Extremal problems on shadows and hypercuts in simplicial complexes

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Abstract

Let $F$ be an $n$-vertex forest. We say that an edge $e \notin F$ is in the shadow of $F$ if $F \cup \{e\}$ contains a cycle. It is easy to see that if $F$ is “almost a tree”, that is, it has $n - 2$ edges, then at least $\lfloor \frac{n^2}{4} \rfloor$ edges are in its shadow and this is tight. Equivalently, the largest number of edges an $n$-vertex cut can have is $\lfloor \frac{n^2}{4} \rfloor$. These notions have natural analogs in higher $d$-dimensional simplicial complexes, graphs being the case $d = 1$. As it turns out in dimension $d > 1$ the corresponding bounds depend on the underlying field of coefficients $\mathbb{F}$. We solve these questions in full for $d = 2$ and for $\mathbb{F} = \mathbb{Q}$ and $\mathbb{F}_2$. The higher-dimensional results remarkably differ from the case in graphs. Thus we construct 2-dimensional “almost-hypertrees” (properly defined below) with an empty shadow for $\mathbb{F} = \mathbb{Q}$. For $\mathbb{F} = \mathbb{F}_2$ we determine the least possible density of the shadow which turns out to be vanishingly small but positive. We also study the least possible size of hypercuts and $k$-hypercuts in $d$-dimensional complexes. Finally, we mention several intriguing open questions.

Keywords  Simplicial Complexes · Extremal Combinatorics · High-Dimensional · Arithmetic progressions · Hypercuts · Hypertrees · Matroids · Homology · Boundary operator

1 Introduction

This article is part of an ongoing research effort to bridge between graph theory and topology (see, e.g. [10, 12, 4, 11, 2, 3, 8, 13]). This research program starts from the observation that a graph can be viewed as a 1-dimensional simplicial complex, and that many basic concepts of graph theory such as connectivity, forests, cuts, cycles, etc., have natural counterparts in the realm of higher-dimensional simplicial complexes. As may be expected, higher dimensional objects tend to be more complicated than their 1-dimensional counterparts, and many fascinating phenomena reveal themselves from the present vantage point. This paper is dedicated to the study of some extremal problems in this domain. We start by introducing some of the necessary basic notions and definitions.

Simplices, Complexes, and the Boundary Operator:

All simplicial complexes considered here have $[n] = \{1, \ldots, n\}$ or $\mathbb{Z}_n$ as their vertex set $V$. A simplicial complex $X$ is a collection of subsets of $V$ that is closed under taking subsets. Namely, if $A \in X$ and $B \subseteq A$, then

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defined for any matroid, we are not aware of any results about pertree in at least \( k \) complement \( \bar{T} \) span of \( X \) such that intersects every hypertree. It is a standard fact in matroid theory that for every hypercut \( C \) labeled trees. A and an acyclic set of size \( d \) and those \( d \)-faces that are not in \( X \). Given a set \( S \) of \( d \)-faces, the complex generated by \( S \) is the \( d \)-complex that contains all the subsets of faces in \( S \).

The permutations on the vertices of a face \( \sigma \) are split in two orientations of \( \sigma \), according to the permutation's sign. The boundary operator \( \partial = \partial_d \) maps an oriented \( d \)-simplex \( \sigma = (v_0, \ldots, v_d) \) to the formal sum \( \sum_{i=0}^{d} (-1)^i (\sigma \setminus v_i) \), where \( \sigma \setminus v_i = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d) \) is an oriented \( (d-1) \)-simplex. We fix some field \( \mathbb{F} \) and linearly extend the boundary operator to free \( \mathbb{F} \)-sums of simplices. We consider the \( \binom{n}{d} \times \binom{n}{d+1} \) matrix form of \( \partial_d \) by choosing arbitrary orientations for \( (d-1) \)-simplices and \( d \)-simplices in \( K_n^d \). Note that changing the orientation of a \( d \)-simplex (resp. \( (d-1) \)-simplex) results in multiplying the corresponding column (resp. row) by \(-1\). Thus the \( d \)-boundary of a weighted sum of \( d \) simplices, viewed as a vector \( z \) (of weights) of dimension \( \binom{n}{d+1} \) is just the matrix-vector product \( \partial_d z \). We denote by \( M_X \) the submatrix of \( \partial_d \) restricted to the columns associated with \( d \)-faces of a \( d \)-complex \( X \).

In this paper the underlying field is always either \( \mathbb{Q} \) or \( \mathbb{F}_2 \). (In the latter case orientation is redundant). It is very interesting to extend the discussion to the case where everything is done over a commutative ring, and especially over \( \mathbb{Z} \), but we do not do this here. We associate each column in the matrix form of \( \partial_d \) with the corresponding \( d \)-simplex \( \sigma \), namely, as a vector of dimension \( \binom{n}{d} \), which is also the boundary \( \partial_d \sigma \). It is standard and not hard to see that for every choice of ground field the matrix \( \partial_d \) has rank \( \binom{n-1}{d} \). A fundamental (easy) fact is that \( \partial_{d-1} \cdot \partial_d = 0 \) for any \( d \).

Rank Function and other notions:
If \( S \) is a collection of \( n \)-vertex \( d \)-simplices, then we define\(^1\) its rank as the \( \mathbb{F} \)-rank of the set of the corresponding columns of \( \partial_d \). (It clearly does not depend on the choice of orientations). The set of all \( d \)-faces of \( K_n^d \) has rank \( \binom{n-1}{d} \), with basis being, e.g., the collection of all \( d \)-simplices that contain the vertex 1.

If \( \text{rank}(S) = |S| \), we say that \( S \) is acyclic over \( \mathbb{F} \). A maximal acyclic set of \( d \)-faces is called a \( d \)-hypertree, and an acyclic set of size \( \binom{n-1}{d} - 1 \) is called an almost-hypertree. Hypertrees over \( \mathbb{Q} \) were studied e.g., by Kalai [10] and others [1, 5] in the search for high-dimensional analogs of Caley’s formula for the number of labeled trees. A \( d \)-dimensional hypercut (or \( d \)-hypercut in short) is an inclusion-minimal set of \( d \)-faces that intersects every hypertree. It is a standard fact in matroid theory that for every hypercut \( C \), there is a hypertree \( T \) such that \( |C \cap T| = 1 \).

The shadow \( SH(S) \) of a set \( S \) of \( d \)-simplices consists of all \( d \)-simplices \( \sigma \notin S \) which are in the \( \mathbb{F} \)-linear span of \( S \), i.e., such that \( \text{rank}(S \cup \sigma) = \text{rank}(S) \). If \( SH(S) = \emptyset \), we say that \( S \) is closed or shadowless. A set of edges in a graph is closed if it is a disjoint union of cliques. The intersection of closed sets is closed. If \( X \) is a \( d \)-complex with a full \( (d-1) \)-skeleton and if its set of \( d \)-faces is closed, we say that the \( d \)-faces in the complement \( \bar{X} \) is an open set. Note that a hypercut is an open set whose complement has co-rank 1.

More generally, a \( \mathit{d} \)-dimensional \( k \)-hypercut (or \( (d, k) \)-hypercut) is a minimal set that intersects every hypertree in at least \( k \) faces. Equivalently, it is an open set whose complement has co-rank \( k \). While this is well defined for any matroid, we are not aware of any results about \( k \)-cuts in any non-graphical case.

