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Note

The Equivalence of Two Problems on the Cube

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Denote by Q_n the graph of the hypercube $C^n = \{+1, -1\}^n$. The following two seemingly unrelated questions are equivalent: 1. Let G be an induced subgraph of Q_n such that $|V(G)| \neq 2^{n-1}$. Denote $\Delta(G) = \max_{x \in V(G)} \deg_G(x)$ and $\Gamma(G) = \max(\Delta(G), \Delta(Q_n - G))$. Can $\Gamma(G)$ be bounded from below by a function of n?; 2. Let $f: C^n \to \{+1, -1\}$ be a boolean function. The *sensitivity* of f at x, denoted $s(f, x)$, is the number of neighbors y of x in Q_n such that $f(x) \neq f(y)$. The sensitivity of f is $s(f) = \max_{x \in C^n} s(f, x)$. Denote by $d(f)$ the degree of the unique representation of f as a real multilinear polynomial on C^n. Can $d(f)$ be bounded from above by a function of $s(f)$?

1. PRELIMINARIES

Denote by Q_n the graph on the n-dimensional cube $C^n = \{+1, -1\}^n$, where any two vertices are adjacent iff they differ in exactly one component. For an induced subgraph G of Q_n, denote the *maximal degree* of G by $\Delta(G)$, i.e.,

$$\Delta(G) = \max_{x \in V(G)} \deg_G(x).$$

In [1], it was shown that if G contains more than 2^{n-1} vertices, then

$$\Delta(G) > \tfrac{1}{2}(\log n - \log\log n + 1)$$

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and there exists a G such that

$$\Delta(G) < \sqrt{n} + 1.$$

The value $\sqrt{n}$ is also conjectured to be the correct order of magnitude for a lower bound on $\Delta(G)$. Denote $\Gamma(G) = \max(\Delta(G), \Delta(Q_n - G))$.

Let $f: C^n \to \{+1, -1\}$ be a boolean function. The *sensitivity* of f at x, denoted by $s(f, x)$, is the number of neighbors y of x for which $f(x) \neq f(y)$. The sensitivity of f is

$$s(f) = \max_{x \in C^n} s(f, x)$$

The sensitivity of f is sometimes called the *critical complexity* of f.

In theoretical computer science, much effort has been expended in the definition of various measures of complexity of boolean functions. Some are derived from an underlying computational model, such as *decision tree depth*. Here the function is computed by repeatedly reading input bits, until the function can be determined from the bits accessed. The *cost* of an algorithm is the number of bits read on the worst case input, and the complexity of a function is the cost of the best algorithm for this function. A similar measure is the *certificate complexity*. A 1-certificate (0-certificate) for f is an assignment to some subset of the variables that forces the value of f to 1 (0). The certificate complexity of f on x, denoted $C(f, x)$, is the size of the smallest certificate that agrees with x. The certificate complexity of f is

$$C(f) = \max_{x \in C^n} C(f, x).$$

Other measures of complexity are of a combinatorial nature, e.g., sensitivity. A related measure is *block-sensitivity*, defined: Denote $[n] = \{1, ..., n\}$ and let $R \subset [n]$. If x is the vector $(x_1, ..., x_n)$, then $x^{(R)}$ is defined as the vector with coordinates:

$$x_i^{(R)} = \begin{cases} x_i, & i \notin R \\ -x_i, & i \in R. \end{cases}$$

The block sensitivity of f at x, denoted $bs(f, x)$, is the largest number t such that there exist t *disjoint* sets $R_1, ..., R_t$ such that for all $1 \leqslant i \leqslant t$, $R_i \subset [n]$, and $f(x) \neq f(x^{(R_i)})$. The block-sensitivity of f is

$$bs(f) = \max_{x \in C^n} bs(f, x).$$

A central activity in this field is determining the relation between various