectral radius of A_s is $O\bigl(\sqrt{d\log^3 d}\bigr)$.*
2. *$\hat{G}$ is $(\gamma(d),1+\log_2 n)$-sparse,*

where A_s is the signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

Proof. Consider a random signing s. For each closed path p in G of length $l=2\lceil\log_2 n\rceil$ define a random variable Y_p equal to the product of the signs of its edges. From lemmas 3.4 and 3.3, the expected value of the trace of A_s^l, which is the expected value of the sum of these variables, is $\bigl(C\sqrt{d\log^3 d}\bigr)^l$, where C is some absolute constant. Note that since l is even the sum is always positive. For each $u,v\in\{0,1\}^n$, with $|S(u,v)|=\log_2 n+1$, and $S(u,v)$ connected, define $Z_{u,v}$ to be d^l if $u^t\hat{A}v\geq\gamma(d)\|u\|\|v\|$, and 0 otherwise. In the proof of Lemma 3.4 we've seen that the probability that $Z_{u,v}$ is not 0 is at most $d^{-6\log_2 n}$, thus the expected value of $Z_{u,v}$ is at most $d^{-4\log_2 n}$. Let Z be the sum of the $Z_{u,v}$'s. Recall that there are at most $n(4d)^{\log_2 n+1}$ pairs (u,v) such that $|S(u,v)|=\log_2 n+1$, and $S(u,v)$ is connected. Hence, the expected value of Z is less than $d^{-2\log_2 n}$.

Let $X=Y+Z$. Note that the expected value of X is approximately that of Y, namely $\bigl(C\sqrt{d\log^3 d}\bigr)^l$. The expectation of Y_p and $Z_{u,v}$ can be easily computed even when the sign of some of the edges is fixed, and that of the other is chosen at random. As there is only a polynomial number of variables, using the method of conditional probabilities (cf. [8]) one can find a signing s such that the value of X is at most its expectation. For this value of X, $tr(A_s^l) = Y \leq \left(2C\sqrt{d\log^3 d}\right)^l$, and $Z = 0$ since if $Z \neq 0$ then $Z \geq d^l$ (which would contradict the fact that $X < d^l$, since $Y \geq 0$). Clearly, the spectral radius of A_s is $O\left(\sqrt{d\log^3 d}\right)$. In the proof of Lemma 3.4 we've seen that if G is $(\gamma(d), \log_2 n)$-sparse then so is $\hat{G}$, for any signing of G. For our choice of s all $Z_{u,v} = 0$, hence $\hat{G}$ is actually $(\gamma(d), \log_2 n + 1)$-sparse. ∎

An alternative method for derandomization, using an almost k-wise independent sample space, is given in section 4.

Recall the construction from the beginning of this section. Start with a d-regular graph G_0 which is an $(n_0, d, \mu)-expander$, for $\mu = 10\sqrt{d\log_2 d}$ and $n_0 > d\log_2^2 n_0$. From the Expander Mixing Lemma (cf. [8]), G_0 is $(\mu, \log_2 n_0)$-sparse. Iteratively chose G_{i+1} to be a 2-lift of G_i according to Corollary 3.1, for $i = 1, \ldots, \log_2(n/n_0)$. Clearly this is a polynomial time algorithm that yields an $\left(n, d, O\left(\sqrt{d\log^3 d}\right)\right)$-expander graph.

3.5. Random 2-lifts

Theorem 3.1 states that for every graph there exists a signing such that the spectral radius of the signed matrix is small. The proof shows that for a random signing, this happens with positive, yet possibly exponentially small, probability. The following example shows the limitations of this argument, and in particular, that there exist graphs for which a random signing almost surely fails to give a small spectral radius.

Consider a graph composed of $n/(d+1)$ disjoint copies of K_{d+1} (the complete graph on $d+1$ vertices). If all edges in one of the components are equally signed, then A_s has spectral radius d. For d fixed and n large, this event will occur with high probability. Note that connectivity is not the issue here – it is easy to modify this example and get a connected graph for which, w.h.p., the spectral radius of A_s is $d - \frac{1}{d+1}$.