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\(^1\)In the language of simplicial homology, consider the \( d \)-complex \( K(S) \) whose set of \( d \)-faces is \( S \). Then, \( \text{rank}(S) \) is just the dimension of \( B_{d-1} \), the linear space of \( (d-1) \)-boundaries of \( K(S) \), and \( |S| - \text{rank}(S) \) is the dimension of the homology group \( H_d(K(S)) \).
Results:

In this paper we study extremal problems concerning the possible sizes of hypercuts and shadows in simplicial complexes. We also consider extremal questions on \((d, k)\)-hypercuts.

Let us start with some trivial observations on graphs. For every cut there is a tree that meets it in exactly one edge, so no cut can have more than \(\binom{n}{2} - n + 2\) edges. Actually the largest number of edges of a cut in an \(n\)-vertex graph is \(\lceil \frac{n^2}{4} \rceil\). We investigate here the 2-dimensional situation and discover that it is completely different from the graphical case. When we discuss 2-complexes, we refer to 2-faces as faces (and keep the terms vertex and edge for 0 and 1 dimensional faces).

A 2-dimensional hypertree has \(\binom{n-1}{2}\) faces. So, by the same reasoning, every hypercut has at most \(\binom{n}{2} - \binom{n-1}{2} - 1\) faces. A hypercut of this size (if one exists) is called perfect. We show that over the rationals perfect hypercuts exist for certain integers \(n\). Also, if a well-known conjecture by Artin in number theory is true, there are infinitely many such \(n\). The construction is based on the 2-complex of length-3 arithmetic progressions in \(\mathbb{Z}_n\), and is of an independent interest.

Over the field \(\mathbb{F}_2\), surprisingly, the situation changes. There are no perfect hypercuts for \(n > 6\), and the largest possible hypercut has \(\binom{n}{2} - \frac{3}{2}n^2 - \Theta(n)\) faces. With some additional technical effort, we give a complete description of the extremal hypercuts.

Similarly, we call a \((2, k)\)-hypercut perfect if it is a complement of an acyclic shadowless set of \(\binom{n-1}{2} - k\) faces. As mentioned, there are no perfect 2-dimensional hypercuts over \(\mathbb{F}_2\). This corresponds to \(k = 1\), but as we show below, for \(k = O(n \log n)\) perfect \((2, k)\)-hypercuts (and, respectively, large acyclic shadowless set) do exist for any field \(\mathbb{F}\).

It seems hard to determine the largest possible \((d, k)\)-hypercuts with \(k > 1\) even for \(d = 2\). On the other hand we find the least possible cardinality of a \((d, k)\)-hypercut for every \(d\). In other words, for every \(r \leq \binom{n-1}{2}\) we find the largest possible set \(S\) of \(d\)-simplices with \(\text{rank}(S) = r\). In graphs the problem is easy, and the optimal graph is an \((r + 1)\)-clique. While the analysis of the higher-dimensional case is harder, the answer is somewhat similar: For \(r = \left(\frac{x-1}{d}\right)\), the extremal set consists of all \(d\)-faces of \(K^d_x \subseteq K^d_n\). The proof is based on a shifting argument, and is related to the Kruskal-Katona Theorem.

The rest of the paper is organized as following. In Section 2 we introduce some additional notions in the combinatorics of simplicial complexes. Section 3 deals with the problem of largest 2-hypercuts over \(\mathbb{Q}\). In Section 4 we study the same problem over \(\mathbb{F}_2\). In Section 5 we deal with perfect \((2, k)\)-hypercuts, or, equivalently, with large acyclic shadowless 2-dimensional sets. In Section 6 we solve the extremal size vs. rank problem in \(K^d_n\). Lastly, in Section 7 we present some of the many open questions in this area.

2 Additional Notions and Facts from Simplicial Combinatorics

Recall that we view the \(d\)-boundary operator as a linear map over \(\mathbb{F}\), that maps vectors supported on oriented \(d\)-simplices to vectors supported on \((d - 1)\)-simplices, given explicitly by the matrix \(\partial_d\), as defined in the Introduction.

The right kernel of \(\partial_d\) is the linear space of \(d\)-cycles. The left image of \(\partial_d\) is the linear space \(B^d(X)\) of \(d\)-coboundaries of \(X\). With some abuse of notation we occasionally call a set of \(d\)-simplices a cycle or a coboundary if it is the support of a cycle or a coboundary. Clearly over \(\mathbb{F}_2\), this makes no difference. In this case each \(d\)-coboundary is associated with a set \(A\) of \((d - 1)\)-faces, and consists of those \(d\)-faces whose boundary has an odd intersection with \(A\).

A \(d\)-coboundary is called simple if its support does not properly contain the support of any other non-empty \(d\)-coboundary. As observed e.g., in [14], a coboundary is simple if and only if its support is a hypercut.

If \(\sigma\) is a face in a complex \(X\), we define its link via \(\text{link}_\sigma(X) = \{\tau \in X : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in X\}\). This is clearly a simplicial complex. For instance, the link of a vertex \(v\) in a graph \(G\) is \(v\)'s neighbour set which we also denote by \(N_G(v)\) or \(N(v)\). For a 2-coboundary \(C\) over \(\mathbb{F}_2\) and a vertex \(v \in [n]\), it is easy to see that the graph \(\text{link}_v(C)\) generates \(C\), i.e. \(C = \text{link}_v(C) \cdot \partial_2\). Namely, the characteristic vector of the 2-faces of \(C\) equals to
the vector-matrix left product of the characteristic vector of the edges of link\(_{(\cdot)}(C)\) with the boundary matrix \(\partial_2\). We recall a necessary and sufficient condition that \(G = \text{link}_{(\cdot)}(C)\) generates a 2-hypercut \(C\) rather than a general coboundary.

Two incident edges \(uv, uw\) in a graph \(G = (V, E)\) are said to be \(\Lambda\)-adjacent if \(vw \notin E\). We say that \(G\) is \(\Lambda\)-connected if the transitive closure of the \(\Lambda\)-adjacency relation has exactly one class.

**Proposition 2.1.** [14] A 2-dimensional coboundary \(B\) is a hypercut if and only if the graph \(\text{link}_{(\cdot)}(B)\) is \(\Lambda\)-connected for every \(v\).

We turn to define \(d\)-collapsibility. A \((d-1)\)-face \(\tau\) in a \(d\)-complex \(K\) is called exposed if it is contained in exactly one \(d\)-face \(\sigma\) of \(K\). An elementary \(d\)-collapse on \(\tau\) consists of the removal of \(\tau\) and \(\sigma\) from \(K\). We say that \(K\) is \(d\)-collapsible if it is possible to eliminate all the \(d\)-faces of \(K\) by a series of elementary \(d\)-collapses. It is an easy observation that the set of \(d\)-faces in a \(d\)-collapsible \(d\)-complex is acyclic over every field.

### 3 Perfect Hypercuts and Shadowless Almost-Hypertrees Over \(\mathbb{Q}\)

As noted above, the number of 2-faces in a 2-dimensional hypercut is at most \(\binom{n}{3} - \binom{n-1}{2} + 1\). A hypercut meeting this bound is called perfect. It is also characterized by the property that its complement is an almost-hypertree which has an empty shadow. The main result of this section is the construction of perfect hypercuts and the corresponding shadowless almost tree.

The following is the main result of this section.

**Theorem 3.1.** Let \(n \geq 5\) be a prime for which \(\mathbb{Z}_n^*\) is generated by \(-1, 2\). Let \(X = X_n\) be a 2-dimensional simplicial complex on vertex set \(\mathbb{Z}_n\) whose 2-faces are arithmetic progressions of length 3 in \(\mathbb{Z}_n\) with difference not in \(\{0, \pm 1\}\). Then,

- \(X_n\) is 2-collapsible. Moreover, it is an almost-hypertree over every field.
- \(\text{SH}(X_n) = \emptyset\) over \(\mathbb{Q}\). Consequently, the complement of \(X_n\) is a perfect hypercut over \(\mathbb{Q}\).

