An approach to the girth problem in cubic graphs

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Abstract

We offer a new, gradual approach to the *largest girth problem for cubic graphs.* It is easily observed that the largest possible girth of all n-vertex cubic graphs is attained by a 2-*connected* graph $G = (V, E)$. By Petersen's graph theorem, E is the disjoint union of a 2-factor and a perfect matching M. We refer to the edges of M as *chords* and classify the cycles in G by their number of chords. We define $\gamma_k(n)$ to be the largest integer g such that every cubic n-vertex graph with a given perfect matching M has a cycle of length at most g with at most k chords. Here we determine this function up to small additive constant for $k = 1, 2$ and up to a small multiplicative constant for larger k.

1 Introduction

The *girth* of a graph G is the shortest length of a cycle in G. Our main concern here is with bounds on $g(n)$, the largest girth of a cubic n-vertex graph. Namely, we seek the best statement of the form "Every n-vertex cubic graphs must have a short cycle". An elementary existential argument [5] shows that $g(n) \geq \log_2 n$ for infinitely many values of n. The best lower bound that we currently have is $g(n) > \frac{4}{3}\log_2 n - 2$. It is attained by sextet graphs, an infinite family of explicitly constructed graphs, first introduced in [2], the girth of which was determined in [12]. On the other hand, the only upper bound that we have is the more-or-less trivial *Moore's Bound* which says that

$$(2 + o_n(1)) \log_2 n \geq g(n).$$

It is our belief that in fact a better upper bound applies

Conjecture 1.1. *There is a constant* $\epsilon_0 > 0$ *such that* $(2 - \epsilon_0) \log_2 n \geq g(n)$

Concretely, Moore's lower bound on n, the number of vertices in a d-regular graph of girth g is as follows. For odd $g = 2k + 1$ it is $n \geq 1 + d \sum_{i=0}^{k-1} (d - 1)^i$, for even $g = 2k$ it is $n \geq 1 + (d - 1)^{k-1} + d \sum_{i=0}^{k-2} (d - 1)^i$.

Moore Graphs which satisfy this with equality were fully characterized about a half century ago [4], [1]. There is an ongoing research effort to determine, or at least estimate $g(n)$, and we initiate here a new approach to this problem. More generally, the search is on for (d, g)-*cages.* These are d-regular graphs of girth g with the smallest possible number of vertices. The record of the best known bounds in this area is kept in [6].

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2 Statement of the problem

As the following observation shows, it suffices to consider 2-connected graphs. We give the simple proof for completeness sake, but actually more is known about the connectivity of cubic cages [11].

Proposition 2.1. *Among all cubic n-vertex graphs that have the largest possible girth $g(n)$, at least one graph is* 2*-connected.*

Proof. It is well-known and easy to show that a cubic graph is 2-connected iff it is bridgeless. So let the edge $xy \in E$ be a bridge in $G = (V, E)$, a cubic n-vertex graph. Let $x' \neq y$ be a neighbors of x, and $y' \neq x$ a neighbor of y. Let G' be obtained from G be deleting the edges xx' and yy' and adding the new edges xy' and yx'. It is easily verified that G' has fewer bridges than G, while $\text{girth}(G') \geq \text{girth}(G)$. □

We next recall Petersen's theorem [10]: The edge set of every 2-connected cubic graph G can be decomposed into a perfect matching M and a 2-factor C. We call this a *Petersen decomposition* of G, and refer to the edges of M as *chords.*

The following is the main focus of this paper:

Definition 2.2. $\gamma_k(n)$ *is the smallest integer g such every cubic n-vertex graph with any Petersen Decomposition has a cycle of length $\leq g$ with at most k chords.*

We start with some elementary observations concerning $\gamma_k(n)$.

