

## LOCALITY IN DISTRIBUTED GRAPH ALGORITHMS\*

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**Abstract.** This paper concerns a number of algorithmic problems on graphs and how they may be solved in a distributed fashion. The computational model is such that each node of the graph is occupied by a processor which has its own ID. Processors are restricted to collecting data from others which are at a distance at most  $t$  away from them in  $t$  time units, but are otherwise computationally unbounded. This model focuses on the issue of *locality* in distributed processing, namely, to what extent a global solution to a computational problem can be obtained from locally available data.

Three results are proved within this model:

- A 3-coloring of an  $n$ -cycle requires time  $\Omega(\log^* n)$ . This bound is tight, by previous work of Cole and Vishkin.
- Any algorithm for coloring the  $d$ -regular tree of radius  $r$  which runs for time at most  $2r/3$  requires at least  $\Omega(\sqrt{d})$  colors.
- In an  $n$ -vertex graph of largest degree  $\Delta$ , an  $O(\Delta^2)$ -coloring may be found in time  $O(\log^* n)$ .

**Key words.** distributed algorithms, graph theory, locality, lower bounds

**AMS(MOS) subject classifications.** 05C35, 68R10, 68Q99

**1. Introduction.** In distributed processing all computations are made based on local data. The aim of this paper is to bring up limitations that follow from this local nature of the computation. Note that within the various computational models for parallel computers this difficulty is specific to the distributed model. Shared memory allows for fast dissemination of data, but no such means exist when dealing with distributed systems.

In the present paper we are mostly interested in proving lower bounds, and therefore assume a powerful version of the distributed model: Each node of the undirected graph  $G = (V, E)$  is occupied by a processor. Computation is completely synchronous and reliable. At each time unit a processor may pass messages to each of its neighbors, and message size is unrestricted. Also, any computations carried out by individual processors take one time unit and are not restricted in any way. This paper is only concerned with the radius of the neighborhood around each node from which data may be collected, this radius being the only significant parameter in this model, as we later elaborate. Of interest is the time complexity of various “global” functions of  $G$ , and the concrete examples are coloring and finding maximal independent sets. Thus the theme of this paper is how local data may be utilized to find globally defined solutions.

Before we proceed, symmetry-breaking has to be addressed (see [JS] and the references therein for literature on symmetry-breaking). It is well known that most functions cannot be computed in a distributed fashion by anonymous processors, even for very simple graphs  $G$ . This impossibility usually results from symmetries that  $G$  may have. Such symmetries are usually broken by means of either randomization or the use of IDs, and the present paper is concerned only with the latter. Thus we assume that there is a mapping ID from the set of vertices  $V$  to the positive integers.

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In most cases ID is a bijection onto  $1, \dots, |V|$ . It is assumed that at time zero the processor occupying a node in  $G$  knows the ID of that node. Incidentally, all our lower bounds hold even if every processor knows in advance what the graph  $G$  is, and only the labeling function ID is unknown.

It is clear that the present model allows us to compute every function of  $G$  in time  $O(\text{diameter}(G))$ . After this amount of time every processor obtains complete knowledge of both  $G$  and ID. The problem is thus solved if, in the memory of each processor a solution for the entire problem is prestored, for every possible labeling. Our concern is therefore only with time complexities below  $\text{diam}(G)$ . The major question we raise is which functions may be computed by a nontrivial algorithm in this model, in the sense that they can be computed faster than  $\text{diam}(G)$ .

The model proposed here is, of course, of purely theoretical interest. Of the many difficulties arising in distributed processing, it focuses only on transforming local data into a global solution. Further research into this model may help classify problems as either locally computable (solvable in time shorter than  $\text{diam}(G)$ ) or not. One may also look for bounds on run times which depend on graph parameters other than the diameter. In accordance with the theory of the complexity class NC, it seems natural to investigate graph problems whose time complexity in this model is polylogarithmic in the number of vertices.

Here are our main results:

(1) Finding a maximal independent set distributively in a labeled  $n$ -cycle, requires time  $\Omega(\log^* n)$ . This bound is tight in view of the  $O(\log^* n)$  algorithm by Cole and Vishkin [CV]. (Technically, their result was stated in the PRAM model, but it extends without change to the distributed model as well.) Our proof relies on the interesting construct of *neighborhood graphs*. An alternative proof based on the Ramsey theorem was found by a number of other investigators in the area [A].