The entire construction and much of the discussion of \(X_n\) is carried out over \(\mathbb{Z}_n\). However, in the following discussion, the boundary operator \(M_X\) of \(X_n\) is considered over the rationals.

We start with two simple observations. First, note that \(X_n\) has a full 1-skeleton, i.e., every edge \((r, q)\), \(0 \leq r < q \leq n - 1\), is contained in some 2-face of \(X\).

Also, we note that the choice of omitting the arithmetic triples with difference \(\pm 1\) is completely arbitrary. For every \(a \in \mathbb{Z}_n^*\), the automorphism \(r \mapsto ar\) of \(\mathbb{Z}_n\) maps \(X_n\) to a combinatorially isomorphic complex of arithmetic triples over \(\mathbb{Z}_n^*\), with omitted difference \(\pm a\). Consequently, Theorem 3.1 holds equivalently for any difference that we omit. In what follows we indeed assume for convenience that the missing difference is not \(\pm 1\), but rather \(\pm 2^{-1} \in \mathbb{Z}_n\).

For \(d \in \mathbb{Z}_n^*\), define \(E_d = E_{d, n} = \left\{ (0, d), (1, d + 1), \ldots, (n - 1, d + n - 1) \right\}\), where all additions are \(\text{mod} n\). This is an ordered subset of directed edges in \(X_n\).

Similarly, we consider the collection of arithmetic triples of difference \(d\),

\[ F_d = F_{d, n} = \left\{ (0, d, 2d), (1, d + 1, 2d + 1), \ldots, (n - 1, d + n - 1, 2d + n - 1) \right\}. \]

Clearly every directed edge appears in exactly one \(E_d\) and then its reversal is in \(E_{-d}\). Likewise for arithmetic triples and the \(F_d\)'s. Since we assume that \(\mathbb{Z}_n^*\) is generated by \{-1, 2\}, it follows that the powers \(\{2^i\} \subset \mathbb{Z}_n^*, i = 0, \ldots, \frac{n-1}{2}, \) are all distinct, and, moreover, no power is an additive inverse of the other. Therefore, the sets \(\{E_{2^i}\}, i = 0, \ldots, \frac{n-1}{2} - 1\), constitute a partition of the 1-faces of \(X_n\). Similarly, the sets \(\{F_{2^j}\}, j = 0, \ldots, \frac{n-1}{2} - 2\), constitute a partition of the 2-faces of \(X_n\). The omitted difference is \(2 \frac{n-1}{2} - 1 \in \{\pm 2^{-1}\}\), as assumed (the sign is determined according to whether \(2 \frac{n-1}{2} = 1\) or \(-1\)).
Lemma 3.2. Ordering the rows of the adjacency matrix $M_X$ by $E_{2i}$’s, and ordering the columns by the $F_{2i}$’s, the matrix $M_X$ takes the following form:

$$M_X = \begin{pmatrix}
I + Q & 0 & 0 & \ldots \\
-I & I + Q^2 & 0 & \ldots \\
0 & -I & \ddots & \ldots \\
0 & 0 & \ddots & I + Q^{2^{n-2}} \\
0 & 0 & \ldots & -I
\end{pmatrix}$$

where each entry is an $n \times n$ matrix (block) indexed by $\mathbb{Z}_n$, and $Q$ is a permutation matrix corresponding to the linear map $b \mapsto b + 1$ in $\mathbb{Z}_n$.

Proof. Consider an oriented face $\sigma \in F_{2i} \subset X_n$. Then, $\sigma = (b, b + 2^i, b + 2^{i+1})$ for some $b \in \mathbb{Z}_n$ and $0 \leq i \leq \frac{n-1}{2}$, i.e., $\sigma$ is the $b$-th element in $F_{2i}$. By definition, $\partial \sigma = (b, b+2^i) + (b+2^i, b+2^{i+1}) - (b, b+2^{i+1})$. The first two terms in $\partial \sigma$ are the $b$-th and $(b+2^i)$-th elements in $E_{2i}$, respectively; the third term corresponds to the $b$-th element in $E_{2i+1}$. Thus, the blocks indexed by $E_{2i} \times F_{2i}$ are of the form $I + Q^{2^i}$, the blocks $E_{2i+1} \times E_{2i}$ are $-I$, and the rest is 0.

We may now establish the main result of this section.

Proof. (of Theorem 3.1) We start with the first statement of the theorem. Let $m = \frac{n-1}{2}$.

Lemma 3.2 implies that the edges in $E_{2m-1}$ are exposed. Collapsing on these edges leads to elimination of $E_{2m-2}$ and the faces in $F_{2m-2}$. In terms of the matrix $M_X$, this corresponds to removing the rightmost "supercolumn". Now the edges in $E_{2m-2}$ become exposed, and collapsing them leads to elimination of $E_{2m-2}$, and $F_{2m-3}$. This results in exposure of $E_{2m-3}$, etc. Repeating the argument to the end, all the faces of $X_n$ get eliminated, as claimed.

To show that $X_n$ is an almost-hypertree we need to show that the number of its 2-faces is $\binom{n-2}{2} - 1$. Indeed,

$$|X_n| = \sum_{j=0}^{\frac{n-1}{2}} |F_{2j}| = \left(\frac{n-1}{2} - 1\right) \cdot n = \left(\frac{n-1}{2}\right) - 1.$$

We turn to show the second statement of the theorem, i.e., that $SH(X_n) = \emptyset$. Let $u \in Q_+^{(2)}$, be a vector indexed by the edges of $X_n$, where $u_e = 2^i$ when $e \in E_{2i}$. Here we think of $2^i$ as an integer (and not an element in $\mathbb{Z}_n$). We claim that for every $\sigma \in K_n^2$, $$\langle u, \partial \sigma \rangle = 0 \iff \sigma \in X_n.$$

Indeed, for every face $\sigma \in K_n^{(2)}$, exactly three coordinates in the vector $\partial \sigma$ are non-zero, and they are $\pm 1$. Since the entries of $u$ are successive powers of 2, the condition $\langle u, \partial \sigma \rangle = 0$ holds if $\partial \sigma$ (or $-\partial \sigma$) has two $1$'s in $E_{2i}$ and one $-1$ in $E_{2i+1}$ for some $0 \leq i \leq \frac{n-1}{2} - 1$. This happens if and only if $\sigma$ is of the form $(b, b + 2^i, b + 2^{i+1})$, i.e., precisely when $\sigma \in X_n$.

This implies that $X_n$ is closed, i.e. $SH(X_n) = \emptyset$, since any 2-face $\sigma \in K_n^{(2)}$ spanned by $X_n$ must satisfy $\langle u, \partial \sigma \rangle = 0$, this being precisely the characterisation of $X_n$. Thus, $X_n$ is a closed set of co-rank 1. Therefore, its complement is a hypercut. Moreover, since $X_n$ is almost-hypertree, this hypercut is perfect.

When the prime $n$ does not satisfy the assumption of Theorem 3.1 we can still say something about the structure of $X_n$. Let the group $G_n = \mathbb{Z}^*_{n}/\{\pm 1\}$, and let $H_n$ be the subgroup of $G_n$ generated by 2. Then,

Theorem 3.3. For every prime number $n$, rank$_0(X_n) = |X_n| - (n - 1) \cdot (|G_n : H_n| - 1)$. In particular, $X_n$ is acyclic if and only if $Z^*_n$ is generated by $\{\pm 1, 2\}$. 

