Discrete Mathematics 64 (1987) 1–11
North-Holland

EXTREMAL PROBLEMS ON PERMUTATIONS UNDER CYCLIC EQUIVALENCE

P. ERDÖS
Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

N. LINIAL
Institute of Mathematics and Computer Science, Hebrew University, Jerusalem, Israel

S. MORAN
Computer Science Department, Technion, Israel Institute of Technology, Haifa, Israel

Received 10 December 1984
Revised 19 March 1986

How much can a permutation be simplified by means of cyclic rotations? For functions $f: S_n \to Z$ which give a measure of complexity to permutations we are interested in finding

$$F(n) = \max \min f(\sigma),$$

where the max is over $\sigma \in S_n$ and the min is over π which are cyclically equivalent to σ.

The measures of complexity considered are the number of inversions and the diameter of the permutation. The effect of allowing a reflection as well as rotations is also considered.

1. Introduction

Let $\sigma = (a_1, \ldots, a_n) \in S_n$ be a permutation and let $[\sigma] = \{(a_j, \ldots, a_n, a_1, \ldots, a_{j-1}), n \geq j \geq 1\}$ be the class of all cyclic permutations of σ. Also for $\pi = (b_1, \ldots, b_n) \in S_n$ denote by π^R the permutation $(b_n, \ldots, b_1) \in S_n$. We also denote by $\langle \sigma \rangle$ the set $[\sigma] \cup \{\tau^R \mid \tau \in [\sigma]\}$. For a real function $f: S_n \to R$, we consider $\bar{f}$ defined by

$$\bar{f}(\sigma) = \min\{f(\tau) \mid \tau \in [\sigma]\},$$

and $\tilde{f}$ given by

$$\tilde{f}(\sigma) = \min\{f(\tau) \mid \tau \in \langle \sigma \rangle\}.$$

Our interest in this article is in finding $\max\{\bar{f}(\sigma) \mid \sigma \in S_n\}$ and $\max\{\tilde{f}(\sigma) \mid \sigma \in S_n\}$, for certain functions f.

Here we deal with two instances of this general problem:

(1) $f(\sigma) = $ number of inversions in $\sigma = |\{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}|$;

(2) $f(\sigma) = \max\{|\sigma(i) - i| \mid i = 1, \ldots, n\}$.

Our interest in those problems was initiated by studies on the design of electrical circuits for parallel computations [1]. Of course, many other problems suggest themselves that we hope to investigate in the future.

2. Counting inversions

As stated in the introduction we investigate here the function

$$F(n) = \max \min I(\sigma),$$

where $I(\sigma)$ is the number of inversions in σ, the max is over $\pi \in S_n$ and the min over $\sigma \in [\pi]$.

Theorem 2.1.

$$0.304^{-}n^2 + \mathrm{O}(n) = \frac{8-\pi}{16}n^2 - \frac{3n}{2} \leqslant F(n) \leqslant \frac{n^2}{3} - \frac{3n-1}{6} = 0.333^{+}n^2 + \mathrm{O}(n).$$

Proof. Let us first remark that $F(n) = \mathrm{O}(n^2)$ is obvious. Since a permutation of S_n can have at most $\binom{n}{2}$ inversions, $F(n) \leqslant \binom{n}{2}$. Also for $\pi = (n, n-1, \ldots, 1) \in S_n$ it is easily verified that

$$\min_{\sigma \in [\pi]} I(\sigma) = \frac{n^2}{4} + \mathrm{O}(n).$$

The upper bound

Let $\sigma = (a_1, \ldots, a_n)$ and let $\tau_k = (a_k, \ldots, a_n, a_1, \ldots, a_{k-1})$, $n \geqslant k \geqslant 1$ be the permutations in $[\sigma]$. Define variables $x_{ij}^{(k)}$, $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n$ as follows:

$$x_{ij}^{(k)} = \begin{cases} 1 & \text{if } a_{k+i} > a_{k+j}, \\ 0 & \text{if } a_{k+i} < a_{k+j}. \end{cases}$$

Whenever reference is made to a_t with $t \notin [1, n]$ we mean $a_{t'}$ where $t' = t - 1 \pmod n + 1$. This convention will be made throughout the article without further notice. Also x_{ij} stands for $x_{ij}^{(n)}$.

