different embeddings (usually only two!). Now we are done by our Main Theorem. ∎

It should be noted that the set of minor minimal embeddings with $\rho(\psi) \geqslant 2$ into the projective plane was determined by Vitray [12].

COROLLARY 5.3. *Let Σ be a closed orientable surface different from the 2-sphere. There are finitely many graphs $G_1, G_2, ..., G_N$ such that if a graph G has an embedding ψ into Σ with $\rho(\psi) \geqslant 2$ then G contains some G_i $(1 \leqslant i \leqslant N)$ as a minor.*

Proof. Having an embedding with representativity at least 2 we reach a minor minimal embedding with $\rho \geqslant 2$ by successive deletions and contractions of edges which do not cause the representativity to drop to 1. ∎

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Group Connectivity of Graphs—A Nonhomogeneous Analogue of Nowhere-Zero Flow Properties

FRANÇOIS JAEGER

Lsd, Imag, bp53x 38041 Grenoble Cedex, France

NATHAN LINIAL

Institute of Mathematics and Computer Science, Hebrew University, Jerusalem 91904, Israel

CHARLES PAYAN

Lsd, Imag, bp53x 38041 Grenoble Cedex, France

AND

MICHAEL TARSI

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Israel

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Let $G=(V, E)$ be a digraph and f a mapping from E into an Abelian group A. Associated with f is its *boundary* ∂f, a mapping from V to A, defined by $\partial f(x) = \sum_{e \text{ leaving } x} f(e) - \sum_{e \text{ entering } x} f(e)$. We say that G is *A-connected* if for every $b: V \to A$ with $\sum_{x \in V} b(x) = 0$ there is an $f: E \to A - \{0\}$ with $b = \partial f$. This concept is closely related to the theory of nowhere-zero flows and is being studied here in light of that theory.

1. INTRODUCTION

Let $G=(V, E)$ be a directed graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions $f: E \to A$. For a vertex $x \in V$ let $E^-(x)$ (resp. $E^+(x)$) be the set of all ingoing (resp. outgoing) edges incident with x. Associated with every function $f \in F(G, A)$ is its *boundary* $\partial f: V \to A$, defined as

$$\partial f(x) = \sum_{e \in E^+(x)} f(e) - \sum_{e \in E^-(x)} f(e),$$

where "$\sum$" refers to addition in the group A.

Visualizing f as some sort of a "flow," ∂f measures the "deficit in material" which "accumulates" at each vertex. Let A^* stand for the set of nonzero elements of A and let $F^*(G, A)$ be the subset of $F(G, A)$ consisting of all functions $f\colon E \to A^*$. An *A-nowhere-zero-flow* (abbreviated as A-NZF) in G is an $f \in F^*(G, A)$ with $\partial f = 0$. Nowhere-zero flow have been studied extensively since they were introduced by *W.* Tutte more than three decades ago [12]. A thorough discussion of previous work on NZFs and a list of references can be found in [6].

Let $G = (V, E)$ be a graph and A an Abelian group. A mapping $b\colon V \to A$ is a *zero sum function on* G if $\sum_{x \in V} b(x) = 0$. The questions that we study here regard the following:

DEFINITION *Let $G = (V, E)$ be an undirected graph and A an Abelian group. G is said to be A-connected if the following holds: Given an orientation G' of G, every zero sum function $b\colon V \to A$ is the boundary ∂f of some function $f \in F^*(G', A)$*

Two obvious facts to note are that for every $f \in F(G, A)$ ∂f is zero sum and that the choice of orientation G' is immaterial: If the above is satisfied for one orientation of G then it holds for every orientation (replace $f(e)$ by $-f(e)$ if the orientation of an edge e is reversed). We also observe that every A-connected graph admits an A-NZF and this is one of the motivations for the present paper. A-connectivity is a property of undirected graphs whose definition calls for an arbitrary orientation, as is also the case with the definition of A-NZF. Throughout this paper a graph G is assumed to be equipped with a fixed arbitrary orientation and the discussion always concerns the undirected underlying graph. This abuse of language helps circumvent long and cumbersome formulations. This convention follows [6].

Another notational convention which we frequently use is the following:

(1.1) A function f defined on a subset H of a set E (of edges) is considered to be defined on the whole set E, where it is assumed that $f(e) = 0$ for every $e \in (E - H)$ (the range of f is an Abelian group).

Here are some reasons why we chose the term "*A-connectivity*": Since every boundary satisfies the zero sum condition for every connected component, A-connectivity obviously implies connectivity. In fact, since every A-connected graph admits an A-NZF, it must also be 2-edge connected. The converse is false, for the cycle C_n is not A-connected if the order of A is $\leqslant n$. A-connectivity is preserved under addition of edges (see Corollary 2.4), as well as under identification of vertices. It is also a local property, highly sensitive to the existence of sparse induced subgraphs. In later sections several less obvious relations between (edge/vertex) connectivity and A-connectivity are explored. Finally, to avoid unnatural restriction to connected graphs, we define a graph to be *locally A-connected* if each of its connected components is A-connected. Accordingly a *locally zero sum function* is a function $b\colon V \to A$ which sums up to zero over every connected component of a graph $G = (V, E)$.

As to the content of this paper, Section 2 contains several equivalent formulations of A-connectivity and develops some tools. Section 3, the main part of the paper, consists of a systematic comparison of problems and theorems from the theory of nowhere-zero flows to their analogues where "admitting an A-nowhere-zero flow" is replaced by "A-connected." We conclude, in Section 4, by some generalizations of the theory to non-graphic regular matroids.