- $\gamma_k(n)$ is a non-increasing function of k.
- In a very loose sense, for fixed k, the function $\gamma_k(n)$ is non-decreasing with n, although this is not literally correct. For example, the Tutte-Coxeter Graph, a well-known Moore graph shows that $\gamma(30) = 8$. Also, clearly, $\gamma_3(30) \geq \gamma(30)$. On the other hand, extensive computer searches show that $\gamma_3(32) = 7$.
- $\gamma_0(n) = n$.
- $\gamma_1(n) = \lfloor \frac{n}{2} \rfloor + 1$. The unique extremal example is a $(2g-2)$-cycle C and a matching M that matches every antipodal pair of vertices in C.
- A lower bound on $\gamma_k(n)$ amounts to a construction of a cubic graph along with a Petersen Decomposition, where every short cycle has more than k chords.
- Upper bounds on $\gamma_k(n)$ are proofs that every cubic graph on n vertices with a given Petersen Decomposition must have a short cycle with few chords.
- For $k \geq \log_2 n$ there holds $\gamma_k(n) = g(n)$.

2.1 Paper Organization

The rest of the paper is organized as follows. In Section 3 we state our upper and lower bounds on $\gamma_k(n)$. In Sections 4 and 5 we prove the upper resp. lower bounds. In Section 6 we make some connection with the girth problem in its general form.

3 Our results

In this paper we present an upper and lower bounds on cycles with at most k chords

Theorem 3.1. *Let n be an even number, then*

1. $\sqrt{2n} - \frac{5}{2} < \gamma_2(n) \leq \sqrt{2n} + 2$. *In fact,* $\sqrt{2n} + \frac{1}{2} < \gamma_2(n)$ *holds for infinitely many n's.*

2. $$\frac{1}{2}(2n + \frac{9}{4})^{\frac{1}{2}} + \frac{5}{4} \leq \gamma_3(n) \leq (2n)^{\frac{1}{2}} + 1$$

3. $$(2n)^{1/3} + O(1) \leq \gamma_5(n) \leq \gamma_4(n) \leq \frac{3}{2}(2n)^{1/3}$$

4. $$2(\frac{n}{4})^{1/4} + O(1) \leq \gamma_7(n) \quad \textit{and} \quad 2(\frac{n}{4})^{1/6} + O(1) \leq \gamma_{11}(n)$$

5. $$\gamma_{2l}(n) \leq 3l + \frac{1}{2}(l + 1 + \ln 2)(n^{\frac{1}{l+1}})$$

   for every $l \geq 3$.

6. $$n^{\frac{4}{3q}} + O(1) \leq \gamma_q(n)$$

   for every prime power q.

4 Upper bounds - proofs

We prove first the upper bounds in Items 1 and 2, starting with some necessary preparations. Let $\alpha_1, \ldots, \alpha_t$ be a sequence of $t \geq 2$ distinct positive integers. The corresponding *cycle* R is comprised of the sequence $(\alpha_1, \ldots, \alpha_t)$ and its cyclic shifts $(\alpha_j, \ldots, \alpha_t, \alpha_1, \ldots, \alpha_{j-1})$, for every $j = 2, \ldots, t$. We say, for $i = 1, \ldots, t-1$ that α_i, α_{i+1} are *adjacent* in R. Also α_t, α_1 are adjacent in R. We denote the set of (unordered) adjacent pairs in R by $A(R)$. Finally we define $w(R) := \sum_{\alpha,\beta \in A(R)} |\alpha - \beta|$.

Proposition 4.1. *Let $\Pi = \{R_1 \ldots R_l\}$ be a partition of $[k]$ into cycles for some even integer k. Then $\sum_i w(R_i) \leq \frac{k^2}{2}$. The bound is tight.*

Proof. We say that a cycle R *covers* the interval $[i, i+1]$ if $i, i+1$ lie between α and β for some adjacent pair $\alpha, \beta \in A(R)$ (allowing $\{i, i+1\}$ and $\{\alpha, \beta\}$ to have non-empty intersection). Let G_i be the number of cycles in Π that cover the interval $[i, i+1]$. Clearly

$$\sum_j w(R_j) = \sum |G_i|.$$

We turn to bound $\sum |G_i|$. For $i < \frac{k}{2}$, the adjacent pair α, β can cover the interval $[i, i+1]$ only if $\min(\alpha, \beta) \leq i$. Every index smaller than or equal to i can contribute a total of 2 to $|G_i|$, once moving left and once moving right along the cycle. Consequently $|G_i| \leq 2i$. By symmetry, the same argument yields $|G_i| \leq 2(k-i)$ when $i > \frac{k}{2}$, and finally $|G_{\frac{k}{2}}| \leq k$. We sum these inequalities and conclude that

$$\sum_{i=1}^{l} |G_i| \leq \frac{k(k-2)}{4} + \frac{k(k-2)}{4} + k = \frac{k^2}{2}.$$

Equality is attained e.g., for the partition $R_1 \dots R_{\frac{k}{2}}$ into $\frac{k}{2}$ cycles with $R_i = \{i, i+\frac{k}{2}\}$. □

The following lemma proves the upper bound on $\gamma_2(n)$ in Theorem 1.