(2) Coloring trees: Let  $T$  be the  $d$ -ary tree of height  $r$ . In time  $2r/3$ , it is impossible to color  $T$  in fewer than  $\sqrt{d}$  colors. Note in contrast the algorithm by Goldberg and Plotkin [GP], which shows that if every node in  $T$  "knows its parent in the tree," i.e., a consistent orientation from the root outwards is given, then a 3-coloring can be found in time  $O(\log^* n)$ . Though stated for the PRAM model, it is easy to see that this result of [GP] applies to the distributed model as well.

(3) In a labeled graph of order  $n$  with maximal degree  $\Delta$ , it is possible to find an  $O(\Delta^2)$ -coloring in time  $(1 + o(1)) \log^* n$ . This was previously shown in [GP] only in the case of constant  $\Delta$ .

Our terminology is standard: a  $k$ -labeling of a graph  $G = (V, E)$  is 1:1 mapping  $f : V \rightarrow \{1, \dots, k\}$ . In case  $k = |V|$  a  $k$ -labeling is called a *labeling*. Given a  $k$ -labeling,  $G$  is said to be  $k$ -labeled, etc. Logarithms are to base 2. The  $k$  times iterated logarithm is denoted by  $\log^{(k)} x$ , i.e.,  $\log^{(1)} x := \log x$  and  $\log^{(k)} x = \log(\log^{(k-1)} x)$ . The least integer  $k$  for which  $\log^{(k)} x < 1$  is denoted by  $\log^* x$ .

**2. Lower bound on finding a maximal independent set in a cycle.** In [CV] a very nice algorithm was presented to find a maximal independent set of vertices (= MIS) in the  $n$ -cycle  $C_n$  in time  $\log^* n$ . In this section we show that this is optimal even in the present model, where computation takes no time. The algorithm presented in §4 also achieves this time bound.

A basic observation is that in the present model there is no loss of generality in assuming that processing proceeds by first collecting all data and then deciding. That is, at time  $t$  each processor knows the labeling of all nodes at distance  $t$  or less away. Also known are all edges between these nodes, except for edges both endpoints

of which are at distance exactly  $t$ . Note that no further information can reach a processor by time  $t$ . This allows us to view the problem in purely combinatorial terms.

Let us state our theorem.

**THEOREM 2.1.** *A synchronous distributed algorithm which finds a maximal independent set in a labeled  $n$ -cycle must take at least  $\frac{1}{2}(\log^* n - 1)$  units of time. An algorithm of the same class which colors the  $n$ -cycle with three colors requires time at least  $\frac{1}{2}(\log^* n - 3)$ . The same bounds hold also for randomized algorithms.*

*Proof.* The proof holds even under the assumption that there is a consistent notion of clockwise orientation common to all processors. Given an algorithm which finds a maximal independent set in the  $n$ -cycle endowed with a clockwise orientation, it is easily seen that in one more timestep, the cycle may be 3-colored. The lower bound is established for 3-coloring.

Coming back to the previous observation, at time  $t$  the data known to a processor  $P$  is an ordered list of  $2t + 1$  labels, starting  $t$  places before it, through its own and on to the next  $t$  labels. Let  $V$  be the set of all vectors  $(x_1, \dots, x_{2t+1})$  where the  $x_i$  are mutually distinct integers from  $\{1, \dots, n\}$ . The algorithm is nothing but a mapping  $c$  from  $V$  into  $\{1, 2, 3\}$ .

Let us denote by  $B_{t,n}$  the graph whose set of vertices is  $V$ . All edges of  $B_{t,n}$  are given by:

$$(x_1, \dots, x_{2t+1}) \quad \text{and} \quad (y, x_1, \dots, x_{2t})$$

are neighbors for all  $y \neq x_{2t+1}$ . So  $B_{t,n}$  has  $n(n-1) \dots (n-2t)$  vertices and is regular of degree  $2(n-2t-1)$ . Note that the mapping  $c: V \rightarrow \{1, 2, 3\}$  is, in fact, a proper 3-coloring of  $B_{t,n}$ . For suppose that  $c$  assigns

$$(x_1, \dots, x_{2t+1}) \quad \text{and} \quad (y, x_1, \dots, x_{2t})$$

the same color. Then the 3-coloring algorithm for the  $n$ -cycle fails in case the labeling happens to contain the segment:

$$y, x_1, x_2, \dots, x_{2t+1}.$$

The proof follows now by standard graph-theoretic arguments which show that the chromatic number  $\chi(B_{t,n})$  of  $B_{t,n}$  satisfies

$$\chi(B_{t,n}) = \Omega(\log^{(2t)} n),$$

the  $2t$  times iterated logarithm of  $n$ . Therefore, for  $\chi(B_{t,n})$  to be at most 3, we must have  $t = \Omega(\log^* n)$ .