2. Preliminary Observations

Without the nowhere zero constaint A-connectivity becomes mere connectivity:

PROPOSITION 2.1. *A graph $G = (V, E)$ is connected if and only if every zero sum function $b\colon V \to A$ is the boundary ∂f of some function $f \in F(G, A)$.*

Proof. The "if" part is obvious. For the "only if" part it clearly suffices to deal with the case where G is a tree. The result is trivial if G has only one vertex. Otherwise, let $x \in V$ be a leaf and $e = (x, y)$ the edge incident with it and let $G' = G - x$. For a zero sum function $b\colon V \to A$ on G, let $b'\colon V(G') \to A$ equal b on $V - \{x, y\}$ and let $b'(y) = b(y) + b(x)$. Now b' is a zero sum function on the tree G', so by induction on the size of the tree, there exists $f' \in F(G', A)$ with $b' = \partial f'$. Assuming that e is directed from x to y, let $f(e) = b(x)$ and $f(E - \{y\}) = f'$ to obtain $b = \partial f$. ▮

An *A-flow* in G is a function $f \in F(G, A)$ where ∂f is identically zero. Accordingly, $\partial f = \partial g$ if and only if f and g are in the same coset of $F(G, A)$ modulo $F_0(G, A)$—the subgroup of all A-flows ($F(G, A)$ is considered a group in the obvious way). Thus the ∂ operator is a bijection from $F(G, A)$ to the set of zero sum functions if and only if G is connected and its group of A-flows $F_0(G, A)$ is trivial, that is, if and only if G is a tree.

A-connectivity can be redefined in terms of A-flows:

PROPOSITION 2.2. *Let $G = (V, E)$ be a graph and A an Abelian group. Then the following three statements are equivalent:*

(i) *G is A-connected.*

(ii) *G is connected and given any $\bar{f} \in F(G, A)$, there exists an A-flow $f \in F_0(G, A)$ such that $f(e) \neq \bar{f}(e)$ for every $e \in E$.*

(iii) *Given a zero sum function* $b: V \to A$ *and* $\bar{f} \in F(G, A)$, *there exists a function* $f \in F(G, A)$ *which satisfies* $\partial f = b$ *and* $f(e) \neq \bar{f}(e)$ *for every* $e \in E$.

Proof. (i) $\Rightarrow$ (ii): If G is A-connected then, given any $\bar{f} \in F(G, A)$, there exists $g \in F^*(G, A)$ such that $\partial g = -\partial \bar{f}$. Define $f = \bar{f} + g$; then f is an A-flow and it differs from $\bar{f}$ on every edge.

(ii) $\Rightarrow$ (iii): Applying Proposition 2.1, there exists, for every zero sum function b, $g \in F(G, A)$ such that $b = \partial g$. Given $\bar{f} \in F(G, A)$, (ii) yields the existence of a flow $f' \in F_0(G, A)$ such that $f'(e) \neq \bar{f}(e) - g(e)$ for every edge e. The function $f = g + f'$ satisfies the assertion of (iii).

(iii) $\Rightarrow$ (i): Take $\bar{f} = 0$. ∎

A similar equivalence holds where "A-connected" in (i) is replaced by "locally A-connected," in which case the connectivity requirement in (ii) should be removed and the function b in (iii) should be a *locally* zero sum function.

Seymour's proof of his 6-NZF theorem [10] is based on his *k-closure* operator, defined in [10], which we now recall:

For a positive integer k, the k-closure is the transitive closure of the operator, $H \to H \cup c$, where c is a circuit with at most k edges not in H. In other words, if H is a subgraph of G, then the k-closure of H in G, denoted by $cl_k(H)$, is the (unique) maximal subgraph of G of the form $H \cup c_1 \cup ,.., \cup c_n$, where for every i, $1 \leqslant i \leqslant n$, c_i is a circuit and $|c_i - (H \cup c_1, ..., \cup c_{i-1})| \leqslant k$. (Throughout this paper subgraphs are referred to as sets of edges).

The main pertinent property of k-closure is the following [10]:

LEMMA 2.1. *Let* $G = (V, E)$ *be a graph and* H *a subgraph of* G *such that* $\mathrm{cl}_k(H) = G$. *Also let* A *be an Abelian group and* $\bar{F}$ *a function,* $\bar{F}: (G - H) \to 2^A$, *such that for every* $e \in G - H$, $|\bar{F}(e)| < |A|/k$. *Then there exists an A-flow* $f \in F_0(G, A)$ *such that* $f(e) \notin \bar{F}(e)$ *for every* $e \in G - H$.

Proof. Let $G = H \cup c_1 \cup ,..., \cup c_n$, where $c_1,..,c_n$ are the circuits which form G as the k-closure of H. Define $G_0 = H$ and for every i, $1 \leqslant i \leqslant n$ $G_i = G_{i-1} \cup c_i$, so $G_n = G$ and $|G_i - G_{i-1}| \leqslant k$. There is nothing to prove if $n = 0$. Otherwise, consider the last circuit c_n. There are $|A|$ different A-flows in $F_0(c_n, A)$. For every $e \in (G_n - G_{n-1})$ there are $|\bar{F}(e)| < |A|/k$ A-flows $f' \in F_0(c_n, A)$, for which $f'(e) \in \bar{F}(e)$. Summing up over at most k edges of $G_n - G_{n-1}$, there remains at least one A-flow $f_1 \in F_0(c_n, A)$, such that $f_1(e) \notin \bar{F}(e)$ for every $e \in G_n - G_{n-1}$. By induction on n, G_{n-1} has an A-flow $f_2 \in F_0(G_{n-1}, A)$, which satisfies $f_2(e) \notin \bar{F}(e)$ for every $e \in (G_{n-1} - c_n) - H$ and $f_2(e) \notin \{a - f_1(e) \mid a \in \bar{F}(e)\}$ for every $e \in (G_{n-1} \cap c_n) - H$.

The required function $f \in F(G, A)$ is defined by $f = f_1 + f_2$ (Recall Convention 1.1; it means $f(e) = f_2(e)$ for $e \in G_{n-1} - c_n$, $f(e) = f_2(e) + f_1(e)$ for $e \in G_{n-1} \cap c_n$, and $f(e) = f_1(e)$ for $e \in c_n - G_{n-1}$.). Then $f \in F_0(G, A)$ and $f(e) \notin \bar{F}(e)$ for every $e \in G - H$ are easily verified ∎.