Note that

$$I(\tau_k) = \sum_{1 \leqslant i < j \leqslant n} x_{ij}^{(k)}.$$

We want to find the average of $I(\tau_k)$ over $n \geqslant k \geqslant 1$; so let us fix $1 \leqslant i < j \leqslant n$ and let us calculate

$$\sum_{k=1}^{n} x_{ij}^{(k)} = \begin{cases} j - i & \text{if } x_{ij} = 0, \\ n - j + i & \text{if } x_{ij} = 1. \end{cases}$$

This is because for $1 \leqslant i < j \leqslant n$ there are $n - j + i$ values of $1 \leqslant k \leqslant n$ for which $i + k \pmod n > j + k \pmod n$. Again let us remark that our residue classes mod n are $1, \ldots, n$ not $0, \ldots, n-1$ as usual. So, we have

$$\sum_{k=1}^{n} x_{ij}^{(k)} = j - i + (n - 2j + 2i)x_{ij}.$$

And therefore

$$\sum_{k=1}^{n} I(\tau_k) = \sum_{k=1}^{n} \sum_{1\leq i<j\leq n} x_{ij}^{(k)} = \sum_{1\leq i<j\leq n} (j-i) + \sum_{1\leq i<j\leq n} (n-2j+2i)x_{ij}$$
$$= \binom{n+1}{3} + \sum_{1\leq i<j\leq n} (n-2j+2i)x_{ij}.$$

For $1 \leq r < s < i \leq n$ we have

$$1 \geq x_{rs} + x_{st} - x_{rt}. \tag{1}$$

Because $x_{rs} = x_{st} = 1$ implies $x_{rt} = 1$.

Let us sum (1) over all triples $1 \leq r < s < t \leq n$. For $1 \leq i < j \leq n$ we count x_{ij} $(j-i-1)$ times in the negative and $i-1+n-j$ times in the positive sign. Altogether we get

$$\binom{n}{3} = \sum_{1\leq r<s<t\leq n} 1 \geq \sum x_{rs} + x_{st} - x_{rt} = \sum_{1\leq i<j\leq n} (n-2j+2i)x_{ij}.$$

Therefore

$$\sum_{k=1}^{n} I(\tau_k) \leq \binom{n+1}{3} + \binom{n}{3} = \frac{(2n-1)n(n-1)}{6}$$

and so the average of $I(\tau)$ over all $\tau \in [\sigma]$ is at most $\frac{1}{6}(2n-1)(n-1)$. It follows that for every $\sigma \in S_n$ there is a $\tau \in [\sigma]$ for which $I(\tau) \leq \frac{1}{6}(2n-1)(n-1) = \frac{1}{3}n^2 - \frac{1}{6}(3n-1)$, proving the upper bound.

The lower bound

We want to find a permutation $\sigma = (a_1, \ldots, a_n) \in S_n$ for which $I(\tau_k)$ is large for all $\tau_k = (a_k, \ldots, a_n, a_1, \ldots, a_{k-1}) \in [\sigma]$. Let us comment first that

$$I(\tau_{k+1}) - I(\tau_k) = n + 1 - 2a_k.$$

Because of moving from τ_k to τ_{k+1}, $a_k - 1$ inversions disappear and $n - a_k$ new inversions are created. Let us assume, for simplicity that $I(\tau_1) \leq I(\tau_k)$ for all $n \geq k > 2$. That means that for all $n-1 \geq k \geq 1$

$$\sum_{j=1}^{k} (n+1-2a_j) \geq 0 \quad (n-1 \geq k \geq 1). \tag{2}$$

To simplify our calculations we assume n to be even, the modifications for odd n are insignificant. We want to find σ for which $a_i < a_j$ for $i < j$ will occur only for $i \leq \frac{1}{2}n < j$. In other words the numbers in $[1, \frac{1}{2}n]$ will appear in reverse order and so will the ones in $[\frac{1}{2}n + 1, n]$. Under this assumption we want to maximize the number of inversions between numbers from these two intervals, while at the same time maintaining (2) valid. This means we set

$$a_i = \tfrac{1}{2}n - i + 1 \quad \text{for } t_1 \geq i \geq 1 \text{ and } a_{t_1+1} = n,$$

for some integer t_1. Of course, we wish to minimize t_1 so as to maximize the number of inversions. This must be done subject to the assumption that (2) must hold, which it certainly does for $t_1 \geqslant k \geqslant 1$. Let us evaluate the left hand side of (2) for $k = t_1 + 1$

$$\sum_{j=1}^{t_1} (n - 2a_j + 1) = \sum_{j=1}^{t_1} (2j - 1) = t_1^2.$$

And therefore

$$\sum_{j=1}^{t_1+1} (n - 2a_j + 1) = t_1^2 - (n - 1) > 0.$$

We, therefore, choose $t_1 = \lceil \sqrt{n-1} \rceil$ to meet our goals.

We continue by letting $a_i = \frac{1}{2}n - i + 2$ for $t_2 + 1 \geqslant i \geqslant t_1 + 2$ and $a_{t_2+2} = n - 1$. The condition (2) reads

$$\sum_{j=1}^{t_2+2} (n - 2a_j + 1) \geqslant 0.$$

We group the terms for $t_1 \geqslant j \geqslant 1$ and those for $t_2 + 1 \geqslant j \geqslant t_1 + 2$ and the $(t_1 + 1)$st and $(t_2 + 2)$nd term arriving at the inequality

$$\sum_{j=1}^{t_2} (2j - 1) = t_2^2 \geqslant (n - 1) + (n - 3) = 2n - 4,$$

and we choose accordingly $t_2 = \lceil \sqrt{2n-4} \rceil$.