However, for a random d-regular graph, it is true that a random 2-lift will, w.h.p., yield a signed matrix with small spectral radius. This follows from the fact that, w.h.p., a random d-regular graph is an $(n, d, O(\sqrt{d}))$-expander ([18, 16, 15]). In particular, by the Expander Mixing Lemma, it is $(O(\sqrt{d}), \log n)$-sparse. By Lemma 3.4, w.h.p., a random 2-lift yields a signed matrix with small spectral radius.

4. A stronger notion of explicitness

In this section we suggest an alternative derandomization scheme to that of section 3.4. We use the construction of Naor and Naor [28] of a small, almost k-wise independent sample space. This derandomization scheme leads to a construction which, in a sense, is more explicit than that in section 3.4.

4.1. Derandomization using an almost k-wise independent sample space

Recall that in the proof of Corollary 3.1 we defined two types of random variables: Let $l = 2\lceil \log_2 n \rceil$. For each closed path p of length l, Y_p is the product of the signs of the edges of p. For each $u,v \in \{0,1\}^n$, with $|S(u,v)| = \log_2 n + 1$, and $S(u,v)$ connected, let $Z_{u,v}$ be d^l if $u^t \hat{A} v \geq \gamma(d)\|u\|\|v\|$, and 0 otherwise. Define X to be the sum of all these random variables.

For brevity it will be convenient to make the following ad-hoc definitions:

Definition 4.1. A signing s of a d-regular graph G is (n,d)-*good*, if the spectral radius of A_s is $O\big(\sqrt{d \log^3 d}\big)$ and $\hat{G}$ is $(\gamma(d), 1+\log_2 n)$-sparse.
A d-regular graph G is an (n,d)-*good expander*, if it is an $\big(n, d, O\big(\sqrt{d\log^3 d}\big)\big)$-expander, and is $(\gamma(d), 1+\log_2 n)$-sparse.

The proof showed that finding a good signing is equivalent to finding a signing for which X does not exceed its expected value. We now show that this conclusion is also true when rather than choosing the sign of each edge uniformly and independently, we choose the signing from an (ϵ,k)-wise independent sample space, with $k = d\log_2 n$ and $\epsilon = d^{-2d\log_2 n}$.

Definition 4.2 ([28]). Let Ω_m be a sample space of m-bit strings, and let $S = s_1 \dots s_m$ be chosen uniformly at random from Ω_m. We say that Ω_m is an (ϵ,k)-wise independent sample space if for any $k' \leq k$ and positions $i_1 < i_2 < \cdots < i_{k'}$,

$$\sum_{\alpha \in \{-1,1\}^{k'}} \left| \Pr[s_{i_1} \dots s_{i_{k'}} = \alpha] - 2^{-k'} \right| < \epsilon.$$

Naor and Naor [28] suggest an explicit construction of such sample spaces. When $k = O(\log m)$ and $1/\epsilon = poly(m)$, the size of the sample space is polynomial in m (other constructions are also given in [6]).

We shall immediately see that the expected value of X does not change significantly when the signing is chosen from such a sample space. Hence, an alternative way of efficiently finding a good signing is to go over the entire sample space. There is at least one point in it for which X does not exceed its expected value, and thus the signing is good.