Lemma 4.2. *Let $G = (V, E)$ be an n-vertex 2-connected cubic graph with Petersen decomposition $E = M \sqcup C$. Then G has a cycle of length $\leq \sqrt{2n} + 2$ with at most 2 chords.*

Proof. Let us mention that actually there always exists a cycle of length $\leq \sqrt{2n} + 1$ with at most 2 chords. We omit the somewhat laborious details of this stronger argument, and turn to prove the theorem.

If $uv \in M$ is a chord, we write $v^* = u$. Let A be an arc of length $k \leq \sqrt{2n}$ along a cycle σ_0 in C, and let $A^* := \{v^* | v \in A\}$. If $A^* \cap A \neq \emptyset$ this yields a cycle of length $\leq \sqrt{2n} + 1$ with only one chord. So, we may and will assume that $A^* \cap A = \emptyset$.

Let $\sigma \neq \sigma_0$ be another cycle of C. If $|A^* \cap \sigma| = r \geq 2$, let us denote the vertices of $A^* \cap \sigma$ by $u_1^*, \dots, u_r^*$ in cyclic order and with indices taken mod r. For every $i = 1, \dots, r$ we define the following 2-chord cycles: It starts with the chord u_i, u_i^*; along σ to u_{i+1}^*; the chord u_{i+1}^*, u_{i+1}; finally along A to u_i.
When $r = 1$ we consider instead the chord-free cycle σ.
This construction applies as well to $\sigma = \sigma_0$, except that the above 2-chord cycles are defined only for $i = 1, \dots, r-1$.

We have mentioned in total either $k-1$ or k such 2-chord cycles depending on whether or not A^* intersects with σ_0.

We turn to bound the total length of those 2-chord cycles, starting with their parts that traverse A. The collection of 2-chord cycles that correspond to any cycle σ in C induces a cycle on A. As we go over all σ we obtain a partition of A into cycles. By Proposition 4.1 the sum total of their lengths does not exceed $\frac{k^2}{2}$. Other than their overlaps on A, the 2-chord cycles traverse every chord twice and are otherwise overlap-free.

Therefore their total length is at most $\leq n + \frac{k^2}{2} + k - 1$. Consequently, the average length of such cycle does not exceed

$$\frac{n + \frac{k^2}{2}}{k-1} + 1.$$

We optimize and take $k = \sqrt{2n+1} + 1$ to conclude that at least one of these 2-chord cycles with length $\leq \sqrt{2n} + 2$.

□

We prove next the upper bound on $\gamma_4(n)$.

Lemma 4.3. *Let $E = M \sqcup C$ be a Petersen decomposition of an n-vertex 2-connected cubic graph $G = (V, E)$. There is a cycle in G of length $\leq L = \lambda n^{\frac{1}{3}} + 3$ with at most four chords, where $\lambda = 3 \cdot 2^{-\frac{2}{3}} \simeq 1.89$.*

Proof. Arguing by contradiction, we assume that all cycles of C are longer than L. We measure distances along C. Thus, the r-*neighborhood* of $v \in V$, is the arc of $2r+1$ vertices centered at v in the cycle of C to which v belongs. The r-neighborhood of $S \subset V$ is the union of the r-neighborhoods of all vertices in S.

Let A_0 be an arc of length $k \leq \frac{2L}{3} - 2$ along a cycle of C. We may assume that any two vertices in A_0^* are at distance at least $\geq \frac{L}{3} - 1$, for otherwise we obtain a cycle of length $< L$ with only two chords. Let $r = \frac{L}{6} - \frac{1}{2}$. It follows that A_1, the r-neighborhood of A_0^* has cardinality $|A_1| = 2kr$. Of course $|A_1^*| = |A_1|$.