In general, where $t_r = \lceil \sqrt{rn - r^2} \rceil$ we set $a_{t_r+r} = n - r + 1$ $(\frac{1}{2}n \geqslant r \geqslant 1)$. This defines $\frac{1}{2}n$ of the a_i $(n \geqslant i \geqslant 1)$ the undefined a_i's are $\frac{1}{2}n, \ldots, 1$ in this order. This construction of σ implies that $I(\sigma) \leqslant I(\tau_k)$ for every $\tau_k \in [\sigma]$. So we have to calculate $I(\sigma)$: The only situation where $i < j$ and $a_i < a_j$ occurs for $i \leqslant \frac{1}{2}n < j$, and $|\{i \mid i \leqslant \frac{1}{2}n \text{ and } a_i < n - r + 1\}| = t_r$ for $r = 1, 2, \ldots, \frac{1}{2}n$. Therefore

$$I(\sigma) = \binom{n}{2} - \sum_{r=1}^{\frac{1}{2}n} t_r.$$

But

$$\sum_{r=1}^{\frac{1}{2}n} t_r \leqslant \sum_{r=1}^{\frac{1}{2}n} (1 + \sqrt{rn - r^2}),$$

and

$$\sum_{r=1}^{\frac{1}{2}n} \sqrt{rn - r^2} \leqslant \int_0^{\frac{1}{2}n} \sqrt{nx - x^2}\, \mathrm{d}x + \tfrac{1}{2}n = \frac{\pi n^2}{16} + \tfrac{1}{2}n.$$

And hence

$$I(\sigma) \geqslant \binom{n}{2} - \frac{\pi n^2}{16} - n = \frac{8 - \pi}{16} n^2 - \tfrac{3}{2}n,$$

establishing the lower bound. □

3. Maximal distance

For $\sigma = (a_1, \ldots, a_n) \in S_n$, let

$$D(\sigma) = \max\{|a_i - i| : i = 1, \ldots, n\}$$

In this section we investigate the functions:

$$G(n) = \max_{\sigma \in S_n} \min_{\tau \in [\sigma]} D(\tau)$$

and

$$H(n) = \max_{\sigma \in S_n} \min_{\tau \in \langle\sigma\rangle} D(\tau).$$

We provide the exact value of $G(n)$ and an approximate value of $H(n)$, as described below: Let

$$\alpha(n) = \min\{k : k^2 + k - 1 \geq n\}, \qquad \beta(n) = \min\{k : k^2 - k - 4 \geq n\},$$

and

$$\gamma(n) = \min\{k : k^2 + \tfrac{1}{2}k \geq n\}.$$

We prove that:

$$G(n) = n - \alpha(n) \quad [n \geq 1], \qquad n - \beta(n) \leq H(n) \leq n - \gamma(n) \quad [n \geq 8].$$

The rest of this section is organized as follows: First we present a general result related to $G(n)$ and $H(n)$ (Proposition 3.1). Then we use this result to prove the upper bounds on $G(n)$ (subsection 3.1) and $H(n)$ (subsection 3.2). We conclude in proving the lower bounds on $G(n)$ and $H(n)$ (subsections 3.3 and 3.4).

In investigating the properties of $D(\sigma)$ it is convenient to deal with the value $k(\sigma) = n - D(\sigma)$. Let $\sigma = (a_1, \ldots, a_n)$ and $k < n$ be given. Then a_i *covers* σ if $|a_i - i| \geq n - k$, and σ is *covered* if some a_i covers it. As in the previous section, we denote the permutations in $[\sigma]$ by $\{\tau_1, \ldots, \tau_n\}$ and the permutations in $\langle\sigma\rangle$ by $\{\tau_1, \ldots, \tau_n, \tau_1^R, \ldots, \tau_n^R\}$, where $\tau_j = (a_j, \ldots, a_n, a_1, \ldots, a_{j-1})$, $\tau_j^R = (a_{j-1}, a_{j-2}, \ldots, a_1, a_n, \ldots, a_j)$. The proof of the following Proposition follows directly from the definitions, and is omitted.

Proposition 3.1.

(a) *If* $a_i = k - w + 1$ *for* $0 < w \leq k$, *then* a_i *covers the* $2w$*-elements set* $\{\tau_{i+1}, \tau_{i+2}, \ldots, \tau_{i+w}, \tau_i^R, \tau_{i-1}^R, \ldots, \tau_{i-w+1}^R\}$. *(Recall that if* t *is not in* $[1, n]$, *then* τ_t *is identified with* $\tau_{t'}$, *where* $t' = t - 1 \pmod n + 1$.)

(b) *If* $a_i = n - k + w$ *for* $0 < w \leq k$, *then* a_i *covers the set* $\{\tau_i, \tau_{i-1}, \ldots, \tau_{i-w+1}, \tau_{i+1}^R, \tau_{i+2}^R, \ldots, \tau_{i+w}^R\}$.