Lemma 4.1. *Let $m=dn/2$, $k=d\log_2 n$ and $\epsilon=d^{-2d\log_2 n}$. Let Ω_m be an (ϵ,k)-wise independent sample space. Let X be as in the proof of Corollary 3.1. Let U_m be the uniform distribution on m bits. Then*

$$\left|\mathbb{E}_{\Omega_m}[X]-\mathbb{E}_{U_m}[X]\right|=o(1).$$

Proof. Recall that $X=\sum_p Y_p+\sum_{u,v} Z_{u,v}$, where Y_p and $Z_{u,v}$ are as above, the first sum is over all closed paths of length $l=2\lceil\log_2 n\rceil$, and the second sum is over all $u,v\in\{0,1\}^n$, with $|S(u,v)|=\log_2 n+1$ and $S(u,v)$ connected. Hence

$$\left|\mathbb{E}_{\Omega_m}[X]-\mathbb{E}_{U_m}[X]\right|\le\sum_p\left|\mathbb{E}_{\Omega_m}[Y_p]-\mathbb{E}_{U_m}[Y_p]\right|+\sum_{u,v}\left|\mathbb{E}_{\Omega_m}[Z_{u,v}]-\mathbb{E}_{U_m}[Z_{u,v}]\right|.$$

Let p be a path of length l, and denote the edges that appear in it an odd number of times by $i_1,\ldots,i_{l'}$, for some $l'<l$. Let $s_{i_1},\ldots,s_{i_{l'}}$ be the signs of these edges. Then the value of Y_p is $\prod_{j=1}^{l'} s_{i_j}$, and (for every distribution)

$$\mathbb{E}[Y_p]=\sum_{\alpha\in\{-1,1\}^{l'}}\Pr[s_{i_1},\ldots,s_{i_{l'}}=\alpha]\cdot\prod_{j=1}^{l'}\alpha_j.$$

Thus,

$$\begin{aligned}
&\left|\mathbb{E}_{\Omega_m}[Y_p]-\mathbb{E}_{U_m}[Y_p]\right|\\
&\quad=\left|\sum_{\alpha\in\{-1,1\}^{l'}}\left(\prod_{j=1}^{l'}\alpha_j\right)\left(\Pr_{\Omega_m}[s_{i_1},\ldots,s_{i_{l'}}=\alpha]-\Pr_{U_m}[s_{i_1},\ldots,s_{i_{l'}}=\alpha]\right)\right|\\
&\quad\le\left|\sum_{\alpha\in\{-1,1\}^{l'}}\left(\Pr_{\Omega_m}[s_{i_1},\ldots,s_{i_{l'}}=\alpha]-2^{-l'}\right)\right|<\epsilon.
\end{aligned}$$

As there are less than d^l closed paths p of length l, $\sum_p|\mathbb{E}_{\Omega_m}[Y_p]-\mathbb{E}_{U_m}[Y_p]|<\epsilon d^l=o(1)$. A similar argument shows that $|\sum_{u,v}|\mathbb{E}_{\Omega_m}[Z_{u,v}]-\mathbb{E}_{U_m}[Z_{u,v}]|=o(1)$ as well. ∎

In fact, it follows that w.h.p. (say, $1-\frac{1}{n^2}$ for an appropriate choice of ϵ), choosing an element uniformly at random from Ω_m leads to a good signing.

4.2. A probabilistic strongly explicit construction

The constructions of section 3.4 and of the previous subsection are explicit in the sense that given n and d they provide a polynomial (in n) time algorithm for constructing an (n,d)-good expander. However, in some applications of expander graphs (e.g. derandomization) a stronger notion of explicitness is required, so called "strongly explicit". Namely, an algorithm that given n, d and $i,j\in[n]$, decides in time polylog(n) whether i and j are adjacent.

We do not know how to achieve such explicitness using the 2-lifts scheme. In this sub-section we suggest the notion of a "probabilistic strongly explicit" construction, and show that this level of explicitness can be obtained. Intuitively, we construct a polynomial number of algorithms which define a graph by deciding adjacency as above. *Most* of these algorithms define graphs which are good expanders.