Clearly a subgraph H is connected and spanning in a graph G if and only if $\mathrm{cl}_1(H) = G$. Therefore

COROLLARY 2.3. *Let* H *be a spanning subgraph of* G *and* $f': (G - H) \to A$. *Then there exists a flow* $f \in F_0(G, A)$, *whose restriction to* $G - H$ *equals* f'.

Proof. Apply Lemma 2.1 with $k = 1$ and $\bar{F}(e) = A - \{f'(e)\}$ for every $e \in G - H$. ∎

Lemma 2.1 also implies that k-closure preserves A-connectivity, for large enough A.

COROLLARY 2.4. *The k-closure of a locally A-connected subgraph is locally A-connected whenever* $|A| > k$.

Proof. Assume $G = (V, E) = \mathrm{cl}_k(H)$, where H is locally A-connected and A is an Abelian group of order $|A| > k$. Given a function $\bar{f} \in F(G, A)$, apply Lemma 2.1 with $\bar{F}(e) = \{\bar{f}(e)\}$ for all $e \in G - H$ to obtain an A-flow $f_1 \in F_0(G, A)$ such that $f_1(e) \neq \bar{f}(e)$ for every $e \in G - H$. The local A-connectivity of H provides, by Proposition 2.2 (ii), the existence of an A-flow $f_2 \in F_0(H, A)$, which satisfies $f_2(e) \neq \bar{f}(e) - f_1(e)$ for every $e \in H$. The function $f = f_1 + f_2$ is an A-flow in G which differs from $\bar{f}$ on every edge. By Proposition 2.2, G is locally A-connected. ∎

3. A-NZF VERSUS A-CONNECTIVITY

3.1. *The Role of* $|A|$ *and Monotonicity*

A fundamental theorem in the theory of NZF's states that the existence of an A-NZF in a graph G depends only on the order of A [12]. Thus, "G admits a k-NZF" is used to indicate that for every Abelian group of order k there exists an A-NZF in G.

We neither prove nor disprove a similar result with regards to A-connectivity. Even for the smallest different groups of the same order, we do not know of any Z_4-connected graph which is not $Z_2 \times Z_2$-connected, or vice versa. Neither can we prove that such graphs do not exist.

Another, closely related, property of NZF's is what we call monotonicity: If G admits a k-NZF then it admits a t-NZF for every $t > k$ [12]. A similar statement with respect to A-connectivity is false. We need the following elementary observation:

PROPOSITION 3.1. *Let P be a cyclic group of prime order, S a proper subset of P, and T a subset of P which contains at least two elements. Then* $|S+T|>|S|$.

Proof. Let a, b be two distinct elements of T. Suppose $|S+T|=|S|$. The $S+a=S+b$, because $|S+a|=|S+b|=|S|$ and both are included in $S+T$. But then $S=S+a-b$ so S contains a coset of the subgroup generated by $a-b$. But P has no proper subgroup so $S=P$, a contradiction. ▮

Let $G=(V, E)$ consist of 4 simple "parallel" paths, each 3 edges long, sharing end vertices x, y and otherwise disjoint (see Fig. 1).

OBSERVATION. *G is Z_5-connected but it is not Z_6-connected.*

Proof. Fix the orientation of G with all paths directed from x to y. Starting with Z_6, define a zero sum function $b: V\to Z_6$ as follows: $b(x)=1$, $b(y)=-1$ and for the other 8 inner vertices v, $b(v)=0$. Let $\bar{f}: E\to Z_6$ assign the odd elements of Z_6, 1, 3, and 5, to the three edges of each path in some order (see Fig. 1). Now check condition (iii) of Proposition 2.2: A function $f\in F(G, Z_6)$ which satisfies the 0-boundary condition on the inner vertices and never agrees with $\bar{f}$ is an even constant on each of the 4 paths. To satisfy the given boundary on x and y, 1 must be expressible as the sum of 4 even numbers, a contradiction.

On the other hand, Z_5-connectivity follows through Proposition 2.2 (ii). The restrictions forced by an assignment $\bar{f}: E\to Z_5$ of "forbidden" values leave each of the 4 parallel paths with a set of at least two allowed values, from which to select the value for the flow f on this path, in such a way to make $\partial(f)(x)=\partial(f)(y)=0$. To complete the proof, it suffices to show that for every four sets $A, B, C, D\subset Z_5$ of 2 elements each, there exist $a\in A$, $b\in B$, $c\in C$, $d\in D$ such that $a+b+c+d=0$. According to Proposition 3.1 the cardinalities of A, $A+B$, $A+B+C$, $A+B+C+D$ form a strictly increasing sequence until they reach the value 5, so $A+B+C+D$ contains all 5 elements of Z_5. In particular it contains 0. ▮

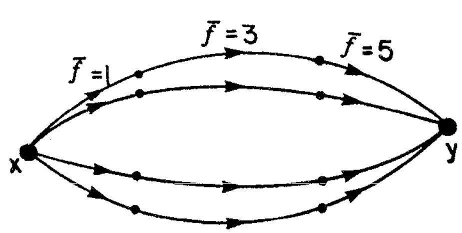

FIGURE 1

A similar construction yields a P-connected graph which is not A-connected, whenever P is of prime order and A has a proper subgroup H, such that $|A|-|H|<|P|-1$. We do not know if monotonicity holds for 3-edge connected graphs. This question will be put in a new perspective by the results to follow.

A main theme in what follows is theorems of the form: "If G is m-edge connected, then it is A-connected for a certain class of groups A."

3.2. $A=Z_2$

Admitting a 2-flow is equivalent to being an Eulerian graph. However, no graph except K_1 is Z_2-connected. There is only one function in $F^*(G, Z_2)$ and its boundary, the degree parity function on V, is not the only zero sum function on G, unless $|V|=1$.