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We only sketch the proof. We saw that the partition of $X_n$’s edges and faces to the sets $E_i$ and $F_i$. We consider also a coarser partition by joining together all the $E_i$’s and $F_i$’s for which $i$ belongs to some coset of $H_n$. This induces a block structure on $M_X$ with $[G_n : H_n]$ blocks. An argument as in the proof of Lemma 3.2 yields the structure of these blocks. Finally, an easy computation shows that one of these blocks is $2$-collapsible, and each of the others contribute precisely $n - 1$ vectors to the right kernel.

We conclude this section by recalling the following well-known conjecture of Artin [9].

**Conjecture 3.4** (Artin’s Primitive Root Conjecture). Every integer other than $-1$ that is not a perfect square is a primitive root modulo infinitely many primes.

This conjecture clearly yields infinitely many primes $n$ for which $\mathbb{Z}_n^*$ is generated by $2$. (It is even conjectured that the set of such primes has positive density). Clearly this implies that the assumptions of Theorem 3.1 hold for infinitely many primes $n$.

### 4 Largest Hypercuts over $\mathbb{F}_2$

In this section we turn to discuss our main questions over the field $\mathbb{F}_2$. The main result of this section is:

**Theorem 4.1.** For large enough $n$, the largest size of a $2$-dimensional hypercut over $\mathbb{F}_2$ is \(\binom{n}{3} - \left(\frac{3}{4}n^2 - \frac{7}{2}n + 4\right)\) for even $n$ and \(\binom{n}{3} - \left(\frac{3}{4}n^2 - 4n + \frac{25}{4}\right)\) for odd $n$.

**Remark 4.2.** The proof provides as well a characterization of all the extremal cases of this theorem.

Since no confusion is possible, in this section we use the shorthand term cut for a $2$-dimensional hypercut.

The first step in proving Theorem 4.1 is the slightly weaker Theorem 4.3. A further refinement yields the tight upper bound on the size of cuts.

Note that since \(\binom{n}{3}\) is a coboundary, the complement $\bar{C} = \left[\binom{n}{3}\right] \setminus C$ of any cut $C$ is a coboundary. Moreover, the complement of the $(n - 1)$-vertex graph, $\text{link}_v(C)$, is a link of $\bar{C}$. In what follows, $\text{link}_v(C)$ is always considered as an $(n - 1)$-vertex graph with vertex set $[n] \setminus \{v\}$. Occasionally, we will consider the graph $\text{link}_v(C) \cup \{v\}$ which has $v$ as an isolated vertex.

**Theorem 4.3.** The size of every $n$-vertex cut is at most $\binom{n}{3} - \frac{3}{4}n^2 + o(n^2)$. In every cut $C$ that attains this bound there is a vertex $v$ for which the graph $G = \text{link}_v(C)$ satisfies either

1. $\bar{G}$ has one vertex of degree $\frac{n}{2} \pm o(n)$ and all other vertices have degree $o(n)$. Moreover, $|E(G)| = n - 1 + o(n)$.

2. $\bar{G}$ has one vertex of degree $n - o(n)$, one vertex of degree $\frac{n}{2} \pm o(n)$, and all other vertices have degree $o(n)$. Moreover $|E(G)| = \frac{3}{4}n \pm o(n)$.

Before proving the theorem we will need some observations.

**Observation 4.4.** Let $G = (V, E)$ be a graph with $n$ vertices, $m$ edges and $t$ triangles and let $C$ be the coboundary generated by $G$. Then $|C| = nm - \sum_{v \in V} d_v^2 + 4t$.

**Proof.** Let $e = (u, v) \in E(G)$. Then $\text{link}_e(C)$ consists of those vertices $x \neq u, v$ that are adjacent to both or none of $u, v$. Namely, $|\text{link}_e(C)| = n - d_u - d_v + 2|N(v) \cap N(u)|$. Clearly $|N(v) \cap N(u)|$ is the number of triangles in $G$ that contain $e$. But $\sum_{e \in E(G)} |\text{link}_e(C)|$ counts every two-face in $C$ three times or once, depending on whether or not it is a triangle in $G$. Therefore

$$|C| + 2t = \sum_{(u,v) \in E} \left(n - d_u - d_v + 2|N(v) \cap N(u)|\right).$$

The claim follows.
Two vertices in a graph are called clones if they have the same set of neighbours (in particular they must be nonadjacent).

**Observation 4.5.** For every nonempty cut $C$ and $x \in V$ the graph $\lambda x (C)$ is connected and has no clones, or it contains one isolated vertex and a complete graph on the rest of the vertices.

**Proof.** Directly follows from the fact that $\lambda x (C)$ is $\Lambda$-connected (Proposition 2.1).

The size of a cut $C$ for which $\lambda x (C)$ is the union of a complete graph on $n - 2$ vertices and an isolated vertex equals to $n - 2$, which is much smaller than the bound in Theorem 4.3. We restrict the following discussion to cuts $C$ for which $\bar{G} = \lambda x (C)$ is connected and has no clones. Let $V = V (\bar{G})$, and we denote by $N(v) := N_{\bar{G}}(v)$. For every $S \subseteq V$, an $S$-atom is a subset $A \subseteq V$ which satisfy: $(u, v) \in E(\bar{G}) \iff (u', v) \in E(\bar{G})$ for every $u, u' \in A$ and $v \in S$.

Claim 4.6 below generalizes Observation 4.5.

**Claim 4.6.** Suppose $C$ is a cut and $G = (V, E) = \lambda x (C)$ for some vertex $x \notin V$. Let $S \subseteq V$, and $G' = \bar{G} \setminus S$. Then, for every non-empty $S$-atom $A$, at least $|A| - 2$ of the edges in $G'$ meet $A$.

**Proof.** Let $H$ be the subgraph of $G'$ induced by an atom $A$. If $H$ has at most two connected components, the claim is clear, since a connected graph on $r$ vertices has at least $r - 1$ edges. We show next that if $H$ has at least three connected components, then for every component but at most one there is an edge in $\bar{E}$ that connects it to $V \setminus (S \cup A)$. This clearly proves the claim.

So let $C_1, C_2, C_3$ be connected components of $H$, and suppose that neither $C_1$ nor $C_2$ is connected in $\bar{G}$ to $V \setminus (S \cup A)$. Let $F := \bigcup_{1 \leq i < j \leq 3} C_i \times C_j \subseteq E$. Since $G$ is $\Lambda$-connected, there must be a $\Lambda$-path connecting every edge in $C_1 \times C_2$ to every edge in $C_2 \times C_3$. However, any path which starts in $C_1 \times C_2$ can never leave it. Indeed, let us consider the first time this $\Lambda$-path exits $C_1 \times C_2$, say $xy$ that is followed by $yw$, where $x \in C_1, y \in C_2, w \notin C_1 \cup C_2$ and $yw \notin E$. By the atom condition, a vertex in $S$ does not distinguish between vertices $x, y \in A$, whence $w \notin S$. Finally, $w$ cannot be in $A$, for $xw \notin E$ would imply that $w \in C_1$. Hence, $C_1$ is connected in $\bar{G}$ to $V \setminus (S \cup A)$, a contradiction.

In the following claims, let $G = (V, E) = \lambda x (C)$, for a cut $C$, and $x \notin V$, and let $G = (V, E) = \lambda x (C)$. Denote by $d = (d_1 \geq d_2 \geq \ldots \geq d_{n-1} \geq 1)$ the sorted degree sequence of $\bar{G}$. Additionally, let $v_1, \ldots, v_{n-1}$ be the corresponding vertices. I.e., $d(v_i) = d_i \geq d_{i+1}$.