Formally:

Definition 4.3. Let $f_n:\{0,1\}^t\times[n]\times[n]\to\{0,1\}$, with $t=O(\log n)$. Given $r\in\{0,1\}^t$, f_n defines a graph $G_{f_n}(r)$, on n vertices, where i and j are adjacent iff $f_n(r,i,j)=1$. We say that f_n is a *δ-probabilistic strongly explicit* description of an (n,d)-good expander graph, if given n, f_n can be computed in time polylog(n), and, with probability at least $1-\delta$ (over a uniform choice of r), $G_{f_n}(r)$ is an (n,d)-good expander graph.

It will be convenient to give a similar definition for a signing of a graph, and for a composition of such functions:

Definition 4.4. Let $h_n:\{0,1\}^t\times[n]\times[n]\to\{-1,1\}$, with $t=O(\log n)$. Given $r\in\{0,1\}^t$, and a graph G on n vertices, h_n defines a signing s_{h_n} of G by $s_{h_n}(r)(i,j)=h_n(r,i,j)$. We say that h_n is a *δ-probabilistic strongly explicit* description of an (n,d)-good signing, if given n, h_n can be computed in time polylog(n), and, for any $(\log_2 n,\gamma(d))$-sparse d-regular graph G on n vertices, with probability at least $1-\delta$ (over a uniform choice of r), h_n defines an (n,d)-good signing.

Definition 4.5. Let $f_n:\{0,1\}^{t_1}\times[n]\times[n]\to\{0,1\}$, and $h_n:\{0,1\}^{t_2}\times[n]\times[n]\to\{-1,1\}$ be as above. Their *composition*, $f_{2n}:\{0,1\}^t\times[2n]\times[2n]\to\{0,1\}$, with $t=\max\{t_1,t_2\}$ is as follows. For $r\in\{0,1\}^t$, let r_1 be the first t_1 bits of r, and r_2 the first t_2 bits in r. f_{2n} is such that the graph $G_{f_{2n}}(r)$ is the 2-lift of $G_{f_n}(r_1)$ described by the signing $s_{h_n}(r_2)$.

The following lemma is easy, and we omit the proof:

Lemma 4.2. *Let f_n be a δ_1-probabilistic strongly explicit description of an (n,d)-good expander, and h_n a δ_2-probabilistic strongly explicit description of an (n,d)-good signing. Then their composition is a $(\delta_1+\delta_2)$-probabilistic strongly explicit description of an $(2n,d)$-good expander.*

We now show a that such explicitness can be achieved for constructions based on 2-lifts. Think of an (ϵ,k)-wise independent space Ω_m as a function $\omega:\{0,1\}^t \to \{-1,1\}^m$, where $|\Omega_m|=2^t$. It follows from the work of Naor and Naor [28]), that not only can ω be computed efficiently, but that given $r\in \{0,1\}^t, p\in[m]$ $\omega(r)_p$ (the p'th coordinate of $\omega(r)$) can be computed efficiently (i.e. in time polylog(m)). Take $m=\binom{n}{2}$, and think of the elements of $\{-1,1\}^m$ as being indexed by unordered pairs $(i,j)\in\binom{[n]}{2}$. Define $h_n(r,i,j)=\omega(r)_{i,j}$. It follows from the above discussion than h_n is a $\frac{1}{n^2}$-probabilistic strongly explicit description of an (n,d)-good signing, for k and ϵ as above.

We now describe how to construct a δ-probabilistic strongly explicit description of an (N,d)-good expander graph. Let G be an (n,d)-good expander, with $n \geq \frac{1}{\delta}$. For $i=0,\dots,l=\log_2(N/n)$, define $n_i = n\cdot 2^i$ and $m_i=\binom{n_i}{2}$. Define $k_i = d\log_2 n_i$. Let $\omega_i:\{0,1\}^{t_i}\to\{-1,1\}^{m_i}$ be a description of an (ϵ_i,k_i)-wise independent space of bit strings of length m_i, where ϵ_i is such that an element chosen uniformly at random from this space yields an (n_i,d)-good signing with probability at least $1-\frac{1}{n_i^2}$.