3.3. 3-NZF *versus* Z_3-*Connectivity*

Tutte [3, Unsolved Problem 48] stated the following conjecture, to which we refer as the strong 3-NZF conjecture:

Conjecture 1. Every 4-edge connected graph admits a 3-NZF.

The conjecture still stands open. Its extension to Z_3-connectivity is, however false. Figure 2 presents a 4-regular, 4-connected graph which is not Z_3-connected.

Denote the graph of Fig. 2 by $G=(V, E)$. The function $b: V\to Z_3$ defined by $b(x)=1$ for every $x\in V$ is a zero sum function. For G to be Z_3-connected, there must exist a function $f\in(G, Z_3)$ with $\partial f=b$. Reversing every edge e for which $f(e)=2$, we obtain an orientation for which b is the boundary of the function which maps all the edges to 1. Hence, in that orientation, the out-degree of each vertex is either 1 or 4. This implies that there are 4 vertices of out-degree 4 and 8 of out-degree 1. But all independent sets in G have at most 3 vertices, so there must be 2 adjacent vertices with out-degree 4, a contradiction.

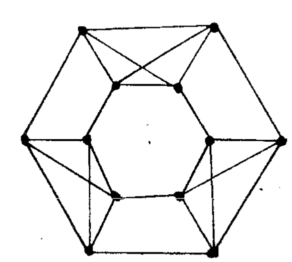

FIGURE 2

As relaxed version of the 3-NZF conjecture, also yet unsolved is the weak 3-NZF conjecture [6]:

Conjecture 2. There exists a constant k such that every k-edge connected graph admits a 3-NZF.

This version is equivalent to its Z_3-connectivity analogue:

Conjecture 3. There exists a constant l such that every l-edge connected graph is Z_3-connected.

Proof of the Equivalence. Let G be a $2(k+1)$-edge connected graph. By a theorem of Nash-Williams [9] and Tutte [13] G contains $k+1$ pairwise edge disjoint spanning trees. (A very short and elegant proof of that result is derived from Edmonds' matroid packing theorem [4].) Let T be one of these trees. By Proposition 1, given a zero sum function $b: V \to A$, there exists $f_1 \in F(T, A)$ with $\partial f_1 = b$. Denote by H the subgraph of G consisting of all the edges e for which $f_1(e) = 0$. Since f_1 was defined on T, H contains the other k edge-disjoint spanning trees and hence it is k-edge connected. Assuming that every k-edge connected graph admits a 3-NZF, there exists in H a Z_3-NZF f_2. Clearly $(f_1+f_2) \in F^*(G, A)$ and $\partial(f_1+f_2) = b$. Every l-edge connected graph, where $l = 2k+2$, is thus Z_3-connected. The other direction is obvious. ∎

By means of the last proof, the strong 3-NZF conjecture implies Z_3-connectivity for every 10-edge connected graph. We do not know, however, of any counterexample to the following stronger statement:

Conjecture 4. Every 5-edge connected graph is Z_3-connected.

Recently, Alon, Linial, and Meshulam [1] have studied the following problem: What is the smallest integer k such that the union (with repetitions) of any k spanning sets $B_1, B_2, ..., B_k$ of the n-dimensional space GF_p^n over the prime field GF_p forms an additive basis of the space; i.e., for any $x \in GF_p^n$ there exist $A_1 \subseteq B_1, ..., A_k \subseteq B_k$ such that $x = \sum_{i=1}^{k} \sum_{y \in A_i} y$? The upper bound presented in their paper depends on both the prime p and the dimension n; however, no counterexample is known to the conjecture that k is upper bounded by p. Focusing on the case where $p = 3$, the following is obtained:

PROPOSITION 3.2. *If there exists an integer k, such that the union with repetitions of any k spanning sets of a finite vector space over GF_3 is an additive basis of the space, then Conjecture* 3 *holds for $l = 2k$. In particular if it holds for $k = 3$ (which may be true for all we know) then every* 6*-edge connected graph is Z_3-connected.*

Proof. Let H be a spanning subgraph of a connected graph $G = (V, E)$. For every $e \in E$, let $f_e \in F(G, Z_3)$ denote the function which maps e to 1 and all the other edges to 0. As a consequence of Proposition 2.1, the space $B_0(G, Z_3)$ of all zero sum functions on G over Z_3 is spanned (as a linear space over GF_3) by $B_H = \{\partial f_e \mid e \in H\}$. Let $\{T_i, i = 1, 2, ..., k\}$ be a collection of pairwise edge-disjoint spanning subgraphs of a graph $G = (V, E)$ (which exists if G is $2k$-edge connected). By hypothesis the union (with repetition) of $B_{T_1}, ..., B_{T_k}$ is an additive basis of $B_0(G, Z_3)$. Since no edge belongs to more than one of the T_i's, every element of $B_0(G, Z_3)$ is the boundary of a function which maps E into $\{0, 1\} \subset Z_3$. Consequently, every element of $b_1 + B_0(G, Z_3)$, where b_1 stands for the boundary of the constant function $f(e) = 1$, is the boundary of a function from $F^*(G, Z_3)$. Clearly $b_1 + B_0(G, Z_3) = B_0(G, Z_3)$ and hence every zero sum function is the boundary of a function $f \in F^*(G, Z_3)$. ∎

3.4. *Groups of Order* 4

Jaeger [5] proved that every 4-edge connected graph admits a 4-NZF. His proof can be modified to obtain the following:

THEOREM 3.1. *Every graph which contains* 2 *edge-disjoint spanning trees (in particular every* 4*-edge connected graph) is A-connected for every Abelian group A of order $|A| \geqslant 4$.*