**Claim 4.7.** Under the above conditions, $d_1 \leq m/2 + 1$.

**Proof.** Apply Claim 4.6 with $S = \{v_1\}$ and $A = N(v_1)$. It yields the existence of at least $|A| - 2$ edges in $\bar{G}$ that meet $A$ but not $v_1$. Since $|A| = d_1, m \geq d_1 + (d_1 - 2)$, implying the claim.

**Claim 4.8.** Under the same condition as in Claim 4.7, $d_1 + d_2 \leq m/2$.

**Proof.** Apply Claim 4.6 with $S = \{v_1, v_2\}$ and $A = N(v_1) \cap N(v_2)$ to conclude that $m \geq d_1 + d_2 + |A| - 3$ (as $(v_1, v_2)$ might be an edge). By inclusion-exclusion, $n - 3 \geq d_1 + d_2 - |A|$. Combining these two inequalities imply the claim.

**Claim 4.9.** Under the same condition as in Claim 4.7, for every $k, \sum_{i=1}^{k} d_i \leq m - n/2 + \frac{k^2}{2} + 2^k$.

**Proof.** There are at most $2^k$ atoms of $S = \{v_1, ..., v_k\}$, and we apply Claim 4.6 to each of them. There are at least $|A| - 2$ edges with endpoints in atom $A$ and other endpoints not in $S$. Consequently, there are at least $\frac{1}{2} (n - k - 2 \cdot 2^k)$ edges in $\bar{G} \setminus S$ (as each edge may be counted twice). In addition, there may be at most $\binom{k}{2}$ edges with endpoints within $S$, hence the total number of edges is bounded by

$$m \geq \sum_{i=1}^{k} d_i - \binom{k}{2} + \frac{1}{2} (n - k - 2 \cdot 2^k)$$
Proof. of Theorem 4.3 Let $C$ be a cut, and assume by contradiction that $|\bar{C}| \leq \frac{\gamma}{2} n^2$, for some $\gamma < \frac{9}{4}$. By averaging, there is a link, say $G = (V, \bar{E}) = \text{link}_\gamma(C)$, of at most $\frac{3|\bar{C}|}{n} \leq \gamma n < 9n/4$ edges, where $V = [n] \setminus \{v\}$. Let $|\bar{E}| = m \leq \gamma n$ for some $\gamma < 9/4$. We will show that $|\bar{C}| \geq 3n^2/4 - o(n^2)$ contradicting the assumption.

Indeed, Observation 4.4 implies that $|\bar{C}| = mn - \sum_{v \in \bar{G}} d_v^2 + 4t$ where $t$ is the number of triangles in $\bar{G}$. Hence it suffices to show that $s(C) := mn - \sum_v d_v^2 \geq \frac{3}{4} n^2 - o(n^2)$.

Given a sequence of reals $d = (d_1 \geq d_2 \geq \ldots \geq d_{n-1} \geq 1)$, we denote $f(d) := mn - \sum_i d_i^2$ where $m = \frac{1}{2} \sum_id_i$. With this notation $s(C) = f(d)$ where $d$ is the sorted degree sequence of $G$ and $v_1, \ldots, v_{n-1}$ the corresponding ordering of the vertices. I.e., $d(v_i) = d_i \geq d_{i+1}$.

We want to reduce the problem of proving a lower bound on $f(d)$ to showing a lower bound on $f_k(d) = mn - \sum_1^k d_i^2$, where $k = k(n)$ is an appropriately chosen slowly growing function. Clearly $f_k(d) - f(d) = \sum_{j=k+1}^{n-1} d_j^2$. But $d_j \leq \frac{2m}{j}$ for all $j$ whence $\sum_{j=k+1}^{n-1} d_j^2 \leq 4m^2 \sum_{j=k+1}^{\infty} \frac{1}{j^2} < \frac{4m^2}{k}$, i.e., $f_k(d) \leq f(d) + \frac{4m^2}{k}$.

Since $m < \frac{9}{4} n$, it suffices to show that $f_k(d) \geq \frac{3}{4} n^2 - o(n^2)$ for an arbitrary $k = \omega(n)(1)$, which is our next goal.

We first note that.

**Claim 4.10.** For every $k = o(\log n)$, $f_k(d) \geq mn - d_1^2 - (m - n/2 - d_1) \cdot d_2 - o(n^2)$.

**Proof.** $f_k(d) = mn - \sum_1^k d_i^2 \geq mn - d_1^2 - (\sum_1^k d_i) \cdot d_2 \geq mn - d_1^2 - (m - n/2 - d_1) \cdot d_2 - o(n^2)$, where the second step is by convexity, and the last step uses Claim 4.9.

We now normalize everything in terms of $n$, namely, write

$$m = \gamma \cdot n, \quad d_1 = x \cdot n, \quad d_2 = y \cdot n.$$

$$g(\gamma, x, y) := \frac{1}{n^2} f_k - o(1) = \gamma - x^2 - \gamma \cdot y + \frac{y}{2} + xy.$$

The problem of minimizing $f_k$ subject to our assumptions on $\gamma, d_2 \leq d_1$, and Claims 4.7, 4.8, 4.9 becomes

**Optimization problem A**

Minimize $g(\gamma, x, y)$, subject to:

1. $1 \leq \gamma \leq \frac{9}{4}$.
2. $0 \leq y \leq x \leq \min\left(\frac{7}{2}, 1\right)$.
3. $x + y \leq \gamma - \frac{1}{2}$.
4. $x + y \leq \frac{1+\gamma}{2}$.

This problem is answered in the following Theorem whose proof is in the appendix.

**Theorem 4.11.** The answer to Optimization problem A is $\min g(\gamma, x, y) = \frac{3}{4}$. The optimum is attained in exactly two points $(\gamma = 1, x = \frac{1}{2}, y = 0)$ and $(\gamma = 2, x = 1, y = \frac{1}{2})$.

Plugging the optimal values on $\gamma, x, y$ back into Claim 4.10 completes the proof of the Theorem.
Proof of Theorem 4.1. Let us recall some of the facts proved so far concerning the largest n-vertex 2-hypercut $C$. Let $v$ be a vertex. As a coboundary, $C$ can be generated by the n-vertex graph which consists of the isolated vertex $v$, and the $(n-1)$-vertex $\Lambda$-connected graph $G = \text{link}_v(C)$. Similarly, $\bar{C}$ can be generated by the disjoint union of $v$ and $\bar{G}$. As we saw, there exists some $v$ for which the corresponding $\bar{G}$ satisfies either

\[
\text{CASE (I)}: \quad m = n - 1 + o(n), \quad d_1 = \frac{n}{2} + o(n) \text{ and } d_2 = o(n),
\]

or

\[
\text{CASE (II)}: \quad m = 2n \pm o(n), \quad d_1 = n - o(n), \quad d_2 = \frac{n}{2} \pm o(n) \text{ and } d_3 = o(n).
\]

where, as before, $m = |E(\bar{G})|$, $d_1 \geq d_2 \geq \cdots \geq d_{n-1}$ is the degree sequence of $\bar{G}$, with $d_i = d(v_i)$. We denote by $t$ the number of triangles in $\bar{G}$. Since $C$ is the largest cut, the graph $\bar{G}$ attains the minimum of $f(\bar{G}) = nm - \sum d_i^2 + 4t$ among all graphs whose complement is $\Lambda$-connected.

We now turn to further analyse the structure of $\bar{G}$, in CASE (I).

Lemma 4.12. Suppose that $\bar{G}$ satisfies CASE (I) and let $H = \bar{G} \setminus v_1$. Then $H$ is either (i) A perfect matching, or (ii) A perfect matching plus an isolated vertex, or (iii) A perfect matching plus an isolated vertex and a 3-vertex path.