We say that $u, v \in A_1^*$ are *consecutive* if the shortest arc of C between them contains no additional vertices of A_1^*. We number the vertices of A_1^* from 1 to $2rk$ cyclically along C, proceeding from a vertex to the consecutive one. If $u \in A_1^*$ is the i-th vertex in this numbering, we associate with it the following 2-chords path P_i which starts at u and ends in A_0. This path begins with a hop from u to $u^* \in A_1$. Then comes a walk along C to the closest vertex in A_0^*. Call this closest vertex β_i and the distance traveled b_i. The path ends with a hop from β_i to $\beta_i^* \in A_0$.

Let $v \in A_1^*$ be the $(i+1)$-st in this order. Consider the following 4-chord cycle:

1. from u to v along C a distance of a_i.
2. along P_{i+1} to $\beta_{i+1}^* \in A_0$.
3. a walk through A_0 to β_i^* a distance of c_i.
4. from β_i^* we traverse P_i to u, in the direction opposite to the above description.

The length of this cycle is thus $a_i + b_i + c_i + b_{i+1} + 4$. There are $|A_1^*| - 1 = 2kr - 1$ cycles in this list, since we exclude the case where u is the $2rk$-th vertex of A_1^* and v is the first. The total length of these cycles is

$$\sum_{i=1}^{2rk-1} (a_i + b_i + c_i + b_{i+1} + 4)$$

We turn to bound the different parts of this sum. The stretches of a_i steps are disjoint from A_0, A_1, and A_0^* hence $\sum_{i=1}^{2rk-1} a_i \leq n - |A_0| - |A_0^*| - |A_1| = n - 2k - 2kr$. Consider next the stretches of walks that go from a vertex in A_1 to the closest vertex in A_0^*. For every $\beta_i \in A_0^*$ the sum of the distances of all vertices in the r-neighborhood of β_i is r^2. It follows that $\sum_{i=1}^{2kr-1} b_i + \sum_{i=2}^{2kr} b_i \leq 2 \sum_{i=1}^{2kr} b_i = 2kr(r+1)$. Finally we turn to bound $\sum c_i$. We argue here in the same way that we did in the proof of Lemma 4.2, except that here the argument is carried out $2r$ times. Consequently, $\sum_{i=1}^{2rk-1} c_i \leq 2r(\frac{k^2}{2})$.

Thus the sum total of the cycles' lengths is at most $n - 2k - 2kr + 2k \cdot r(r+1) + rk^2 + 4(2kr)$. This gives the following upper bound on their average length $\leq \frac{n}{2rk-1} + r + \frac{k}{2} + 4$. We get the tightest upper bound on the length by letting $k = \sqrt{\frac{n}{r}}$ and $r = \frac{(2n)^{1/3}}{2}$. We conclude that there is at least one cycle of length $\leq 3(\frac{n}{4})^{\frac{1}{3}} + 4$. □

We finally present an upper bound on $\gamma_k(n)$ for every even k. This yields an upper bound also for the case of odd k, since $\gamma_k(n)$ is a decreasing function of k. The proof is just an adaptation of the upper bound proof for $\gamma_4(n)$.

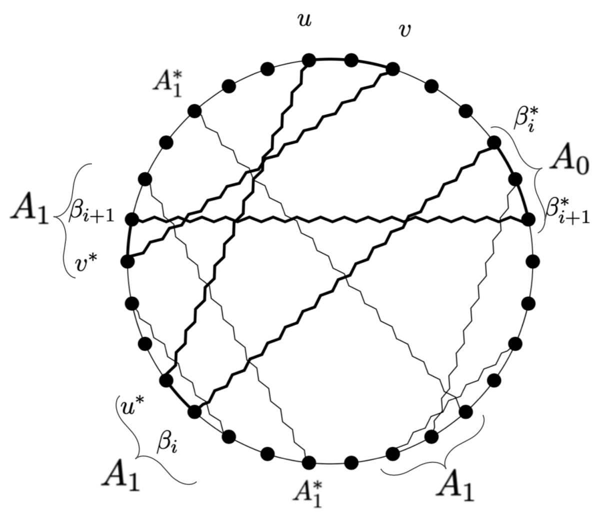

Figure 1: An illustration of the expansion of A_0. Zigzag edges represent chords.