Proof. Let T_1 and T_2 be two edge-disjoint spanning trees of $G = (V, E)$. Pick a nonzero element x of A. By Proposition 2.3 there exists an A-flow $f_1 \in F_0(G, A)$ with $f_1(e) = x$ for every $e \in G - T_1$. Given a zero sum function b on G, construct, by Proposition 2.1, a function $f_2 \in F(T_1, A)$ with $\partial(f_1+f_2) = b$. Clearly $\partial(f_1+f_2) = b$ and $(f_1+f_2)(e) = x$ for every $e \notin T_1$. Consider now the edges $e \in T_1$ for which $(f_1+f_2)(e) = 0$. Take the modulo 2 sum of all the elementary circuits which those edges form with the spanning tree T_2 (an edge belongs to that sum if and only if it is contained in an odd number of those circuits). This sum, denoted by C, is an edge-disjoint union of circuits. There exist a $y \in A - \{0, x, -x\}$ (since $|A| \geqslant 4$), and an $f_3 \in F_0(G, A)$ which equals either y or $-y$ (according to the orientation of each edge) on C and 0 elsewhere. The sum $f = f_1+f_2+f_3$ belongs to $F^*(G, A)$ and its boundary equals b. ∎

3.5. *The Analogue of Seymour's* 6-NZF *Theorem*

Seymour [10] proved that every 2-edge connected graph admits a 6-NZF. As already mentioned, 2-edge connectivity does not imply A-connectivity, regardless of the order of A. However, 3-edge connectivity (in fact even a slightly weaker condition) yields the following analogue to Seymour's 6-NZF Theorem.

THEOREM 3.2. *Let G be a 3-edge connected graph and v a vertex of degree 3 of G. Then $G-v$ is A-connected for any Abelian group A of order at least 6. In particular, every 3-edge connected graph is A-connected for such A.*

In the sequel we assume without loss of generality that all graphs are loopless and we fix an additive group A or order at least 6. The proof of Theorem 3.2 needs the following three lemmas:

LEMMA 3.1. *Theorem 3.2 is equivalent to its restriction to cubic graphs.*

Proof. It is clear from the formulation of A-connectivity given in Proposition 2.2 (ii) that a connected graph is A-connected if and only if each of its blocks is A-connected. Also note that a vertex v of degree 3 in a 3-edge connected graph cannot be a cut vertex. Hence in the statement of Theorem 3.2 we may assume that G is a 2-vertex connected and 3-edge connected graph. Consider a vertex u of G of degree d, and let $e_1, ..., e_d$ be the edges incident to u. We delete u, introduce d new vertices $u_1, ..., u_d$, d new edges forming a cycle $C(u)$ on these vertices and for $i=1, ..., d$ the edge e_i is now made incident to u_i instead of u. Performing this operation on every vertex u of G of degree at least 4, we clearly obtain a 2-edge connected cubic graph H. Moreover an edge-cut of size 2 of H cannot be disjoint from all cycles $C(u)$ (it would correspond to an edge-cut of size 2 of G) and hence it is contained in such a cycle $C(x)$. But then x would be a cut-vertex of G, a contradiction. Hence H is 3-edge connected. Note that if v is a vertex of degree 3 of G, v is still a vertex of degree 3 of H. Then if Theorem 3.2 holds for cubic graphs, $H-v$ is A-connected, and since $G-v$ can be obtained from $H-v$ by contraction of edges, $G-v$ is A-connected. ∎

In the sequel we call a *truncated cubic* 3-*edge connected graph* any graph obtained from a cubic 3-edge connected graph with at least 4 vertices by the deletion of a single vertex, and we denote by TC3 the class of these graphs. Note that every graph in TC3 is simple and has an odd number of vertices, exactly three of which have degree 2. We now give a simple constructive characterization of TC3, based on the following two basic operations:

Sticking. Let G, G' be two (disjoint) graphs of TC3, v a vertex of G and v'_1, v'_2, v'_3 the three vertices of degree 2 of G'. If v has degree 3, let e_1, e_2, e_3 be the three edges of G incident to v. A 3-sticking of G, G' at v is obtained by deleting v from G and making e_i incident to $v'_{\sigma(i)}$ $(i=1, 2, 3)$ instead of v for an arbitrary permutation σ of $\{1, 2, 3\}$ (see Fig. 3a). Similarly if v has degree 2, denoting by e_1, e_2 the two edges of G incident to v, a 2-sticking

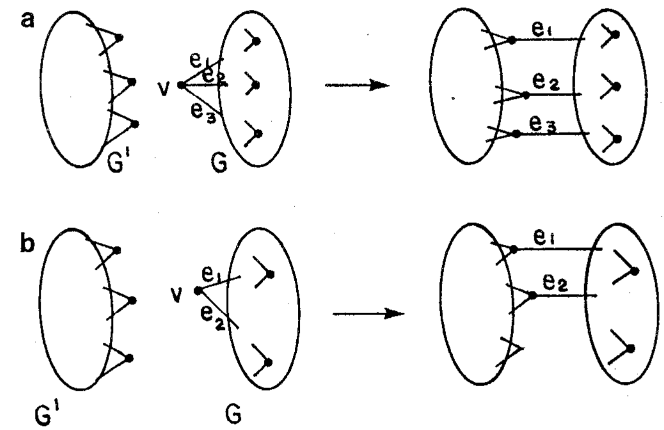

FIGURE 3

of G, G' at v is obtained by deleting v from G and making e_i incident to $v'_{\sigma(i)}$ $(i=1, 2)$ instead of v for an arbitrary permutation σ of $\{1, 2, 3\}$ (see Fig. 3b).

Growing. Let G be a graph of TC3, e an edge, and u a vertex of degree 2 of G (e and u might be incident). The growing of G at e and u is obtained by replacing e by a path of length 2 with the same ends (thus creating a new vertex v of degree 2) and then introducing a new vertex w joined by two new edges to u and v (see Fig. 4).

LEMMA 3.2. TC3 *is the smallest class of graphs which contains the triangle and is closed under sticking and growing.*

Proof. The triangle is obtained from K_4 by deletion of a vertex and hence belongs to TC3. By adding to each graph involved in a sticking or growing operation a new vertex joined by new edges to the 3 vertices of degree 2 (see Fig. 3′, 4′), these operations can be identified with well known operations which preserve the class of cubic 3-edge connected graphs. Hence TC3 is closed under sticking and growing.