The functions $h_{n_i}(r,p,q)=\omega_i(r)_{p,q}$ are $\frac{1}{n_i^2}$-probabilistic strongly explicit descriptions of (n_i,d)-good signings. Let f_n be a description of G. For simplicity, assume that adjacency in G can be decided in time polylog(n). Thus, f_n is, trivially, a 0-probabilistic strongly explicit description of an (n,d)-good expander. Define f_{n_i} as the composition of $f_{n_{i-1}}$ and $h_{n_{i-1}}$. It follows from this construction and Lemma 4.2 that:

Lemma 4.3. *f_{n_l} is a $\frac{1}{n}$-probabilistic strongly explicit description of an (N,d)-good expander graph.*

5. A converse to the Expander Mixing Lemma

So far, we discussed an algebraic definition of expansion in graphs. Namely, we said a graph is an (n,d,λ)-expander if all eigenvalues but the largest are, in absolute value, at most λ. A seemingly unrelated combinatorial definition says that a d-regular graph on n vertices is an (n,d,c)-edge expander if every set of vertices, W, of size at most $n/2$, has at least $c|W|$ edges emanating from it.

Surprisingly, the two notions are closely related. Thus (cf. [8]), an (n,d,λ)-expander is also an $(n,d,\frac{d-\lambda}{2})$-edge expander. Conversely, an (n,d,c)-edge expander is also an $(n,d,d-\frac{c^2}{2d})$-expander[2]. Thus, though the two notions of expansion are qualitatively equivalent, they are far from being quantitatively the same. While algebraic expansion yields good bounds on edge expansion, the reverse implications are very weak. It is also known that this is not just a failure of the proofs and indeed this estimate is nearly tight [2]. Is there, we ask, another combinatorial property that is equivalent to spectral gaps? We next answer this question.

For two subsets of vertices, S and T, let $e(S,T)$ denote the number of edges between them. We follow the terminology of [31]:

Definition 5.1. A d-regular graph G on n vertices is (d,α)-*jumbled*, if for every two subsets of vertices, A and B,

$$\left|e(A,B) - d|A||B|/n\right| \leq \alpha\sqrt{|A||B|}.$$

A very useful property of (n,d,λ)-expanders, known as the Expander Mixing Lemma (cf. [8]), is that a an (n,d,λ)-expander is (d,λ)-jumbled. Lemma 3.3 implies the promised converse to this well known fact:

Corollary 5.1. *Let G be a d-regular graph on n vertices. Suppose that for any $S,T \subset V(G)$, with $S \cap T = \emptyset$*

$$\left|e(S,T) - \frac{|S||T|d}{n}\right| \leq \alpha\sqrt{|S||T|}.$$

Then all but the largest eigenvalue of G are bounded, in absolute value, by $O(\alpha(1+\log(d/\alpha)))$.

Note 5.1. In particular, this means that for a d-regular graph G, $\lambda(G)$ is a $\log_2 d$ approximation of the "jumbleness" parameter of the graph.

Proof of Corollary 5.1. Let A be the adjacency matrix of G. Denote $B = A - \frac{d}{n}J$, where J is the all ones $n \times n$ matrix. Clearly B is symmetric, and the sum of the absolute value of the entries in each row is at most $2d$. Observe that A and B have the same eigenvectors. The all ones vector is an eigenvector for eigenvalue d in A and 0 in B; all other eigenvectors correspond to the same eigenvalue in both A and B. Thus, for the corollary

[2] A related result, showing that *vertex* expansion implies spectral gap appears in [3]. The implication from edge expansion is easier, and the proof we are aware of is also due to Noga Alon.

to follow from Lemma 3.3 it suffices to show that for any two vectors, $u, v \in \{0,1\}^n$:

$$\left|u^t B v\right| = \left|u^t \frac{d}{n} J v - u^t A v\right| \le \alpha \|u\| \|v\|.$$

This is exactly the hypothesis for the sets $S(u)$ and $S(v)$. ∎

Note 5.2. For a *bipartite* d-regular graph $G = (L, R; E)$ on n vertices, if for any $S \subset R, T \subset L$, with $S \cap T = \emptyset$

$$\left|e(S,T) - \frac{|S||T|d}{2n}\right| \le \alpha \sqrt{|S||T|}.$$

Then all but the largest eigenvalue of G are bounded, in absolute value, by $O(\alpha(1+\log(d/\alpha)))$.