Lemma 4.4. *Let $E = M \sqcup C$ be a Petersen decomposition of an n-vertex 2-connected cubic graph $G = (V, E)$. There is a cycle in G of length $\leq 3l + (l-1)(2^{-\frac{l}{l+1}} n^{\frac{1}{l+1}}) + n^{\frac{1}{l+1}} 2^{\frac{1}{l+1}}$ with at most $2l$ chords.*

Proof. We only sketch the proof, since it is similar to the proof of lemma 4.3. Let A_0 be an arc of length $k < \frac{1}{2}(l+1)(n^{\frac{1}{l+1}})$. For every $0 < i \leq l-1$, we let A_i be the r-neighborhood of A_{i-1}^*. It follows that A_i has cardinality $|A_i| = (2r)^i k$. We number the vertices of A_{l-1}^* from 1 to $(2r)^{l-1}k$ cyclically along C. Let $u \in A_{l-1}^*$ be the i-th vertex in this numbering. We associate with it the following l-chords path P_i which starts at u and ends in A_0. It begins with a hop from u to $u^* \in A_{l-1}$ then a walk along C to the closest vertex in A_{l-2}^* call it β_{l-2} a distance of b_{l-2}^i then a hop to β_{l-2}^* and in general from vertex $\beta_j^* \in A_{j-1}$ a walk along C to the closest a distance b_i^j to a vertex $\beta_{j-1} \in A_{j-2}^*$. Let $v \in A_{l-1}^*$ be the $(i+1)$-st in this order and consider the following $2l$-chord cycle:

1. from u to v along C a distance a_i.
2. along P_{i+1} to $\beta^{i+1*} \in A_0$.
3. a walk along A_0 to β_0^{i*} distance c_i.
4. from β_0^{i*} we traverse P_i to u in the opposite direction of the path.

The length of this cycle is thus $a_i + \sum_{j=1}^{l-1} b_i^j + b_{i+1}^j + c_i + 2l$. There are $|A_{l-1}|^* - 1 = (2r)^{l-1}k - 1$ cycles to consider. The total length is $\sum_{i=1}^{(2r)^{l-1}k-1}(a_i + c_i + \sum_{j=1}^{l-1} b_i^j + b_{i+1}^j + 2l)$, we bound all the steps as we did in proving Lemma 4.3. This yields that $\frac{n}{(2r)^{l-1}k-1} + (l-1)r + \frac{k}{2} + 3l - 2$ is an upper bound on their average length, and this for any two integers $r, k \geq 1$. The optimal choice is $k = \sqrt{\frac{2n}{(2r)^{l-1}}}$ and $r = \frac{(2n)^{\frac{1}{l+1}}}{2}$. With this choice we conclude that there is at least one cycle of length $\leq 3l + (l-1)(2^{-\frac{l}{l+1}} n^{\frac{1}{l+1}}) + n^{\frac{1}{l+1}} 2^{\frac{1}{l+1}}$ □

5 Lower bounds

We prove the lower bound on $\gamma_2(n)$ by providing an explicit construction. In the Petersen Decomposition of our graph, C is a Hamilton cycle.

Lemma 5.1. *$\gamma_2(n) \geq \sqrt{(2n + \frac{9}{4})} + \frac{1}{2}$ whenever $n = 8l^2 + 6l$ with l a positive integer. Namely, we construct a cubic n-vertex graph $G = (V, E)$ with a Petersen Partition $E = M \sqcup C$, where C is a Hamilton cycle and where the shortest cycle with 2 chords or less has length $4l + 2$.*

Proof. All the indices mentioned below are taken mod n. The vertices of C are numbered $0, \ldots, n-1$ in this order. Note that n is divisible by $2l$. We divide C into $n/2l$ equal-length *blocks* and construct a graph that is invariant under rotation by $2l$. The first block is comprised of vertices $0, \ldots, 2l-1$. Let $A = (a_0, a_1 \ldots a_{l-1})$ be a sequence of l odd integers. For every $l > j \geq 0$, we introduce the chord $(2j, 2j + a_j)$, and, as mentioned, we rotate these chords with steps of $2l$. Stated differently, we (uniquely) express every even integer x in the range $0, \ldots, n-1$ as $x = 2lk + 2j$, where $0 \leq k < \frac{n}{2l}$, $0 \leq j < l$ and connect vertex x by a chord to $x + a_j$. Since the integers a_i are all odd, every chord connects a vertex of even index to one of odd index. Also, every even vertex is incident with exactly one chord. In order for G to be cubic, also every odd vertex must be incident with a single chord. Namely, we need to choose the a_i so that

$$2lk + 2j + a_j = 2lr + 2s + a_s \Rightarrow k = r, j = s \text{ for every } 0 \leq k, r < \frac{n}{2l},\ 0 \leq j, s < l. \quad (1)$$