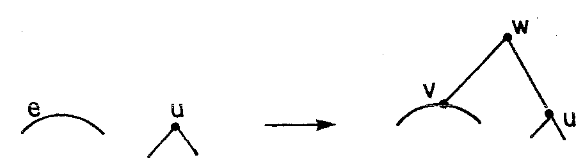

FIGURE 4

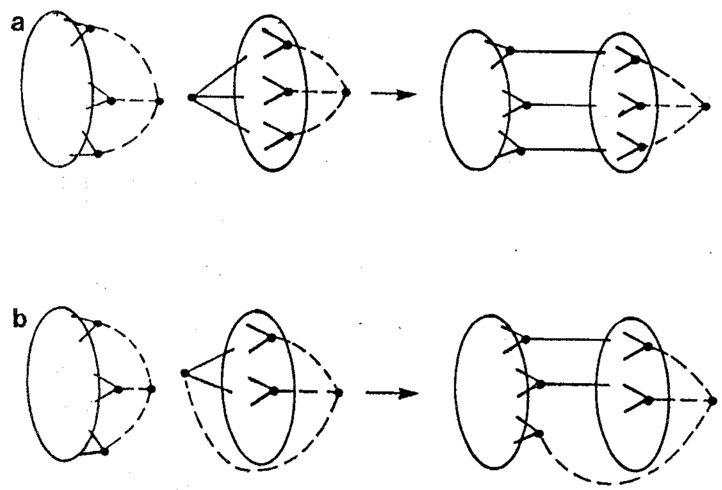

FIGURE 3′

Conversely, let G be a graph in TC3 distinct from the triangle and let e be an edge of G incident to a vertex of degree 2. Let G^+ be the cubic 3-edge connected graph obtained from G by adding a new vertex u joined to each of its 3 vertices of degree 2.

Assume first that the vertex-set of G^+ can be partioned into V_1, V_2 with $|V_i| \geqslant 2$ $(i=1,2)$ in such a way that G^+ has exactly three edges joining V_1, V_2. Since G^+ is 3-edge connected these three edges are mutually disjoint. If one (respectively none) of them is incident to u, it is easy to see that G can be obtained from two smaller graphs of TC3 by a 2-sticking (respectively 3-sticking) operation.

Now if no such partition exists, the graph H^+ obtained from G^+ by deleting the edge e and "erasing" its two ends (see Fig. 5) is a 3-edge connected cubic graph with at least 4 vertices. Hence $H^+ - u$ belongs to TC3 and it is easy to see that G can be obtained from this graph by a growing operation. ∎

Remark. One of the referees of this paper kindly pointed out that the 2-sticking operation is not needed in the above construction of TC3. Our approach avoids the proof of this non-trivial result.

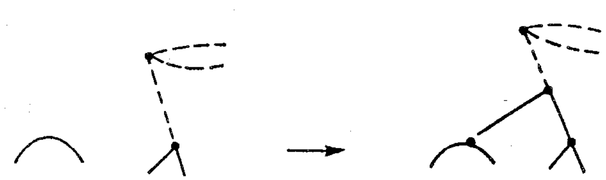

FIGURE 4′

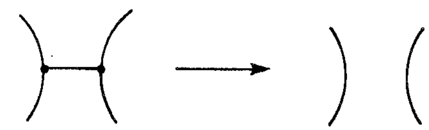

FIGURE 5

We now introduce another class of graphs defined inductively. A 2-join of the graphs G, G' is obtained by taking disjoint copies of G, G' and adding exactly two new edges joining these two copies. We denote by 2C the smallest class of graphs which contains the isolated-vertex graph K_1 (one vertex and no edges) and is closed under 2-joins. We call the members of 2C *2-constructible* graphs. Note that any 2-constructible graph on p vertices is connected and has exactly $2(p-1)$ edges.

We shall also need the following definition. Let G, G' be two graphs of which we take disjoint copies, and v be a vertex of G. A *replacement of v by G' in G* is obtained by first "splitting" the vertex v of G into new vertices $v_1, ..., v_k$, each edge previously incident to v being now incident to one of these new vertices, and then identifying each of these new vertices with some vertex of G' (see Fig. 6).

LEMMA 3.3. *Every truncated cubic 3-edge connected graph G has a spanning tree T such that the contraction of the edges of G not in T yields a 2-constructible graph.*

Proof. It will be convenient to formulate the property of G stated in Lemma 3.3 as follows. G will be said to be *well colored* if its edges are colored blue and red in such a way that

(i) the blue edges form a spanning tree,

(ii) contracting the red edges yields a 2-constructive graph.

We need the following observation. Let $G=(V, E)$ be a well colored graph of TC3 with $|V| = n$, so that $|E| = 3(n-1)/2$. Let p be the number of

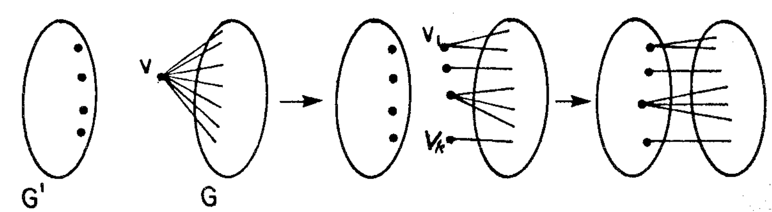

FIGURE 6

connected components formed on V by the red edges. The graph obtained from G by contracting the red edges has p vertices and since it is 2-constructible, G has $2(p-1)$ blue edges. Then by (i) $2(p-1)=n-1$. It follows that G has $\frac{1}{2}(n-1)=n-p$ red edges and hence the graph formed on V by the red edges is a forest.