The proof is essentially identical to the one above, taking $B = A - \frac{d}{n}C$, instead of $B = A - \frac{d}{n}J$, where $C_{i,j}$ is 0 if i, j are on the same side, and 2 otherwise.

The corollary is actually tight, up to a constant multiplicative factor, as we now show:

Theorem 5.1. *For any large enough d, and $7\sqrt{d} < \alpha < d$, there exist infinitely many (d, α)-jumbled graphs with second eigenvalue $\Omega(\alpha(\log(d/\alpha) + 1))$.*

It will be useful to extend Definition 5.1 to unbalanced bipartite graphs:

Definition 5.2. A bipartite graph $G = (U, V, E)$ is (c, d, α)-*jumbled*, if the vertices in U have degree c, those in V have degree d, and for every two subsets of vertices, $A \subset U$ and $B \subset V$,

$$\left|e(A,B) - d|A||B|/|U|\right| \le \alpha \sqrt{|A||B|}.$$

We note that such bipartite graphs exit:

Lemma 5.1. *For $c|d$ and $\alpha = 2\sqrt{d}$, there exist (c, d, α)-jumbled graphs.*

Proof. Let $G' = (U', V', E')$ be a c-regular Ramanujan bipartite graph, such that $|U'| = |V'| = n$. Let $G = (U, V, E)$ be a bipartite graph obtained from G' by partitioning V' into subsets of size d/c, and merging each subset into a vertex, keeping all edges (so this is a multi-graph).

Let $A \subset U$ and $B \subset V$. Let $A' = A$, and let B' be the set of vertices whose merger gives B. Clearly $e(A,B) = e(A',B')$, $|B'| = d/c|B|$. As G' is Ramanujan, by the expander mixing lemma

$$\left|e(A',B') - c|A'||B'|/n\right| \le 2\sqrt{c|A'||B'|},$$

or:

$$\big|e(A,B) - d|A||B|/|U|\big| \leq 2\sqrt{d|A||B|}. \qquad \blacksquare$$

The following inequality can be easily proven by induction:

Lemma 5.2. *For $i=0,\ldots,t$ let a_i be numbers in $[0, 2^{2i}\cdot N]$, for some $N>0$. Then*

$$\Big(\sum a_i 2^{-i}\Big)^2 \leq 3N \sum a_i.$$

Proof of Theorem 5.1. Fix d, and $\sqrt{d} < \Delta < d$. Set $t = \frac{1}{2}\log_2(\frac{3d}{4\Delta})$, $\tau = \sum_{i=0}^{t} 4^i = \frac{d}{\Delta} < \Delta$. Let N be some large number and $n = \tau N$ (this will be the number of vertices). For $i,j = 0,\ldots,t$ set $d_{i,j} = \frac{d}{\tau}2^{2j} + \Delta 2^{j-i}$ when $i,j<t$ or $i=j=t$, and $d_{i,j} = \frac{d}{\tau}2^{2j} - \Delta 2^{j-i}$ otherwise. Note that in this case $\frac{d}{\tau}2^{2j} - \Delta 2^{j-i} \geq 2^j(\Delta 2^j - \Delta 2^{-i}) > 0$. Set $\alpha_{i,j} = 2\sqrt{\min\{d_{i,j}, d_{j,i}\}}$.

Let V_i be subsets of size $4^i N$, and G a graph on vertices $V = \cup_{i=0}^{t} V_i$ (hence, $|V|$ = n). For $0 < i,j \leq t$ construct a $(d_{j,i}, d_{i,j}, \alpha_{i,j})$-jumbled graph between V_i and V_j (or $(d_{i,i}, \alpha_{i,i})$-jumbled if $i=j$).