In order that every cycle with exactly one chord has length $\geq g = 4l + 2$ we must satisfy

$$n - g + 1 \geq a_j \geq g - 1 \text{ for every } 0 \leq j \leq l - 1 \quad (2)$$

We choose the parameters so that $a_i \equiv a_j \bmod 2l$ for all $i, j = 0, \ldots, l-1$. This yields Condition (1) by considering the equations mod $2l$. Concretely, we let A be the arithmetic progression $a_j = 4l(j+1) + 1$. Condition (2) is easily seen to hold.

Consider the shortest 2-chord cycle with chords

$$(2kl + 2j, 2kl + 2j + a_j) \text{ and } (2rl + 2s, 2rl + 2s + a_s).$$

We need to show that its length is $\geq g$. There are two cases to consider, as the arc of this cycle which starts at $2kl + 2j$ can end at either $2rl + 2s$ or $2rl + 2s + a_s$. Let us introduce the notation $\|x\| := \min\{x, n - x\}$ for an integer $n - 1 \geq x \geq 0$. We extend the definition to all integers x by first taking the residue of $x \bmod n$. Using this terminology, we need to show that if $(k, j) \neq (r, s)$, then

$$\|2kl + 2j - (2rl + 2s)\| + \|2kl + 2j + a_j - (2rl + 2s + a_s)\| \geq g - 2$$

and

$$\|2kl + 2j - (2rl + 2s + a_s)\| + \|2kl + 2j + a_j - (2rl + 2s)\| \geq g - 2.$$

Due to the cyclic symmetry of our construction no generality is lost if we assume that $r = 0$ and $0 \leq k < 2l + 2$. We also spell out the values of a_j, a_s and g and now we have to show that if $k \neq 0$ or $j \neq s$, then

$$\|2kl + 2(j - s)\| + \|2kl + (4l - 2)(j - s)\| \geq 4l \quad (3)$$

and

$$\|2l(k-2s-2)+2(j-s)-1\| + \|2l(k+2j+2)+2(j-s)+1\| \geq 4l. \tag{4}$$

The inequality $\|x\| \geq y$ for an integer $n > x \geq 0$ and a positive integer y is equivalent to the conjunction of the inequalities $x \geq y$ and $n - x \geq y$. For negative arguments we use the fact that $\|-x\| = \|x\|$, then apply the above. Let us refer to the left hand side of Equation (3) as $\|A_1\| + \|A_2\|$ and to Equation (4) as $\|B_1\| + \|B_2\|$. Note that $n - B_1 \geq n - A_1 \geq 4l$. It suffices therefore to prove that $|A_1| + \|A_2\| \geq 4l$ and $|B_1| + \|B_2\| \geq 4l$.

We start with the case $k = 0$, which implies $j \neq s$. In Equation (3) this yields $|A_1| \geq 2$ and $n - 4l > 4l^2 - 6l + 2 \geq |A_2| \geq 4l - 2$ hence $|A_1| + \|A_2\| \geq 4l$. In Equation (4) when $k = 0$ either $j \geq 1$ and then $n - 4l > 4l^2 + 2l - 1 > |B_2| \geq 4l$ or $s \geq 1$ and then $|B_1| \geq 4l$ hence $|B_1| + \|B_2\| \geq 4l$.

We turn our attention to Equation (3) with $k \geq 1$. It is easy to see that $A_1 > 4l$ when $k \geq 3$. On the other hand, $A_2 > n - 4l$ when $k \leq 2$. It remains to show that $|A_1| + |A_2| \geq 4l$ when $k \in \{1, 2\}$. This is done in the following case analysis.

