

and there exists a G such that

$$d(G) < \sqrt{n} + 1.$$

The value \sqrt{n} is also conjectured to be the correct order of magnitude for a lower bound on $d(G)$. Denote $I(G) = \max(d(G), d(Q_n - G))$.

Let $f: C^n \rightarrow \{+1, -1\}$ be a boolean function. The *sensitivity* of f at x , denoted by $s(f, x)$, is the number of neighbors y of x for which $f(x) \neq f(y)$. The sensitivity of f is

$$s(f) = \max_{x \in C^n} s(f, x)$$

The sensitivity of f is sometimes called the *critical complexity* of f .

In theoretical computer science, much effort has been expended in the definition of various measures of complexity of boolean functions. Some are derived from an underlying computational model, such as *decision tree depth*. Here the function is computed by repeatedly reading input bits, until the function can be determined from the bits accessed. The *cost* of an algorithm is the number of bits read on the worst case input, and the complexity of a function is the cost of the best algorithm for this function. A similar measure is the *certificate complexity*. A 1-certificate (0-certificate) for f is an assignment to some subset of the variables that forces the value of f to 1 (0). The certificate complexity of f on x , denoted $C(f, x)$, is the size of the smallest certificate that agrees with x . The certificate complexity of f is

$$C(f) = \max_{x \in C^n} C(f, x).$$

Other measures of complexity are of a combinatorial nature, e.g., sensitivity. A related measure is *block-sensitivity*, defined: Denote $[n] = \{1, \dots, n\}$ and let $R \subset [n]$. If x is the vector (x_1, \dots, x_n) , then $x^{(R)}$ is defined as the vector with coordinates:

$$x_i^{(R)} = \begin{cases} x_i, & i \notin R \\ -x_i, & i \in R. \end{cases}$$

The block sensitivity of f at x , denoted $bs(f, x)$, is the largest number t such that there exist t disjoint sets R_1, \dots, R_t , such that for all $1 \leq i \leq t$, $R_i \subset [n]$, and $f(x) \neq f(x^{(R_i)})$. The block-sensitivity of f is

$$bs(f) = \max_{x \in C^n} bs(f, x).$$

A central activity in this field is determining the relation between various

Note

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Denote by Q_n the graph of the hypercube $C^n = \{+1, -1\}^n$. The following two seemingly unrelated questions are equivalent: 1. Let G be an induced subgraph of Q_n such that $|V(G)| \neq 2^{n-1}$. Denote $d(G) = \max_{x \in V(G)} \deg_G(x)$ and $I(G) = \max(d(G), d(Q_n - G))$. Can $I(G)$ be bounded from below by a function of n ? 2. Let $f: C^n \rightarrow \{+1, -1\}$ be a boolean function. The sensitivity of f at x , denoted $s(f, x)$, is the number of neighbors y of x in Q_n such that $f(x) \neq f(y)$. The sensitivity of f is $s(f) = \max_{x \in C^n} s(f, x)$. Denote by $d(f)$ the degree of the unique representation of f as a real multilinear polynomial on C^n . Can $d(f)$ be bounded from above by a function of $s(f)$? © 1992 Academic Press, Inc.

1. PRELIMINARIES

Denote by Q_n the graph on the n -dimensional cube $C^n = \{+1, -1\}^n$, where any two vertices are adjacent iff they differ in exactly one component. For an induced subgraph G of Q_n , denote the *maximal degree* of G by $d(G)$, i.e.,

$$d(G) = \max_{x \in V(G)} \deg_G(x).$$

In [1], it was shown that if G contains more than 2^{n-1} vertices, then

$$d(G) > \frac{1}{2}(\log n - \log \log n + 1)$$

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measures. The measures of complexity s_1 and s_2 are equivalent if they are polynomially related; i.e., there exist polynomials $p_1(x)$ and $p_2(x)$ such that

$$\forall f, \quad s_1(f) \leq p_2(s_2(f)), \quad s_2(f) \leq p_1(s_1(f)).$$

Nisan [3] showed that decision tree depth, certificate complexity, and block-sensitivity are equivalent. Nisan first considered the more natural measure of sensitivity (which is block-sensitivity restricted to singletons), but was unable to prove equivalence to decision tree depth and certificate complexity. However, only after introducing block-sensitivity was equivalence obtained.