Now let us show that every graph of TC3 can be well colored. This property is immediate for the triangle. By Lemma 3.2 it is enough to show that it is preserved under sticking and growing operations.

Consider first two disjoint well colored graphs $G=(V, E)$, $G'=(V, E')$ of TC3 and let v be a vertex of G. Let us perform a sticking of G, G' at v (see Fig. 3). We let each edge retain its color in this process. Clearly property (i) is still satisfied in the resulting graph G''.

Now let H'' be the graph obtained from G'' by contracting its red edges. The graph H'' could also be obtained by first contracting the red edges in both G and G', which yields 2-constructible graphs H and H', and then performing a replacement of the vertex w of H by H', where w corresponds to the red connected component of v in G. To see this, let us view the sticking operation as a replacement of v by G' in G: v is split into d vertices $v_1, ..., v_d$ of degree one, where $d \in \{2, 3\}$ is the degree of v, and these vertices are then identified with some vertices of degree 2 of G'. We observe that since G has no red cycles, the vertices $v_1, ..., v_d$ belong to different red connected components $C_1, ..., C_d$. Then when v_1 is identified with some vertex of G', the component C_1 which contains v_i is attached to the corresponding red component of G'. This shows that each red component of G'' which is not contained in $V-\{v\}$ or V' is formed by attaching some C_i's to a red component of G', and hence H'' can be obtained by performing an appropriate replacement of the vertex w of H corresponding to v by H'.

We now show that the replacement of a vertex w of a graph H of 2C by a graph H' of 2C yields a graph H'' which is also in 2C. Indeed let us consider a construction of H starting from isolated vertices (one for each vertex of H) and using the 2-join operation (with connected operands only). In the first 2-join operation involving w, the isolated vertex w is one of the operands. Replacing this operand by H' and modifying the remaining 2-join operations in the appropriate way will yield a construction which shows that H'' is in 2C. Thus we have shown that property (ii) is also satisfied in the graph G''.

Consider now a well colored graph G of TC3 and let e be an edge and u be a vertex of degree 2 of G. Let us perform a growing operation of G at e and u (see Fig. 4). Each edge of G distinct from e will retain its color in the resulting graph G'. If e is red we color red the two edges of the path replacing e and blue the two new edges incident to the new vertex. Then (i) is clearly satisfied. Moreover the graph obtained from G' by contracting its red edges can also be obtained by first contracting the red edges of G and then performing a 2-join operation with one new vertex. Hence (ii) is also satisfied. Finally if e is blue, coloring blue the two new edges incident to the new vertex, it is possible to color one edge of the path replacing e blue and the other red such that (i) is satisfied. Then (ii) will also be satisfied as before. ∎

Proof of Theorem 3.2. By Lemma 3.1 it is enough to show that every graph $G=(V, E)$ of TC3 is A-connected. Let T be a spanning tree of G such that the contraction of the edges of $R := E-T$ yields a 2-constructible graph. It is clear from the definition of 2-constructible graphs that $\text{cl}_2(R)=G$. Let $b: V \to A$ be a zero sum function. First apply Proposition 2.1 to obtain a function $f_1 \in F(T, A)$ with $\partial f_1 = b$. By Lemma 2.1 construct an A-flow $f_2 \in F_0(G, A)$, which avoids a set $\bar{F}(e)$ on each edge of $T=G-R$. The set of forbidden values $\bar{F}(e)$ for every $e \in T$ is $\{f_1(e), f_1(e)-x, f_1(e)+x\}$, where x is a fixed nonzero element of $|A|$. By Lemma 2.1 three values may be forbidden, provided $|A|/2>3$, i.e., $|A| \geqslant 7$. If $A=Z_6$ then set $x=3$. In that case $x=-x$ and there are only 2 forbidden values. The function $g=f_1-f_2$ has boundary b and it satisfies $g(e) \notin \{0, x, -x\}$ for $e \in T$.

As in the proof of Theorem 3.1, now take all the edges for which g equals 0 and form the modulo 2 sum C of their elementary circuits with respect to T. Let f_3 be an *A-flow* which equals either x or $-x$ (according to the orientation of each edge) on C and 0 elsewhere. $f=f_3+g$ belongs to $F^*(G, A)$ and satisfies $\partial f=b$ and hence G is A-connected. ∎

Remarks. 1. By simple induction, the edge-set of any 2-constructible graph G can be partitioned into two spanning trees of G. It then follows from Lemma 3.3 (and the fact that there are no red cycles) that the edge-set of any graph of TC3 can be partitioned into 3 cotrees (a cotree is the complement of a spanning tree). This refines a result used in [5] to establish the 8-flow theorem; the proof given there was matroid-theoretical.

2. In addition to the detailed proof of the 6-NZF theorem in [10] there is an alternative proof, briefly sketched in the last paragraph of that article. The core of the detailed proof establishes the following property: Every cubic 3-connected graph G contains an Eulerian subgraph C such that $\text{cl}_2(C)=G$. This structure implies the existence of a 6-NZF, but is insufficient for Z_6-connectivity (for instance if $G=C$ is a circuit of length $\geqslant 6$). The sketched proof, however, can be modified to provide the tools to prove A-connectivity for $|A| \geqslant 6$. The proof presented here is different.

3. It seems natural to propose the following:

Conjecture 5. Every 3-edge connected graph is Z_5-connected.

Note that this would imply Tutte's 5-flow conjecture.

4. Extensions to Regular Matroids

The notion of flows in general, as well as that of NZF's, is naturally extended to the wider framework of regular matroids [11, 7, 8]. The terminology we use in this section is that of [11] and the reader is referred to that article for the relevant terms and definition. In particular, as agreed for graphs, a regular matroid is assumed to be equipped with an arbitrary orientation.

Although the boundary of a function on a non-graphic matroid cannot be easily defined (being based on the existence of vertices), A-connectivity is naturally defined in terms of flows, by means of Proposition 2.2 (ii), as follows:

Definition. Let M be a regular matroid and A a non-trivial Abelian group. M is locally *A-connected* if and only if for every function $\bar{f}: M \to A$ there exists an A-flow $f \in F_0(M, A)$, such that for every $e \in M$, $f(e) \neq \bar{f}(e)$.