(c) *If* $k < a_i \leq n - k$, *then* a_i *covers* ϕ. □

3.1. *Upper bound on* $G(n)$

We now use Proposition 3.1 to obtain an upper bound on $G(n)$. To simplify the notations we denote $\min_{\tau \in [\sigma]} D(\tau)$ by $G(\sigma)$, (thus, $G(n) = \max_{\sigma \in S_n} G(\sigma)$). Let

$\sigma=(a_1,\ldots,a_n)$ and $k\leqslant n$ be given. We shall show that if $G(\sigma)\geqslant n-k$ then $k^2+k-1\geqslant n$, which, by the definition of $a(n)$, proves the upper bound.

For $i=1,\ldots,n$, let $v(a_i)=|\{\tau\in[\sigma]\colon a_i \text{ covers } \tau\}|$. Then, by Proposition 3.1,

$$\begin{aligned} &v(n)=v(1)=k,\\ &v(n-1)=v(2)=(k-1),\\ &\\ &v(k)=v(n-k+1)=1,\\ &v(j)=0 \quad \text{for } k<j\leqslant n-k. \end{aligned}$$

Let $\mathrm{FAR}([\sigma])=\{\tau\in[\sigma]\colon \tau \text{ is covered}\}$. Then if $G(\sigma)\geqslant n-k$, $|\mathrm{FAR}([\sigma])|=\|[\sigma]\|=n$. On the other hand, $|\mathrm{FAR}([\sigma])|\leqslant\sum_{i=1}^n v(a_i)=2(1+2+\cdots+k)=k(k+1)$. This means that if $G(\sigma)\geqslant n-k$, then $k(k+1)\geqslant n$, which gives the upper bound

$$G(n)\leqslant n-\min\{k\colon k^2+k\geqslant n\}.$$

To improve this bound to

$$G(n)\leqslant n-\alpha(n) \quad (\text{recall that } \alpha(n)=\min\{k\colon k^2+k-1\geqslant n\}),$$

we show that if $G(\sigma)\geqslant n-k$ then for some τ in $[\sigma]$ there are i_1 and i_2, $i_1\neq i_2$, such that both a_{i1} and a_{i2} cover τ. Such a permutation τ is said to be *over covered.* Clearly, if some τ in $[\sigma]$ is over covered then

$$|\mathrm{FAR}([\sigma])|\leqslant\sum_{i=1}^n v(a_i)-1=k^2+k-1,$$

which implies the upper bound on $G(n)$. The next lemma proves that such an over covered permutation must exist.

Lemma 3.1.1. *If $G(\sigma)\geqslant n-k$, then there is a permutation $\tau\in[\sigma]$ which is over covered.*

Proof. Assume the contrary. Then each permutation in $[\sigma]$ is covered by a unique a_i $(1\leqslant i\leqslant n)$. Hence $n=\sum_{i=1}^n v(a_i)=k^2+k$, which implies that $k\leqslant\frac{1}{2}n$. We say that a permutation τ in $[\sigma]$ is of type (S) if the unique a_i that covers it is not larger than k, and of type (L) otherwise (that is: if that a_i is larger than $n-k$). There are exactly $\frac{1}{2}(k^2+k)$ permutations of each type, and hence for some j in $\{1,\ldots,n\}$, τ_j is of type (L) and τ_{j+1} if of type (S). Let $a_i>n-k$ cover τ_j and $a_{i'}\leqslant k$ cover τ_{j+1}. Note that since $k\leqslant\frac{1}{2}n$ we must have that $a_i\neq a_{i'}$, and hence $i\neq i'$.

By Proposition 3.1(b) we have that $i-v(a_i)+1\leqslant j\leqslant i$, and a_i covers τ_l for $j\leqslant l\leqslant i$. Since τ_{j+1} is covered by $a_{i'}$, it cannot be covered by a_i. Hence, i cannot be greater than j, which implies that it must be equal to j. By similar reasons, using Proposition 3.1(a), we get that $i'=j$. Thus we get that $i=j=i'$, a contradiction. The lemma follows. □

3.2. *Upper bound on $H(n)$*

Let $H(\sigma)=\min_{\tau\in\langle\sigma\rangle} D(\tau)$. Like in the proof of the upper bound on $G(n)$, we shall show that if for some $\sigma=(a_1,\ldots,a_n)$ and k it holds that $H(\sigma)\geqslant n-k$, then $k^2+\frac{1}{2}k\geqslant n$. For $i=1,\ldots,n$ let $w(a_i)=|\{\tau\in\langle\sigma\rangle: a_i \text{ covers } \tau\}|$ $(=2v(a_i))$. Then, by Proposition 3.1.

$$
\begin{aligned}
&w(n)=w(1)=2k,\\
&w(n-1)=w(2)=2(k-1),\\
&\vdots\\
&w(k)=w(n-k+1)=2,\\
&w(j)=0 \quad \text{for } k<j\leqslant n-k.
\end{aligned}
$$