- There holds $A_1 < 0$ exactly when $s > kl + j$. But then $A_2 < 0$ as well and we get $|A_1| + |A_2| = -A_1 - A_2 = 4l(s - k - j)$. Clearly $s > kl + j \geq k + j$ so that $|A_1| + |A_2| \geq 4l$ as claimed.

- When $A_1 > 0 > A_2$, there holds $|A_1| + |A_2| = A_1 - A_2 = (4l - 4)(s - j)$. We are in this range exactly when when $\frac{kl}{2l-1} < s - j < kl$, in this case we would compute and $A_1 - A_2 \geq 4l$, because $j + 1 < s$, and $k \geq 1$.

- In the last remaining case $A_1, A_2 > 0$. Here $s < \frac{kl}{2l-1} + j$ and $A_1 + A_2 = 4l(k + j - s) > 4l$.

Inequality (3) follows.

We proceed to Equation (4). Note that $B_2 > 0$. Also, $B_2 \leq 4l$ only when $k = j = 0$ in this case $|B_1| = |-4ls - 4l - 2s - 1| > 4l$. Hence we are left to consider the case where $\|B_2\| = n - B_2$, the only case when $n - B_2 < 4l$ is when $k = 2l + 1$ and $j = l - 1$ but then $|B_1| > 4l$. Inequality (4) follows. With this we can conclude that the construction is valid.

□

Remark 5.2. *For $l = 1$, this construction yields the Heawood graph which is a Moore graph.*

We proceed to bound $\gamma_3(n)$ by giving an appropriate explicit construction.

The main idea of the construction is to transform the graph from Lemma 5.1 into a "bipartite" version. This relies on the fact that in the original construction n is even and every chord connects a vertex of even index to one with an odd index. In this construction no cycle has exactly 3 chords, while the length of every 2-chord cycle in the original graph is only cut in half at worst.

Lemma 5.3. *$\gamma_3(n) \geq \frac{1}{2}\sqrt{(2n + \frac{9}{4})} + \frac{5}{4}$ for every $n = 8l^2 + 6l$ with l a positive integer. Namely, we construct a cubic n-vertex graph $G = (V, E)$ with a Petersen Partition, where the shortest cycle with 3 chords or less has length $2l + 2$.*

Proof. The graph that we construct has the same vertex set V and the same set of chords M as in the graph G of Lemma 5.1. What changes is the 2-factor C of the Petersen Partition. In the original construction C was a Hamilton Cycle that traverses the vertices in order. In the present construction C is the disjoint union of two cycles, C_{even} and C_{odd} that traverse all the even-indexed resp. odd vertices in order.

It is clear that H is cubic, since every vertex is incident to exactly one chord and has two neighbors in the 2-factor in which it resides. As shown in Lemma 5.1, every chord connects an even-indexed vertex to an odd one. It follows that every cycle in H has an even number of chords. Therefore we only need to consider cycles with 2 chords. Also, the only chordless cycles are C_{odd}, C_{even} whose length is $\frac{n}{2} \gg 2l+2$.

Finally we come to cycles $\mathcal{C}$ with exactly 2 chords in H. Such a cycle is composed of an arc from v_1 to v_2 in C_{odd}, a chord v_2v_3 to $v_3 \in C_{even}$, an arc from v_3 to v_4 in C_{even} and a chord v_4v_1 back to C_{odd}. We can consider the cycle $\tilde{\mathcal{C}}$ in G that traverses C from v_1 to v_2, takes the chord v_2v_3, then traverses C from v_3 to v_4 and finally the chord v_4v_1. This is a 2-chord cycle in G. and length$(\tilde{\mathcal{C}}) = 2 \cdot$ length$(\mathcal{C}) - 2$. As Lemma 5.1 shows length$(\tilde{\mathcal{C}}) \geq 4l-2$. The conclusion follows. □

The lower bounds on $\gamma_4, \gamma_5, \gamma_7, \gamma_{11}$ are proved by an explicit construction as well. We start with a general construction method from which these claims easily follow

Lemma 5.4. *Given an n-vertex d-regular graph $H = (V', E')$ with girth m, there is an explicit $2nd$-vertex cubic graph $G = (V, E)$ with Petersen Partition $M \sqcup C$, where the shortest length of a cycle in G with at most $m-1$ chords is $2d$.*