It is straightforward to generalize the relevant definitions (that of the k-closure for example) and then to state and prove the extensions for regular matroids of Lemma 2.1, Propositions 2.3, 2.4, as well as Theorem 3.1. A restricted version of Theorem 3.2 can also be proved, referring to regular matroids which contain disjoint subsets T and R such that $\mathrm{cl}_1(T) = \mathrm{cl}_2(R) = M$.

Let us pay now some attention to the case where M is a cographic matroid, $M = M^*(G)$. In that case, A-connectivity of M is equivalent to the following property of G, defined in terms of vertex coloring:

Definition. Let $G = (V, E)$ be a graph and A a non-trivial Abelian group. Then G is *A-colorable* if and only if for every $\bar{f} \in F(G, A)$ there exists an "A-coloring" $c: V \to A$ such that for every $e = (x, y) \in E$ (assumed to be directed from x to y), $c(x) - c(y) \neq \bar{f}(e)$.

Clearly an A-colorable graph is $|A|$-colorable (take $\bar{f} = 0$) and A-colorability is the dual of local A-connectivity, in the same way that k-colorability is the dual of admitting a k-NZF.

Let the dual of the k-closure operator (that is the k-closure in the cocycle matroid $M^*(G)$ of a graph G) be referred to as the k^*-closure. We observe that a graph G is the k^*-closure of a subgraph H if and only if every non-empty subgraph of $G - E(H)$ contains an edge-cut of cardinality at most k. (This is the dual form of the following easy observation: G is the k-closure of H if and only if, for every proper subgraph $\bar{H}$ which contains H, the graph obtained from G by the contraction of $\bar{H}$, has a circuit of cardinality at most k). In particular, G is the k^*-closure of the empty set if and only if every non-empty subgraph of G contains an edge-cut of cardinality at most k.

In light of the last observation, Proposition 2.4, when applied to cographic matroids, takes the following form:

Proposition 4.1. *Let G have an A-colorable subgraph H such that every non-empty subgraph of $G - H$ contains an edge-cut of cardinality less than $|A|$. Then G is A-colorable.*

A direct consequence of Proposition 4.1 is the well known fact (e.g., [2, p. 221]) that if every non-empty subgraph of G has a vertex of degree less than k then G is k-colorable. In fact such a graph is also A-colorable for every A of order at least k. Another immediate implication of Proposition 4.1 is the following:

Proposition 4.2. *Every simple planar graph is A-colorable for every Abelian group of order ≥ 6.*

This last proposition leads to asking whether 6 can be improved to 5 or 4. Note that Proposition 4.2 also follows by duality from Theorem 3.2. Similarly the replacement of 6 by 5 in Proposition 4.2. follows from Conjecture 5 and the replacement of 6 by 4 (which obviously would imply the Four Color Theorem) is allowed for triangle-free graphs by Theorem 3.1.

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The Bounded Chromatic Number for Graphs of Genus g

KENNETH A. BERMAN AND JEROME L. PAUL

Department of Computer Science, University of Cincinnati, Cincinnati, Ohio 45221-0008

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For $\mathscr{G}$ a collection of finite graphs, the *bounded chromatic number* $\chi_B(\mathscr{G})$ is the smallest number of colors c for which there exists an integer N such that every graph $G \in \mathscr{G}$ can be vertex c-colored without forcing more than N monochromatic edges. The *bounded (simple) path chromatic number* $\chi_P(\mathscr{G})$ ($\chi_{SP}(\mathscr{G})$) is the smallest number of colors c for which there exists an integer N such that every graph $G \in \mathscr{G}$ can be c-colored without forcing a monochromatic (simple) path of length more than N. For the set $\mathscr{S}_g$ of all graphs of genus g it is known that $4 \leqslant \chi_B(\mathscr{S}_g) \leqslant 6$, and $\chi_{SP}(\mathscr{S}_g) = 4$. In this paper we show that $\chi_B(\mathscr{S}_g) \leqslant 5$, and $\chi_P(\mathscr{S}_g) = 4$. For $g \geqslant 1$, let $\mu_5(g)$ ($\pi_4(g)$) denote the smallest integer x such that every graph $G \in \mathscr{S}_g$ can be 5-colored without forcing more than x monochromatic edges (4-colored without forcing a monochromatic path of length more than x). We also show that $2g \leqslant \mu_5(g) \leqslant 74g - 36$ and $(3/8)g - (1/2)\sqrt{3g} + 3/32 \leqslant \pi_4(g) \leqslant 224g - 106$.

1. INTRODUCTION

Let G be a graph with vertex set V and edge set E. By a vertex *c-coloring* of G we mean any partition of V into c classes. An edge is *monochromatic* if both of its end vertices are colored the same. A *path P of length p* joining vertices u and v is an alternating sequence of $p+1$ vertices and p edges $u_0 e_1 u_1 e_2 \cdots e_p u_p$ such that $u = u_0$, $v = u_p$, and such that e_i joins u_{i-1} and u_i, $i = 1, 2, ..., p$. We denote the length of P by $\lambda(P)$, i.e., $\lambda(P) = p$. Vertices in a path can be repeated but edges must be distinct. If $u = v$ the path is called *closed.* We call the path *simple* if the vertices in the path are all distinct. A *simple closed* path, called a *circuit,* is defined similarly. A *monochromatic* path is a path all of whose edges are monochromatic.

The genus of G, denoted by $\gamma(G)$, is defined to be the smallest genus of all surfaces (compact orientable 2-manifolds) on which G can be embedded. For g a non-negative integer, let $\mathscr{S}_g$ denote the set of all graphs of genus g.

Now consider a collection $\mathscr{G}$ of (finite) graphs. The *bounded chromatic*