The lower bound on  $\chi(B_{t,n})$  is proved, using a family of digraphs  $D_{s,n}$  closely related to  $B_{t,n}$ . The vertices of  $D_{s,n}$  are all sequences  $(a_1, \dots, a_s)$  with  $1 \leq a_1 < a_2 < \dots < a_s \leq n$ . The outneighbors of  $(a_1, \dots, a_s)$  are all vertices of the form  $(a_2, \dots, a_s, b)$  with  $a_s < b \leq n$ . Note that  $B_{t,n}$  contains the underlying graph of  $D_{2t+1,n}$ , so in particular  $\chi(B_{t,n}) \geq \chi(D_{2t+1,n})$ .

Given a digraph  $H = (V, E)$  its *dilinegraph*  $DL(H)$  is a digraph whose vertex set is  $E$  with  $(u, w)$  an edge if  $head_H(u) = tail_H(w)$ . The relation between the digraphs  $D_{s,n}$  is given by Proposition 2.1.

**PROPOSITION 2.1.**  *$D_{1,n}$  is obtained from the complete graph of order  $n$  by replacing each edge by a pair of edges, one in each direction, and  $D_{s+1,n} = DL(D_{s,n})$  for all  $s \geq 1$ .*

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*Proof.* The statement concerning  $D_{1,n}$  is just the definition. For the other claim, identify the edge connecting  $(x_1, \dots, x_s)$  and  $(x_2, \dots, x_s, y)$  in  $D_{s,n}$  with the vertex  $(x_1, \dots, x_s, y)$  in  $V(D_{s+1,n})$  and check that the adjacency relationship in  $D_{s+1,n}$  is that of  $DL(D_{s,n})$ .  $\square$

The bound on  $\chi(D_{s,n})$  is derived from the following simple and well-known proposition.

PROPOSITION 2.2. For a digraph  $G$ ,

$$\chi(DL(G)) \geq \log \chi(G).$$

*Proof.* A  $k$ -coloring of  $DL(G)$  may be thought of as a mapping  $\Psi : E(G) \rightarrow \{1, \dots, k\}$  such that if  $u, w \in E(G)$  and  $head(u) = tail(w)$ , then  $\Psi(u) \neq \Psi(w)$ . Now vertex-color  $G$  by associating with node  $x$  the set

$$c(x) = \{\Psi(u) \mid x = tail(u)\}.$$

This is easily seen to be a  $2^k$  vertex-coloring of  $G$ . Indeed if  $u = (x, y) \in E(G)$ , then  $\Psi(u) \in c(x)$  but  $\Psi(u) \notin c(y)$ , or else  $\Psi$  is improper. Therefore  $c(x) \neq c(y)$ .  $\square$

The main claim can be derived now. If an MIS can be found in time  $t - 1$ , then necessarily

$$3 \geq \chi(B_{t,n}).$$

But

$$\chi(B_{t,n}) \geq \chi(D_{2t+1,n}) \geq \log^{(2t)} n.$$

So

$$2t \geq \log^* n - 1, \quad t \geq \frac{1}{2}(\log 2^* n - 1),$$

as claimed.

The claim on randomized algorithms is proved in Corollary 2.1 below.  $\square$

In contrast with the low time complexity of 3-coloring, we can show that for an even  $n$ , finding a 2-coloring of  $C_n$  requires time  $\Omega(n)$ .

THEOREM 2.2. A synchronous distributed algorithm which 2-colors a labeled  $2n$  cycle with labels from  $\{1, \dots, 2n\}$  must take at least  $n - 1$  units of time.

*Proof.* Now the lowest  $t$  has to be found such that  $B_{t,2n}$  is bipartite. But even for  $t = n - 2$ , the graph  $B_{t,2n}$  contains an odd cycle:

$$(1, \dots, 2t + 1), (2, \dots, 2t + 2), (3, \dots, 2t + 3), (4, \dots, 2t + 3, 1),$$

$$(5, \dots, 2t + 3, 1, 2), \dots, (2t + 3, 1, \dots, 2t), (1, \dots, 2t + 1).$$

The claim follows.  $\square$

Let us point out that for an even  $n$  the last theorem implies that finding a maximum independent set requires time  $\lceil n/2 \rceil - 1$ . It is easily verified that the same is true for odd  $n$  as well.

We want to elaborate on the method developed here and point out its general features. Given a graph  $G = (V, E)$  of order  $n$ , and  $t \geq 1$ , the  $t$ -neighborhood graph of

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