Let $\text{FAR}(\langle\sigma\rangle)=\{\tau\in\langle\sigma\rangle: \tau \text{ is covered}\}$ and $\text{OVER}(\langle\sigma\rangle)=\{\tau\in\langle\sigma\rangle: \tau \text{ is over covered}\}$. Since each permutation in $\text{OVER}(\langle\sigma\rangle)$ is covered by at least two distinct a_i's, we have that $\text{FAR}(\langle\sigma\rangle)\leqslant\sum_{i=1}^n w(a_i)-|\text{OVER}(\langle\sigma\rangle)|=2k(k+1)-|\text{OVER}(\langle\sigma\rangle)|$. Also, if $H(\sigma)\geqslant n-k$, then $|\text{FAR}(\langle\sigma\rangle)|=|\langle\sigma\rangle|=2n$. Thus we have

Lemma 3.2.1. *If $H(\sigma)\geqslant n-k$, then $2k(k+1)\geqslant 2n+|\text{OVER}(\langle\sigma\rangle)|$.* □

By the above lemma, the upper bound of $n-\gamma(n)$ on $H(n)$ follows from the following lemma.

Lemma 3.2.2. *If $H(\sigma)\geqslant n-k$, where $n\geqslant 2k$, then $|\text{OVER}(\langle\sigma\rangle)|\geqslant k$.*

Proof. Let $\sigma\in S_n$ be such that $H(\sigma)\geqslant n-k$. Consider the list of indices $1\leqslant i_1<i_2<\cdots<i_{2k}\leqslant n$ for which $w(a_{i_j})>0$, and let w_{i_j} denote the number $\frac{1}{2}w(a_{i_j})$. Consider now the following partition of $\langle\sigma\rangle$ to the $2k$ sets $S_{i_1},\ldots,S_{i_{2k}}$ defined by:

$$S_{i_j}=\{\tau_l,\tau_l^R: l\in[i_j,i_{j+1})\}\quad (j=1,\ldots,2k).$$

In the definition above, and throughout this lemma, $[i_{2k},i_1)$ means $[i_{2k},n]\cup[1,i_1)$ if $i_1>1$, and $[i_{2k},n]$ if $i_1=1$. Also, for $t>2k$, i_t means $i_{t'}$, where $t'=t-2k$. We denote by $c(S_{i_j})$ the number of distinct permutations in S_{i_j} which are over covered. The following claim is the main tool used in the proof of this lemma. Though the claim is not surprising, its proof is rather tedious.

Claim 1. If for some j, $c(S_{i_j})+c(S_{i_{j+1}})=0$, then $c(S_{i_{j+2}})+c(S_{i_{j+3}})\geqslant 2$.

Proof of Claim 1. Let j satisfy the hypothesis of the claim, and denote i_j, i_{j+1}, i_{j+2} and i_{j+3} by i, i', i'' and i''' respectively. We prove the claim only for the case $a_i\leqslant k$, since the proof of the case $a_i>n-k+1$ is similar. Let $a_i=k-w_i+1$, where $w_i=\frac{1}{2}w(a_i)$. Then by Proposition 3.1, a_i covers τ_l for $i<l\leqslant i+w_i$ and τ_m^R

for $i - w_i < m \leqslant i$. In particular, τ_{i+w_i} and τ_i^R are covered by a_i, but τ_{i+w_i+1} and τ_{i+1}^R are not (since $k<n$). Let l be such that a_l covers τ_{i+1}^R. We consider three cases:

(1) $a_i \geqslant n-k+1$. Then, by Proposition 3.1 and the fact that $2k \leqslant n$, we have that $l<i<l+w_l$, which implies that a_l covers also τ_i^R, in contradiction with the assumption that τ_i^R is not over covered (since $\tau_i^R \in S_i$).

(2) $a_i \leqslant k$ and $l \neq i'$. This means that $i = l - w_i < i' < l$. We distinguish between two subcases:

(2.1) $a_{i'} \leqslant k$. Then $\tau_{i'}^R$ is over covered (by $a_{i'}$ and a_l), which contradicts the assumption that $c(S_{i'}) = 0$.

(2.2) $a_{i'} \geqslant n-k+1$. In this case $\tau_{i'+1}^R$ is over covered (by a_l and $a_{i'}$), and hence $\tau_{i'+1}^R = \tau_{i''}^R$ cannot be in $S_{i'}$ (since $c(S_{i'}) = 0$). This means that $\tau_{i'+1}^R$ is in $S_{i''}$, and hence that $i'' = i'+1 \leqslant l$. Since $a_{i'} \geqslant n-k+1$, $a_l \leqslant k$ and $l - w_l < i' < i'' \leqslant l$, none of $a_{i'}$ and a_l covers $\tau_{i''}$. We shall use this last fact to show that there is another permutation in $S_{i''} \cup S_{i'''}$, beside $\tau_{i'+1}^R = \tau_{i''}^R$, which is over covered. This will prove the claim. We consider three subcases, according to the value of m for which a_m covers $\tau_{i''}$.