The theorem follows from the following two lemmata.

Lemma 5.3. *G is $(d, 7\Delta)$-jumbled.*

Proof. It is not hard to verify that G is indeed d regular. Take $A,B \subset V$, and denote their size by a and b. Denote $A_i = A\cap V_i$, $B_i = B\cap V_i$, and their size by a_i and b_i. We want to show that:

$$|e(A,B) - dab/n| \leq 3\Delta\sqrt{ab}.$$

For simplicity we show that $e(A,B) \leq dab/n + 7\Delta\sqrt{ab}$. A similar argument bounds the number of edges from below. From the construction, $\big|e(A_i,B_j) - d_{i,j}|A_i||B_j|/|V_j|\big| \leq \alpha_{i,j}\sqrt{|A_i||B_j|}$, or:

$$e(A_i,B_j) \leq d_{i,j}a_i b_j/(4^j N) + \alpha_{i,j}\sqrt{a_i b_j}.$$

Summing up over $i,j=0,\ldots,t$ we get:

$$\begin{aligned} e(A,B) &\leq \sum d_{i,j}a_i b_j/(4^j N) + \alpha_{i,j}\sum \alpha_{i,j}\sqrt{a_i b_j} \\ &\leq d/n \sum a_i b_j + \Delta/N \sum a_i b_j 2^{-(i+j)} + \sum \alpha_{i,j}\sqrt{a_i b_j}. \end{aligned}$$

$\sum a_i b_j = ab$, so it remains to bound the error term. As $a_i, b_i \in [0, N\cdot 2^{2i}]$, by Lemma 5.2:

$$\begin{aligned} \Delta/N \sum a_i b_j 2^{-(i+j)} &= \Delta/N \Big(\sum a_i 2^{-i}\Big)\Big(\sum b_i 2^{-i}\Big) \\ &\leq \Delta/N \Big(\sqrt{3N\sum a_i}\Big)\Big(\sqrt{3N\sum b_i}\Big) = 3\Delta\sqrt{ab}. \end{aligned}$$

It remains to show that $\sum \alpha_{i,j}\sqrt{a_i b_j} \le 4\Delta\sqrt{ab}$, and hence it's enough to show that:

$$\sum 2\sqrt{\frac{d}{\tau}a_i b_j 2^{2\max\{i,j\}}} + \sum 2\sqrt{\Delta 2^{|j-i|}a_i b_j} \le 4\Delta\sqrt{ab}.$$

Indeed, as $i,j \le t$, and $2^{2t} < \tau$,

$$\sum \sqrt{\frac{d}{\tau}a_i b_j 2^{2\max\{i,j\}}} < \sqrt{d}\sum \sqrt{a_i b_j} < \Delta\sqrt{ab}.$$

Similarly, $2^{|j-i|} \le 2^t < \sqrt{\tau} < \sqrt{\Delta}$, and so:

$$\sum \sqrt{\Delta 2^{j-i}a_i b_j} < \Delta\sum \sqrt{a_i b_j} = \Delta\sqrt{ab}. \qquad \blacksquare$$

Lemma 5.4. $\lambda(G) \ge \Delta(t+1)$.