Yet another complexity measure is obtained from the unique representation of the boolean function f as a real multilinear polynomial over the cube:

$$f(x) = \sum_{I \subseteq [n]} \left[\alpha_I \prod_{i \in I} x_i \right].$$

The coefficient α_I (which satisfies $-1 \leq \alpha_I \leq 1$ for all $I \subseteq [n]$) is also called $\hat{f}(I)$, the Fourier transform of f at I . Denote by $d(f)$ the degree of this polynomial, i.e.,

$$d(f) = \max_{I \subseteq [n]} \{|I| : \alpha_I \neq 0\}.$$

Nisan and Szegedy [4] show that $d(f)$ is also equivalent to the three complexity measures mentioned above. As for the relation between sensitivity and degree, Szegedy [6] showed that

$$d(f) \geq \sqrt{s(f)}.$$

This can easily be shown to be tight. Whether $s(f)$ is also equivalent to all of the above is still unknown. In particular, an upper bound on $d(f)$ in terms of $s(f)$ is sought and is conjectured to be $s^2(f)$. Such a bound would mean that sensitivity is equivalent to all the previously mentioned quantities. In the next section we show that this upper bound is equivalent to a lower bound on $T(\cdot)$.

2. THE EQUIVALENCE THEOREM

THEOREM 2.1. The following are equivalent for any function $h: N \rightarrow R$:

1. For any induced subgraph G of Q_n such that $|V(G)| \neq 2^{n-1}$, $T(G) \geq h(n)$.
2. For any boolean function f , $d(f) < h^{-1}(s(f))$.

Proof. We first transform 1 into a statement concerning boolean functions: Associate with the subgraph G a boolean function g such that $g(x) = 1$ iff $x \in V(G)$. Note that $\deg_G(x) = n - s(g, x)$ for $x \in V(G)$ and the same holds in $Q_n - G$ for $x \notin V(G)$. Denote by $E(g)$ the average value of g on C^n . Now 1 and 2 are clearly equivalent to the following:

- 1'. For any boolean function g , $E(g) \neq 0$ implies $\exists x: s(g, x) \leq n - h(n)$.
- 2'. For any boolean function f , $s(f) < h(n)$ implies $d(f) < n$.

To see the equivalence of 1' and 2', define

$$g(x) = f(x) p(x),$$

where $p(x)$ is the parity function of x : $p(x) = \prod_{i=1}^n x_i$. Note that for all $x \in C^n$, $s(g, x) = n - s(f, x)$ and for all $I \subseteq [n]$, $\hat{g}(I) = \hat{f}([n] - I)$, therefore $E(g) = \hat{g}(\emptyset) = \hat{f}([n])$, where $\hat{f}([n])$ is the Fourier transform of f at $[n]$, i.e., the highest order coefficient in the representation of f as a polynomial. $1' \rightarrow 2'$. Assume that $d(f) = n$, i.e., $\hat{f}([n]) \neq 0$. This is equivalent to $E(g) \neq 0$. By 1', $\exists x: s(g, x) \leq n - h(n)$; therefore $\exists x: s(f, x) \geq h(n)$, contradicting the premise.

$2' \rightarrow 1'$. Assume that $\forall x, s(g, x) > n - h(n)$. This implies that $s(f) < h(n)$. By 2', $d(f) < n$, which is equivalent to $\hat{f}([n]) = \hat{g}(\emptyset) = E(g) = 0$, contradicting the premise. ■

3. CONCLUSION

Substituting $h(x) = \sqrt{x}$ in Theorem 2.1 shows that the two bounds $d(G) \geq \sqrt{n}$ and $d(f) < s^2(f)$ are equivalent. The example from [1] which shows that there exists G such that $d(G) \leq \sqrt{n} + 1$ can be used to show that the upper bound on $d(f)$ would be tight if it were true. All of this means that a proof of $d(G) \geq \sqrt{n}$ would imply that boolean function sensitivity is equivalent to all other complexity measures mentioned in [3].

The sensitivity complexity measure $s(f)$ is especially important, since it also lower bounds $T(f)$ —the time needed by a parallel RAM to compute f (a parallel RAM is a collection of synchronous parallel processors sharing a global memory with no write-conflicts allowed). Cook and Dwork [2] have shown that $T(f) \geq \log s(f)$. In fact, Nisan [3] later improved this to $T(f) \geq \log bs(f)$ (this is a stronger inequality, since for any f , $bs(f) \geq s(f)$). Simon [5] has also shown that a n -variable boolean function which depends on all its variables must have sensitivity at least $\Omega(\log n)$.

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