(2.2.1) $m = i''$. Then it must hold that $a_{i''} \geqslant n-k+1$, and hence $\tau_{i''+1}^R$ is over covered (by $a_{i''}$ and a_l), and clearly $a_{i''+1} \in S_{i''} \cup S_{i'''}$.

Note that the above argument is valid when ever $a_{i''} \geqslant n-k+1$, and hence we may assume now that $a_{i''} \leqslant k$.

(2.2.2) $m = i$. Then we have that $i < i' < i'+1 = i'' \leqslant i + w_{i'}$, and hence $\tau_{i'}$ is over covered (by a_i and $a_{i'}$)—a contradiction to the assumption that $c(S_{i'}) = 0$.

(2.2.3) $m \notin \{i, i', i''\}$. Then either $m < i < i'' \leqslant m + w_m$ and $a_m \leqslant k$, or $m - w_m < i'' \leqslant m$ and $a_m \geqslant n-k+1$. In the first case τ_{i+1} and $\tau_{i'}$ are over covered (the first by a_m and a_i, the second by a_m and $a_{i'}$), which contradicts the assumption. In the second case $\tau_{i''+1}$ is over covered (by a_m and $a_{i''}$), and clearly $\tau_{i''+1}$ is in $S_{i''} \cup S_{i'''}$.

(3) $a_l \leqslant k$ and $l = i'$. Since $c(S_i) = 0$, $a_{i'}$ covers τ_{i+1}^R but not τ_i^R. Thus, $i' = i + w_{i'}$. Since $w_i \neq w_{i'}$ and $1 \leqslant w_i, w_{i'} \leqslant k$, we have that $w_{i'}$ can be either strictly smaller or strictly larger than w_i. We consider each of these two possibilities here:

(3.1) $w_{i'} < w_i$. Then $\tau_{i'+1}$ is over covered (by a_i and $a_{i'}$). This means, by the assumption that $c(S_{i'}) = 0$, that $i'+1 \notin S_{i'}$, hence $i'' = i'+1$. We consider two subcases, according to the value of $a_{i''}$.

(3.1.1) $a_{i''} \leqslant k$. Then if $w_{i''} = 1$ (i.e. $a_{i''} = k$), $w_{i'}$ must be larger than 1, hence $\tau_{i''+1}$ is over covered (by $a_{i'}$ and $a_{i''}$), and thus both $\tau_{i''}$ and $\tau_{i''+1}$ are over covered, and the claim follows. If $w_{i''} > 1$ then $\tau_{i'}^R$ is over covered (by $a_{i'}$ and $a_{i''}$), in contradiction with the assumption of the claim.

(3.1.2) $a_{i''} \geqslant n-k+1$. Then $\tau_{i''}^R$ is not covered by any of a_i, $a_{i'}$, $a_{i''}$, and hence it must be covered by some a_m where $m \notin \{i, i', i''\}$. If $a_m \geqslant n-k+1$, then $m < i < i'' < m + w_m$, and hence a_m covers also τ_i^R, a contradiction.

Hence $m - w_m < i'' < m$ and $a_m \leq k$. This implies that a_m covers also $\tau^R_{i''+1}$, which is covered also by $a_{i''}$. Hence both $\tau_{i''}$ and $\tau^R_{i''+1}$ (which are in $S_{i''} \cup S_{i'''}$) are over covered.

(3.2) $w_{i'} > w_i$ (i.e., $i + w_i < i + w_{i'} = i'$). Then $\tau_{i'}$ is not covered by a_i, neither by $a_{i'}$. The assumption that $c(S_i) = 0$ implies that $\tau_{i'}$ is covered by some a_m, where $a_m \geq n - k + 1$ and $m - w_m < i' < m$, which means that $\tau_{i'+1}$ is over covered (by $a_{i'}$ and a_m). Since $c(S_{i'}) = 0$, this implies that $i'' = i' + 1$, and that $\tau^R_{i''}$ is not covered by $a_{i'}$, neither by a_m. We shall use this last fact to show that there must be another permutation in $S_{i''} \cup S_{i'''}$, beside $\tau_{i'+1} = \tau_{i''}$, which is over covered. We consider two cases, according to the value of i'':

(3.2.1) $i'' = m$ (hence $a_{i''} \geq n - k + 1$). Then $\tau^R_{i''}$ must be covered by some a_p where $p \notin \{i, i', i''\}$. If $a_p \geq n - k + 1$, then $p < i' < p + w_p$ and $\tau^R_{i'}$ is over covered (by $a_{i''}$ and a_p): a contradiction. If $a_p \leq k$ then $p - w_p < i'' < p$, and $\tau^R_{i''+1}$ is over covered (by a_p and $a_{i''}$), and the claim holds.