Proof. Take $x \in \mathbb{R}^n$ to be -2^{-t} on vertices in V_t, and 2^{-i} on vertices in V_i, for $i<t$. It is easy to verify that $x \perp \vec{1}$, and that $\|x\|^2 = N\cdot(t+1)$. Let M be the adjacency matrix if G. Since $\vec{1}$ is an eigenvector of M corresponding to the largest eigenvalue, by the variational characterization of eigenvalues, $\lambda(G) \ge \frac{x^t M x}{\|x\|^2}$. Hence, to prove the lemma it suffices to show that $x^t M x \ge \Delta N(t+1)^2$. Indeed:

$$\begin{aligned} x^t M x &= \sum_{i,j=0}^{t} d_{i,j}4^i N 2^{-(i+j)} - 4\sum_{i=0}^{t-1} d_{i,t}4^i N 2^{-(i+t)} \\ &= \frac{d}{\tau}N\sum_{i,j=0}^{t} 2^{i+j} + \Delta N\sum_{i,j=0}^{t} 1 - 4\frac{d}{\tau}N\sum_{i=0}^{t-1} 2^{i+t} + 4\Delta N\sum_{i=0}^{t-1} 1 \\ &= \frac{d}{\tau}N\big(2^{2(t+1)} - 4\cdot 2^{2t}\big) + \Delta N\big((t+1)^2 + 4t\big) > \Delta N(t+1)^2. \qquad \blacksquare \end{aligned}$$

6. Reflections on Lemma 3.3

6.1. Finding the proof: LP-duality

As the reader might have guessed, the proof for Lemma 3.3 was discovered by formulating the problem as a linear program. Define $\Delta_{i,j} = |(x^i)^t A x^j|$. Our assumptions translate to:

$$\forall 1 \le i \le j \le k : |\Delta_{i,j}| \le \alpha\sqrt{s_i s_j},$$
$$\forall 1 \le i \le k : \sum_j |\Delta_{i,j}| \le d s_i.$$

We want to deduce an upper bound on $|x^tAx|$. In other words, we are asking, under these constraints, how big

$$\frac{|x^tAx|}{\|x\|^2} \leq \frac{\sum_{i,j=1}^{k} \Delta_{i,j} 2^{-(i+j)}}{\sum_i 2^{-2i} s_i}$$

can be.

The dual program is to minimize:

$$\frac{\alpha \sum_{i\leq j} b_{i,j}\sqrt{s_i s_j} + d\sum_i c_i s_i}{\sum_i 2^{-2i} s_i}$$

under the constraints:

$$\begin{aligned} \forall 1 \leq i < j \leq k, &\quad b_{i,j} + c_i + c_j \geq 2^{-(i+j)+1} \\ \forall 1 \leq i \leq k, &\quad b_{i,i} + c_i \geq 2^{-2i} \\ \forall 1 \leq i \leq j \leq k, &\quad b_{i,j} \geq 0 \\ \forall 1 \leq i \leq k, &\quad c_i \geq 0\,. \end{aligned}$$

The following choice of b's and c's satisfies the constraints, and gives the desired bound. These indeed appear in the proof of Lemma 3.3:

$$\begin{aligned} \forall 1 \leq i < j \leq k, j < i+\gamma, &\quad b_{i,j} = 2^{-(i+j)+1} \\ \forall 1 \leq i < j \leq k, j \geq i+\gamma, &\quad b_{i,j} = 0 \\ \forall 1 \leq i \leq k, &\quad b_i = 2^{-2i} \\ \forall 1 \leq i \leq k, &\quad c_i = 2^{-2i-\gamma+1}. \end{aligned}$$

6.2. Algorithmic aspect

Lemma 3.3 is algorithmic, in the sense that given a matrix with a large eigenvalue, we can efficiently construct, from its eigenvector, a pair $u,v \in \{0,1\}^n$ such that $S(u)\cap S(v)=\emptyset$, and $|u^tAv| \geq \alpha\|u\|\|v\|$. (There is a small caveat – in the proof we used a probabilistic argument for rounding the coordinates. This can be easily derandomized using the conditional probabilities method.) Taking into consideration Note 5.1, given a d-regular graph G where $\lambda(G)$ is large, one can efficiently find disjoint subsets S and T, such that $e(S,T) - \frac{d}{n}|S||T| \geq c\cdot\lambda(G)/\log d\sqrt{|S||T|}$ (for some constant c). It is conceivable that this might be useful in designing graph partitioning algorithms.

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