Proof. The proof proceeds as follows: for every $H$ other than the above, we find a local variant $\bar{G}_1$ of $\bar{G}$ with $f(\bar{G}_1) < f(\bar{G})$. We then likewise modify $G_1$ to $G_2$ etc., until for some $k \geq 1$ the graph $G_k$ is $\Lambda$-connected. We modify the graph as follows.

For every connected component $U$ of $H$ of even size $|U| \geq 4$, we replace $H|_U$ with a perfect matching on $U$, and connect $v_1$ to one vertex in each of these $\frac{|U|}{2}$ edges. Now all connected components of $H$ are either an edge or have an odd size.

Consider now odd-size components. Note that $H$ can have at most one isolated vertex. Otherwise $\bar{G}$ is disconnected or it has clones, so that $\bar{G}$ is not $\Lambda$-connected. As long as $H$ has two odd connected components which together have 6 vertices or more, we replace this subgraph with a perfect matching on the same vertex set, and connect $v_1$ to one vertex in each of these edges. If the remaining odd connected components are a triangle and an isolated vertex, remove one edge from the triangle, and connect $v_1$ only to one endpoint of the obtained 3-vertex path. In the last remaining case $H$ has at most one odd connected component $U$.

If no odd connected components remain or if $|U| = 1$, we are done.

In the last remaining case $H$ has a single odd connected component of order $|U| \geq 3$. We replace $H|_U$ with a matching of $\frac{|U| - 1}{2}$ edges, connect $v_1$ to one vertex in each edge of the matching and to the isolated vertex. If, in addition, there is a connected component of order 2 with both vertices adjacent to $v_1$ (Note that by the proof of Claim 4.6 there is at most one such component.), we remove as well one edge between $v_1$ and this component.

All these steps strictly decrease $f$. We show this for the first kind of steps. The other cases are nearly identical.

Recall that $|E(H)| = \frac{n}{2} + o(n)$ and that $H$ has at most one isolated vertex. Therefore every connected component in $H$ has only $o(n)$ vertices. Let $U$ be a connected component with $2u \geq 4$ vertices of which $0 < r \leq 2u$ are neighbours of $v_1$, and let $\beta = |E(H|_U|) - (2u - 1) \geq 0$. Let $\bar{G}'$ be the graph after the aforementioned modification w.r.t. $U$. We denote its number of edges and triangles by $m'$ and $t'$ resp., and its degree sequence by $d_i'$. Then,

\[
f(\bar{G}) - f(\bar{G}') = n(m - m') - \sum_i (d_i'^2 - d_i^2) + 4(t - t') \geq \]

\[
n(\beta + r - 1) - (d_i'^2 - (d_i - r + u)^2) - \sum_{i \in U} d_i^2 \geq \]

\[
n(\beta + r - 1) + (u - r)(2d_1 + u - r) - 2u(4u - 2 + 2\beta + r).
\]
In the second row we use $t \geq t'$, which is true since the modification on $U$ creates no new triangles. In the third row we use $\sum_{i \in U} d_i^2 \leq (\max_{i \in U} d_i) \left( \sum_{i \in U} d_i \right)$.

Let us express $d_1 = \frac{n-w}{2}$ where $w = o(n)$. What remains to prove is that

$$n(\beta + u - 1) + (u-r)(u-r-w) \geq 2u(4u - 2 + 2\beta + r).$$

Or, after some simple manipulation, and using the fact $r \leq 2u$, that

$$\beta n + (u-1)n \geq 4\beta u + u(7u + w + 3r - 4).$$

This is indeed so since $u = o(n)$ implies that $\beta n \gg \beta u$ and $2 \leq u \leq o(n)$ implies $(u-1)n \gg u(7u + w + 4r - 4)$.

The other cases are done very similarly, with only minor changes in the parameters. In the case of two odd connected components which together have $2u \geq 6$ vertices, in the final step the main term is $(\beta + u - 2)n \geq n + \beta u$ since $u > 2$. In the case of changing a triangle to a 3-vertex path the main term in the final inequality is $(\beta + u - 1)n = n$.

The structure of $\bar{G}$ for CASE (I) is almost completely determined by Lemma 4.12. Since $G$ is $\Lambda$-connected, in $\bar{G}$ $v_1$ must have a neighbour in each component of $H$, and can be fully connected to at most one component. In addition, if $P$ is a 3-vertex path in $H$, then $v_1$ has exactly one neighbour in $P$ which is an endpoint. Otherwise we get clones. Therefore the only possible graphs are those that appear in Figure 1. The first row of the figure applies to odd $n$, where the optimal $G$ satisfies $f(\bar{G}) = \frac{3}{4}n^2 - 4n + \frac{25}{4}$. The other rows correspond to $n$ even, with four optimal graphs that satisfy $f(\bar{G}) = \frac{3}{4}n^2 - \frac{7}{2}n + 4$.

![Figure 1: The graphs $\bar{G}$ that are considered in the final stage of the proof of CASE (I).](image)

This concludes CASE (I), and we now turn to Case (II). Our goal here is to reduce this back to CASE (I), and this is done as follows.
Claim 4.13. Let \( \bar{G} = \text{link}_v(\bar{C}) \) be a graph on \( n - 1 \) vertices with parameters as in CASE (II). If \( H = \bar{G} \setminus \{v_1, v_2\} \) has an isolated vertex \( z \) that is adjacent in \( \bar{G} \) to \( v_1 \) then \( f(\bar{G}) \) is bounded by the extremal examples found in CASE (I).

Proof. Let \( S \) be the star graph on vertex set \( V \cup \{v\} \) with internal node \( v_1 \) and \( n - 1 \) leaves. Consider the graph 
\( F := \bar{G} \oplus S \) on the same vertex set, whose edge set is the symmetric difference of \( E(S) \) and \( E(\bar{G}) \). Since every triplet meets \( S \) in an even number of edges, the coboundary that \( F \) generates equals to the coboundary that \( \bar{G} \) generates, which is \( \bar{C} \).

In addition, \( z \) is an isolated vertex in \( F \) since its only neighbor in \( \bar{G} \) is \( v_1 \). Consequently, \( F = \text{link}_v(\bar{C}) \), and the claim will follow by showing that \( F \) agrees with the conditions of CASE (I). Indeed, \( \deg_F(v_1) = n - 1 - \deg_{\bar{G}}(v_1) = o(n) \), and \( |\deg_F(u) - \deg_{\bar{G}}(u)| \leq 1 \) for every other vertex \( u \). Hence, \( \deg_F(v_2) = \frac{n}{2} + o(n) \), and \( \deg_F(u) = o(n) \) for every other vertex \( u \).

If \( H \) has no isolated vertex that is adjacent in \( \bar{G} \) to \( v_1 \), we show how to modify \( \bar{G} \) to a graph \( \bar{G}_1 \), such that (i) \( \bar{G}_1 \) is \( \Lambda \)-connected, (ii) \( \bar{G}_1 \setminus \{v_1, v_2\} \) has an isolated vertex which is adjacent to \( v_1 \) in \( \bar{G}_1 \), and (iii) \( f(\bar{G}_1) < f(\bar{G}) \).

Since \( \bar{G} \) is \( \Lambda \)-connected and using the proof of claim 4.6, \( H \) has at most one connected component \( U_1 \) in \( H \) where all vertices are adjacent to \( v_1 \) and not to \( v_2 \) in \( \bar{G} \). Similarly, it has at most one connected component \( U_2 \) where all vertices are adjacent to both \( v_1 \) and \( v_2 \). Also, since \( d_1 = n - o(n) \), \( d_2 = \frac{n}{2} + o(n) \) and \( H \) has at most 3 isolated vertices, there exists an edge \( xy \in E(H) \) such that \( xv_1, xv_2, yv_1 \in E(\bar{G}) \), but \( yv_2 \notin E(\bar{G}) \).