(3.2.2) $i'' \neq m$. Hence $m - w_m < i'' < m$. If $a_{i''} \leq k$ then $\tau_{i''+1}$ is over covered (by $a_{i''}$ and a_m). If $a_{i''} \geq n - k + 1$, then $\tau^R_{i''}$ is not covered by any of a_i, $a_{i'}$ and $a_{i''}$. Let p be such that a_p covers $\tau^R_{i''}$. If $a_p \leq k$ then $p - w_p < i'' < p$ and $\tau^R_{i''+1}$ is over covered (by $a_{i''}$ and a_p), and the claim holds. If $a_p \geq n - k + 1$ then $p < i' < p + w_m$ and $\tau^R_{i'}$ is over covered (by a_m and $a_{i'}$): a contradiction. This completes the proof of the claim.

We need one more claim for the proof of the lemma:

Claim 2. *Let* $B_1, \ldots, B_{2k}$ be $2k$ boxes, each containing c_i balls, and assume that for each $i = 1, \ldots, 2k$, if $c_i + c_{i+1} = 0$, then $c_{i+2} + c_{i+3} \geq 2$. Then $\sum_{i=1}^{2k} c_i \geq k$.

Proof. By induction on the number t of indices i such that $c_i + c_{i+1} = 0$. If $t = 0$, then there are at least k i's such that $c_i \geq 1$ and the claim holds. So assume that for some i $c_i + c_{i+1} = 0$. By the hypothesis of the claim, $c_{i+2} + c_{i+3} \geq 2$. Relocate two balls from boxes B_{i+2} and/or B_{i+3} in B_{i+1} and B_{i+3}. This does not change the sum $\sum_{i=1}^{2k} c_i$, and reduce the number of indices i with the above property by at least one, thus the claim follows by induction.

Proof of Lemma 3.2.2. Let $c(S_{i_j}) \equiv c_j$. Then $|\mathrm{OVER}(\langle\sigma\rangle)| = \sum_{i=1}^{2k} c_i$, and by Claim 1 the assumption of Claim 2 holds. Hence, by Claim 2, $\sum_{i=1}^{pk} c_i \geq k$. □

3.3. *Lower bound on* $G(n)$

First we show that if $n = k^2 + k - 1$ for some positive integer k, then there is a permutation σ in S_n for which $G(\sigma) = n - k$.

Let $(f_1, \ldots, f_k)$ be the sequence defined by:

$$f_1 = 1, \qquad f_{i+1} = f_i + k - i + 1 \quad (1 \leq i \leq k - 1).$$

(i.e., $f_i = k(i-1) + \frac{1}{2}(3i - i^2)$). In particular, $f_k = \frac{1}{2}(k^2 + k)$. Similarly, let $(g_1, \ldots, g_k)$ be the sequence defined by:

$$g_1 = f_k + 2 = \tfrac{1}{2}(k^2 + k) + 2, \quad g_{i+1} = g_i + k - i + 1 \quad (1 \leqslant i \leqslant k-2), \quad g_k = 2.$$

(i.e., for $1 \leqslant i < k, g_i = k(i-1) + \frac{1}{2}(3i - i^2)$. In particular, $g_{k-1} = k^2 + 1 = n - k + 2$.

Note that for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k-1, f_i < g_j$, and also that $g_k = 2 \neq f_i$; it follows that $f_i \neq g_j$ for all i, j in $\{1, 2, \ldots, k\}$.

Let $\sigma = (a_1, \ldots, a_n)$ be any permutation in S_n that satisfies the condition:

$$\text{For } 1 \leqslant i \leqslant k, \quad a_{f_i} = i \quad \text{and} \quad a_{g_i} = n - k + i. \qquad (*)$$

Then $G(\sigma) = n - k$. This follows by the following facts, that are easily verified by Proposition 3.1:

(1) For $f_i < j \leqslant f_{i+1}$, τ_j is covered by a_{f_i} $(i = 1, \ldots, k-1)$;
(2) $\tau_{f_k} + 1\ (= \tau_{g_1 - 1})$ is covered by a_{f_k} and τ_{g_1} is covered by a_{g_1};
(3) For $g_i < j \leqslant g_{i+1}$, τ_j is covered by $a_{g_{i+1}}$ $(i = 1, \ldots, k-1)$;
(4) For $j \in [g_{k-1}, n] \cup \{1\}$, τ_j is covered by $a_{g_k} (= a_2)$. (Note that τ_2 is also covered by a_2, and is the unique permutation in $[\sigma]$ which is over covered.)