Proof. To construct G, associate with every vertex v in H a $2d$-cycle $C^{(v)}$. The 2-factor C of G's Petersen's decomposition is the union of $C^{(v)}$ over all $v \in V'$. For every edge $v_1v_2 \in E'$ we introduce two edges: One between a vertex $v \in C^{(v_1)}$ and a vertex $u \in C^{(v_2)}$. The second edge is $u'v'$, where v' the antipode of v in $C^{(v_1)}$ and u' is the antipode of u in $C^{(v_2)}$. Since girth$(H) = m$, a cycle in G with fewer than m chords can be either one of the $2d$-cycles $C^{(v)}$ or it must include some pairs of antipodal vertices, as described above. However, in the latter case it must include both chords uv and $u'v'$, and hence its length must be at least $2d+2$. □

Corollary 5.5.

$$\gamma_4(n) \geq \gamma_5(n) \geq 2(\frac{n}{4})^{1/3} + O(1),$$

$$\gamma_7(n) \geq 2(\frac{n}{4})^{1/4} + O(1)$$

and

$$\gamma_{11}(n) \geq 2(\frac{n}{4})^{1/6} + O(1).$$

Proof. Let q be a prime power, these statements follows from three infinite families of Moore Graphs where $d = q+1$, and $g = 6, 8, 12$. All these constructions come from projective planes. For the case where $g = 6, d = q+1$ we recall the construction of H, the points vs. lines graph over the finite field of order q. This is a bipartite $(q+1)$-regular graph, whose two sides are called P and L for "points" and "lines", with $|P| = |L| = q^2 + q + 1$. The girth of H is 6. As for the cases where $g = 8, 12$, and $d = q+1$ these graphs are similarly incidence graphs of generalized Quadrangles and generalized Hexagons [3].

□

Corollary 5.6. *Let q be a prime power then*

$$\gamma_q(n) \geq (n)^{\frac{4}{3q}}$$

Proof. This is based on a family of graphs due to Lazebnik, Ustimenko and Woldar [7]. For q a prime power, they construct a q-regular graph $G(V,E)$ of order $2q^{k-t+1}$ and girth $g \geq k+5$. Here $k \geq 1$ is an odd integer, and $t = \lfloor \frac{k+2}{4} \rfloor$. Namely, G is a q-regular graph with n vertices and $g \geq \frac{4}{3}\log_q(q-1)\log_{q-1}(n)$. □

6 Something on the general girth problem

Some of our results have a bearing on the girth problem for d-regular graphs also for $d > 3$. As shown in [11] and [8], every (d,g)-cage is d-edge-connected, whence there is no loss in generality in considering only $(d-1)$-edge-connected d-regular graphs $G = (V,E)$. As shown in [9], every such graph has a 2-factor C. In this view, we ask again about short cycles with few chords in G, where a *chord* is an edge in $M := E \setminus C$.

Definition 6.1. *$\gamma_k^d(n)$ is the smallest integer g such every d-regular n-vertex graph with a given 2-factor C has a cycle of length $\leq g$ with at most k chords.*

We illustrate the connection by proving the following lemma.

Lemma 6.2. $\gamma_2^d(n) \leq \sqrt{\frac{2n}{d-2}}$.

Proof. Again, for any set of vertices $S \subseteq V$, we denote

$$S^* := \{v \in V | \text{ there is some } u \in S \text{ such that } uv \in M \text{ is a chord}\}.$$

Let A be an arc of length $k \leq \frac{2n}{d-2}$ then $|A^*| = (d-2)k$. Let $u_1, \ldots, u_{k(d-2)}$ be the vertices of A^* in cyclic order. For every $i = 1, \ldots, k(d-2)$ we define the following 2-chord cycle. It starts with the chord u_i, u_i^*, then proceeds along C to u_{i+1}^*, traverses the chord u_{i+1}, u_{i+1}^* and continues along C to u_i. There are $k(d-2)$ such cycles in total and as in the proof of Lemma 4.2, their total length does not exceed $\leq n + \frac{k^2}{2} \cdot (d-2)$. Consequently the average length of such cycle is bounded from above by

$$\frac{n}{k(d-2)} + \frac{k}{2}.$$

We take $k = \sqrt{\frac{2n}{d-2}}$ and conclude that at least one cycle has length $\leq \sqrt{\frac{2n}{d-2}}$.

□

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