\( \bar{G}_1 \) is constructed as following:

1. If neither components \( U_1 \) nor \( U_2 \) exist, remove the edge \( xy \) and the edge \( v_1v_2 \), if it exists. Otherwise, let \( r := |U_1 \cup U_2| \).

2. If \( r \) is even, replace it in \( H \) with a perfect matching on \( u - 2 \) vertices and two isolated vertices. Connect \( v_1 \) to every vertex in \( U_1 \cup U_2 \). Make \( v_2 \) a neighbor of one of the isolated vertices, and one vertex in each of the edges of the matching. Additionally, remove the edge \( v_1v_2 \) if it exists.

3. If \( r \) is odd, replace it in \( H \) by a perfect matching on \( u - 1 \) vertices and one isolated vertex. Connect \( v_1 \) to every vertex in \( U_1 \cup U_2 \), and \( v_2 \) to one vertex in each edge of the matching.

The fact that value of \( f \) decreased is shown similarly to the calculation in CASE (I).

5 Maximum Acyclic Closed 2-Complexes and Perfect \((2, k)\)-Hypercuts

As we saw, perfect 2-dimensional cuts exist over \( \mathbb{Q} \) and so the complements of the largest possible cuts coincide with the largest shadowless acyclic complexes. In contrast, we now know that no perfect 2-dimensional cuts exist over \( \mathbb{F}_2 \) for \( n > 6 \). Thus, over \( \mathbb{F}_2 \) these two concepts need to be considered separately. In Section 4 we studied the complements of the largest possible cuts, and here we investigate the largest shadowless acyclic complexes over \( \mathbb{F}_2 \).

Recall that a set \( S \) of \( d \)-simplices is closed if \( \text{span}(S) = S \). Thus a cut is a complement of a closed set of rank \( \binom{n - 1}{d} - 1 \). The following is a generalization of cuts that is well studied for graphs, mostly in an algorithmic context (e.g., [15, 16]).

Definition 5.1. A set of \( d \)-simplices is \((d, k)\)-hypercut if it is the complement of a closed set of rank \( \binom{n - 1}{d} - k \).

It is not hard to show that a \((d, k)\)-hypercut is synonymous with a union of \( k \) distinct \( d \)-hypercuts. Also its size is at most \( \binom{n - 1}{d} - k \) since it shares at least \( k \) \( d \)-faces with every \( d \)-hypertree. A \((d, k)\)-hypercut that attains this bound is called perfect.

The main result of this section is a construction of a large-sized acyclic closed 2-complex:
Theorem 5.2. For every integer \( n \), there exists a 2-collapsible 2-complex \( A_n \) with \( \binom{n-1}{2} - O(n \log n) \) faces that remains 2-collapsible after the addition of any new face. Thus, this complex is acyclic and closed over every field, and perfect \((2, O(n \log n))\)-hypercuts exist.

Proof. For simplicity of presentation, we restrict the discussion to \( n \)'s of the form \( n = 2^k - 1 \). The vertex set \( V = V(A_n) \) is the set of all binary strings of length \( \leq k - 1 \), including the empty string. Hence \( |V| = 2^k - 1 \), as needed. We denote strings by lowercase letters. The faces of \( A_n \) are defined as follows: Let \( x, y \in V \) be such that neither one is a prefix of the other. Associated with every such pair is the face \( \sigma_{xy} = (x, y, z) \) where \( z \) is the longest common prefix of \( x, y \).

Clearly every edge \((x, y)\) as above is exposed, so that all the faces of \( A_n \) can be simultaneously collapsed.

It is easy to verify that \( |\{(x, y)\mid x \text{ is a prefix of } y\}| = \Theta(n \log n) \). This yields the claim on the number of faces in \( A_n \).

We still need to show that \( A_n \) remains 2-collapsible when a new face \((a, b, c)\) is added. Almost all faces of \( A_n \cup \{(a, b, c)\} \) get collapsed right away, as before. The only faces that survive the first round of collapse are \( S = \{(a, b, c), \sigma_{ab}, \sigma_{ac}, \sigma_{bc}\} \), where \( \sigma_{xy} \) is null if there is a prefix relation between \( x, y \). But a 2-complex with 4 faces or less that is non-2-collapsible must be the boundary of a tetrahedron. So suppose that \( S \) is the boundary of the tetrahedron \((a, b, c, z)\). For this to happen, no two strings among \( a, b, c, z \) can have a prefix relation. In addition, the string \( z \) is the longest common prefix of each pair among \( a, b, c \). Consider the first position \( i \) for which \( a_i, b_i, c_i \) are not all equal to see that this is impossible.

\[ \blacksquare \]

6  Largest Shadows and Smallest \((d, k)\)-Hypercuts

In this section we do not restrict the underlying field \( \mathbb{F} \) nor the dimension \( d \). We ask how small \( \text{rank}(Y) \) can be given its size \( |Y| \). Equivalently, what is the largest possible size of a \( d \)-complex of rank \( r \)? Clearly, the extremal \( Y \) is closed, and the largest acyclic subset in \( Y \) has the maximum possible shadow for any acyclic set of size \( r \).

A-priori the answers to these questions may depend on the number of vertices \( n \). However, as we shall see, this is not so, and if \( r \leq \binom{n-1}{d} = \text{rank}(K_n^d) \), there exists an optimal \( n \)-vertex \( d \)-complex of rank \( r \).

Since a \((d, r)\)-hypercut is the complement of a closed set of co-rank \( r \), it follows that a \((d, r)\)-hypercut in \( K_n^d \) has the least possible size if and only if it is the complement of an extremal \( Y \) as above of co-rank \( r \). That is, \( \text{rank}(Y) = \binom{n-1}{d} - r \). As it turns out, minimum \( r \)-cuts in \( K_n^d \) are much simpler objects than maximum cuts, let alone maximum \( r \)-cuts.

We need to recall some notions related to Kruskal-Katona Theorem (e.g., [6]). For every two integers \( m \) and \( d \), there is a unique representation

\[ m = \sum_{i=s}^{d+1} \binom{m_i}{i}, \quad \text{where} \quad m_{d+1} > m_d > \ldots > m_s \geq s \geq 1. \]

It is called the cascade form. Let

\[ m_{\{d\}} = \sum_{i=s}^{d+1} \binom{m_i}{i-1}. \]

Kruskal-Katona Theorem: The number of \((d-1)\)-faces in a \( d \)-complex of size \( m \) is at least \( m_{\{d\}} \). Equality holds e.g., for the \( d \)-complex generated by the first \( m \) members in the co-lexicographic order on \((d + 1)\)-sets over \( \mathbb{N} \).

This clearly implies monotonicity.

Fact 6.1. If \( m > m' \) then \( m_{\{d\}} \geq m'_{\{d\}} \) for any \( d \).

Our result can be stated as following:
Theorem 6.2. Every $d$-complex of size $m = \sum_{i=s}^{d+1} \binom{m_i}{i}$, (in cascade form) has rank at least

$$\geq \sum_{i=s}^{d+1} \binom{m_i - 1}{i - 1},$$

where $\binom{0}{0} = 1$. The bound is attained e.g., for a co-lexicographic initial segment as above.

The proof is based on a shifting argument.