To prove that the lower bounds of $n - \alpha(n)$ on $G(n)$ holds also for $n \neq k^2 + k - 1$ we make the following observations:

Lemma 3.3.1. *If $n \neq k^2 + k - 1$ for all positive integers k, then $\alpha(n-1) = \alpha(n)$ [similarly, if $n \neq k^2 - k - 4$, then $\beta(n-1) = \beta(n)$].*

Proof. By the definition of $\alpha(n)$ $[\beta(n)]$. □

Lemma 3.3.2. *For all positive integers $n, G(n-1) \geqslant G(n) - 1$ $[H(n-1) \geqslant H(n) - 1]$.*

Proof. Define a mapping $\mu : S_{n+1} \to S_n$ by:

$$\mu(\sigma') = \mu(b_1, \ldots, b_{n+1}) = \sigma = (a_1, \ldots, a_n),$$

where a_i is defined as follows: let i_0 be such that $b_{i_0} = n + 1$. Then for $i < i_0$ $a_i = b_i$, and for $i_0 < i < n$ $a_i = b_{i+1}$. It is straight forwards to verify that this mapping satisfies the following conditions:

(a) $\mu([\sigma']) = [\mu(\sigma')]$ $[\mu(\langle\sigma'\rangle) = \langle\mu(\sigma')\rangle]$. (For a subset T of S_{n+1}, $\mu(T) = \{\mu(\tau) : \tau \subset T\}$.)
(b) $D(\mu(\sigma')) \geqslant D(\sigma') - 1$.

By (a) and (b) above, for all $\sigma' \in S_{n+1}$ we have that $G(\mu(\sigma')) \geqslant G(\sigma') - 1$ $[H(\mu(\sigma')) \geqslant H(\sigma) - 1]$, which implies the lemma. □

Thus, the lower bound on $G(n)$ for $k^2 + k - 1 \geqslant n > (k-1)^2 + (k-1) - 1$ is proved inductively, where the base of the induction is $n = k^2 + k - 1$ and the correctness for $n-1$ follows from the correctness for n by the inequalities:

$$\begin{aligned} G(n-1) &\geq G(n)-1 && \text{(by Lemma 3.3.2)} \\ &= n-\alpha(n)-1 && \text{(by the induction hypothesis)} \\ &= (n-1)-\alpha(n-1) && \text{(by Lemma 3.3.1).} \end{aligned}$$

3.4. *Lower bound on $H(n)$*

Like in the previous subsection, we show first that if $n=k^2-k-4$ for some positive integer k, then there is a permutation σ in S_n for which $D(\sigma)=n-k$.

Let $(f_1, \ldots, f_{k-1})$ be the sequence defined by:

$$f_1=1, \qquad f_{i+1}=f_i+k-i \quad (1 \leq i \leq k-2).$$

(i.e., $f_i=1+k(i-1)+\frac{1}{2}(i-i^2)$). In particular, $f_{k-1}=\frac{1}{2}(k^2-k)$.

Similarly, let $(g_1, \ldots, g_{k-1})$ be the sequence defined by:

$$g_1=f_{k-1}-1=\tfrac{1}{2}(k^2-k-2), \qquad g_{i+1}=g_i+k-i \quad (1 \leq i \leq k-3), \qquad g_{k-1}=2.$$

(i.e., for $1 \leq i < k-1$, $g_i=\frac{1}{2}(k^2-3k+2ki-i^2+i-2)$.) In particular, $g_{k-2}=k^2-k-4=n$.

It is not hard to verify that for $1 \leq i, j \leq k-1$, $f_i \neq g_j$. Like in the lower bound for $G(n)$, we claim here that for any permutation $\sigma \in S_n$ which satisfies the condition below, $H(\sigma)=n-k$:

$$\text{For } i=1, \ldots, k-1, \quad a_{f_i}=i \quad \text{and} \quad a_{g_i}=n+1-i. \tag{**}$$

To see this, observe that:

(1) For $f_i<j \leq f_{i+1}$, τ_j is covered by a_{f_i} $(i=1, \ldots, k-2)$;
(2) For $g_{i-1}<j \leq g_i$, τ_j is covered by a_{g_i} $(i=2, \ldots, k-2)$;
(3) τ_{f_1} $(=\tau_1)$ is covered by a_{f_k};
(4) For $f_{i-1}<j \leq f_i$, τ_j^R is covered by a_{f_i} $(i=2, \ldots, k-1)$;
(5) For $g_i<j \leq g_{i+1}$, τ_j^R is covered by a_{g_i} $(i=1, \ldots, k-2)$;
(6) $\tau_{f_1}^R$ $(=\tau_1^R)$ is covered by a_{f_1} $(=a_1)$.

The proof of the lower bound for $H(n)$ for all $n>7$ follows by Lemmas 3.3.1 and 3.3.2, along the same line of the proof of the lower bound on $G(n)$. The details are omitted.

We conjecture that $H(n)$ is equal to $n-\beta(n)$ (for $n>7$), though a simple proof of that conjecture may not exist.

Reference

[1] P. Erdős, S. Moran, S. Zaks, I. Koren and G. Silberman, Mapping data flow graphs on VLSI processing arrays, TR 306, Department of Computer Science, Technion, Haifa.