A King in two tournaments

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Abstract

According to Landau’s theorem every tournament has a King, i.e., a vertex that can reach every other vertex in two steps or less. Here we extend this result by considering two not-necessarily-equal tournaments $T_1, T_2$ on the same vertex set. We show that there is a vertex which can reach all vertices via (i) A step in $T_1$, or (ii) A step in $T_2$ or (iii) A step in $T_1$ followed by a step in $T_2$.

1 Introduction

Directed graphs play a major role in modeling computer systems. This is particularly the case in the domain of distributed computing. In such a model, a directed edge $(i, j)$ in a digraph indicates that processor $P_i$ is sending a message to processor $P_j$. In what follows, communication is governed by an $n$-vertex tournament $T = ([n], E)$. Initially each processor $P_i$ holds a data item $m_i$ that is known only to it. In a round of communication processor $P_i$ tells every processor $P_j$, with $(i, j) \in E$, everything it knows. An old theorem of Landau [2] asserts that every tournament has a vertex that 2-dominates all other vertices. Consequently, after two rounds of communication some data item $m_\nu$ is known to all processors $P_j$. Notice that likewise in a dual way, there is also a vertex $\mu$ such that, after two rounds processor $P_\mu$ gets to know all data items $m_j$.

In the study of certain fundamental aspects of distributed computing [1] the following question came up. What happens if different

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tournaments govern the communication in different rounds? As before, communication takes place in rounds, each of which is governed by an arbitrary tournament. Thus, we start again with each processor \( P_i \) holding its private data item \( m_i \) for \( n \geq i \geq 1 \). There is a sequence of tournaments \( T_1 = ([n], E_1), T_2 = ([n], E_2), \ldots \). At round 1 the communication pattern is dictated by tournament \( T_1 \), at round 2 by \( T_2 \), etc. Do the consequences of Landau’s theorem still hold in this more general setting? If so, how many rounds of communication are necessary to yield the same conclusions as above? We find it satisfactory and somewhat surprising that as in Landau’s case, two rounds of communication suffice.

## 2 Two rounds suffice

Let \( T_1 = ([n], E_1), T_2 = ([n], E_2) \) be two tournaments. Suppose that in a round of \( T_1 \) followed by a round of \( T_2 \) processor \( P_j \) gets to know data item \( m_i \). We denote this by \( i \Rightarrow j \). Clearly, this is equivalent to

\[
i = j \lor (i, j) \in E_1 \lor (i, j) \in E_2 \lor \exists k \text{ s.t. } (i, k) \in E_1 \text{ and } (k, j) \in E_2
\]

It is useful to note that the negation of this condition \( i \not\Rightarrow j \) is equivalent to

\[
i \neq j \land (j, i) \in E_1 \land (j, i) \in E_2 \land \Gamma_2(j) \supset \Gamma_1(i) \quad (1)
\]

where \( \Gamma_\delta(x) \) is the set of out-neighbors of \( x \) in tournament \( T_\delta \).

**Theorem 1.** Let \( T_1 = ([n], E_1), T_2 = ([n], E_2) \) be two tournaments. Then there is \( \nu \in [n] \) such that \( \nu \Rightarrow j \) for every \( j \).

**Proof.** By induction on \( n \). The statement is easily verified for \( n = 3 \). Let \( n \) be the smallest integer for which the theorem does not hold and let \( T_1 = ([n], E_1), T_2 = ([n], E_2) \) be a counterexample. By minimality of \( n \), for every \( n \geq j \geq 1 \) there is some \( n \geq i \geq 1 \) such that \( i \Rightarrow j k \) for every \( k \neq i, j \), where \( \Rightarrow j \) indicates that the relation is defined with respect to tournaments \( T_1 \setminus \{j\} \) and \( T_2 \setminus \{j\} \). When this happens we say that \( i \) is singled out by \( j \). Clearly \( i \not\Rightarrow j \), or else the theorem holds with \( \nu = i \), since \( i \Rightarrow j \) clearly implies \( i \Rightarrow k \). Consequently, no vertex is singled out more than once. We denote \( \pi(j) = i \) and conclude that \( \pi \) is a permutation on \([n]\), since \( \pi(j) \) is defined for every \( n \geq j \geq 1 \) and \( \pi \) is an injective mapping.

However, this is impossible. By Condition (1), every \( j \) satisfies \( |\Gamma_2(j)| > |\Gamma_1(\pi(j))| \). But \( \sum_j |\Gamma_2(j)| = \sum_j |\Gamma_1(\pi(j))| = \binom{n}{2} \), since \( \pi \) is a permutation. This contradiction completes the proof. \( \square \)
The same proof yields as well

**Corollary 2.** Let \( T_1 = ([n], E_1), T_2 = ([n], E_2) \) be two tournaments. Then there is \( \mu \in [n] \) such that \( i \Rightarrow \mu \) for every \( i \).

**Proof.** The same proof works. Just switch between \( T_1, T_2 \) and reverse all edges in the two tournaments.

\[\square\]

**References**