Definition 6.3. Let $Y \subseteq K^d_n$, and $u < v \in [n]$. For every $\sigma \in Y$, define

$$S_{uv}(\sigma) = \{ \sigma \setminus \{ v \} \cup \{ u \} \mid u \in \sigma, \ u \notin \sigma, \ \sigma \setminus \{ v \} \cup \{ u \} \notin Y \}
 \text{otherwise}$$

In addition, $S_{uv}(Y) = \{ S_{uv}(\sigma) \mid \sigma \in Y \} \subseteq K^d_n$. A complex $Y$ is called shifted if $S_{uv}(Y) = Y$ for every $u < v$.

It is well known and easy to prove that every complex can be made shifted by repeatedly applying shifting operations $S_{uv}$ with $u < v$ (See [7]). We prove Theorem 6.2 by showing that it holds for shifted complexes, and that shifting does not increase the rank.

Claim 6.4. Every shifted complex satisfies Theorem 6.2.

Proof. Let $Y$ be a shifted complex with $m = \sum_{i=s}^{d+1} \binom{m_i}{i}$, $d$-faces, (in cascade form). Let $\text{st}_1(Y)$ be the star of vertex 1, namely, the set of all faces of $Y$ containing it. The $d$-complex $Z$ generated by $\text{st}_1(Y)$ is clearly $d$-collapsible, since every $d$-face $\sigma$ in $Z$ contains the vertex 1, and hence $\sigma \setminus \{1\}$ is exposed. Therefore $F$, the set of $Z$’s $d$-faces, is acyclic. In fact, $F$ is maximal acyclic, i.e., it spans every $d$-face $\sigma \in Y$. Indeed, if $\sigma \notin F$, let $\sigma^+ = \{ 1 \} \cup \sigma$ be the $(d+1)$-simplex obtained by augmenting $\sigma$ by vertex 1. It is easy to verify that

$$\partial \sigma^+ = \sigma + \sum_{v \in \sigma} \epsilon_v \cdot (\sigma \setminus \{ v \} \cup \{ 1 \}).$$

where $\epsilon_v \in \{-1, 1\}$ are the corresponding signs. Since $\partial_1 \partial_{d+1} = 0$, we conclude that

$$\partial_1 \sigma = -\sum_{v \in \sigma} \epsilon_v \cdot \partial_1(\sigma \setminus \{ v \} \cup \{ 1 \}),$$

and since $Y$ is shifted, all $(\sigma \setminus \{ v \} \cup \{ 1 \})$ are in $F$. Hence, $F$ spans $\sigma$. We showed that $\text{rank}(Y) = |F|$, but what is the size of $F$?

Following [6], we show that $|F| \geq \sum_{i=s}^{d+1} \binom{m_i - 1}{i - 1}$. Indeed, assume to the contrary that $|F| < \sum_{i=s}^{d+1} \binom{m_i - 1}{i - 1}$. Let $\text{ast}_1(Y)$, the antistar of vertex 1, namely the $d$-complex of all faces in $Y$ that do not contain the vertex 1. Then,

$$|\text{ast}_1(Y)| = m - |F| > \sum_{i=s}^{d+1} \binom{m_i}{i} - \sum_{i=s}^{d+1} \binom{m_i - 1}{i - 1} = \sum_{i=s}^{d+1} \binom{m_i - 1}{i}.$$ 

Let $q$ be the largest $i$ such that $m_i = i$, or $q = 0$ if no such $i$ exist, and let $a := |\text{ast}_1(Y)|$. By the above,

$$a \geq 1 + \sum_{i=s}^{d+1} \binom{m_i - 1}{i} \geq \binom{q}{q} + \sum_{i=q+1}^{d+1} \binom{m_i - 1}{i},$$

By monotonicity (Fact 6.1)

$$a_{(d)} \geq \binom{q}{q-1} + \sum_{i=q+1}^{d+1} \binom{m_i - 1}{i-1}$$

Also

$$\binom{q}{q-1} + \sum_{i=q+1}^{d+1} \binom{m_i - 1}{i-1} \geq \sum_{i=s}^{d+1} \binom{m_i - 1}{i-1}$$
since each term for \( i \leq q \) contributes 1 to the right hand side and there are at most \( q \) such terms.

By applying the Kruskal-Katona Theorem to \( ast_1(Y) \), the number of its \((d-1)\)-faces is at least \( a_{d(d)} \). But 
\( Y \) is shifted, so for every \((d-1)\)-face \( \tau \) of \( ast_1(Y) \), the \( d \)-simplex \( \tau \cup \{1\} \in F \), whence \( |F| \geq \sum_{i=s}^{d+1} \binom{m_i-1}{i-1} \), contrary to our assumption.

The initial co-lexicographic segment of length \( m \) clearly satisfies \( |F| = \sum_{i=s}^{d+1} \binom{m_i-1}{i-1} \). ■

We show next that shifting does not increase rank.

**Lemma 6.5.** \( \text{rank}(Y) \geq \text{rank}(S_{uv}(Y)) \) for every \( d \)-complex \( Y \) and every two vertices \( v, u \in V(Y) \).

**Proof.** Let \( Z_d \) be the vector space of \( d \)-cycles spanned by \( Y \), namely, the right kernel of the matrix \( \partial_d \) restricted to the columns of \( Y \). Since \( \text{rank}(Y) + \dim(Z_d(Y)) = |Y| \), and shifting preserves the size \( |Y| \), it suffices to show that \( \dim(Z_d(Y)) \leq \dim(Z_d(S_{uv}(Y))) \). This will be achieved by means of a mapping that may be viewed as “forced shifting”.

Let \( C_i(Y) \) be the free vector space of \( i \)-chains of \( Y \) over \( \mathbb{F} \). Namely the collection of all formal linear combinations with coefficients from \( \mathbb{F} \) of oriented \( i \)-faces of \( Y \). For \( 0 \leq j \leq d \), define the map \( \Phi = \Phi_{v \rightarrow u} \) from \( C_j(K^d_n) \) to \( C_j(K^d_n) \) as the linear extension of

\[
\Phi(\sigma) = \begin{cases} 
0 & \text{if } u, v \in \sigma \\
\sigma \setminus \{v\} \cup \{u\} & \text{if } u \notin \sigma, v \in \sigma \\
\sigma & \text{if } v \notin \sigma
\end{cases}
\]

In the second line we need to specify an orientation of \( \Phi(\sigma) \). The orientation of \( \sigma \) is specified by an ordered list of its vertices. The orientation of \( \Phi(\sigma) \) is determined by replacing \( v \) by \( u \) in that list. Note that this is a proper definition of \( \Phi(\sigma) \) which does not depend on the choice of the permutation used to specify \( \sigma \)’s orientation.

The main point is that \( \Phi \) maps \( d \)-cycles to \( d \)-cycles, i.e., \( \partial Z = 0 \) implies that \( \partial \Phi(Z) = 0 \). In fact, as we now show, something stronger is true and \( \Phi \) is a chain map. In words, \( \Phi \) commutes with the boundary operator \( \partial \), i.e., \( \Phi \partial = \partial \Phi \).

By linearity, it suffices to verify it for \( d \)-simplices. Let \( \sigma = (\sigma_0, \ldots, \sigma_d) \) be an oriented \( d \)-simplex. Recall that \( \partial \sigma = \sum_{i=0}^{d} (-1)^{i-1} \sigma_i \), where \( \sigma_i \) is the oriented \((d-1)\)-face of \( \sigma \) obtained by omitting the \( i \)-th entry from \( \sigma \).

1. We start with the case where both \( u, v \in \sigma \), say \( u = \sigma_j \), \( v = \sigma_k \). We need to show that \( \Phi(\partial \sigma) = 0 \). Clearly, \( \Phi(\sigma^i) = 0 \) for every \( i \neq j, k \), and thus

\[
\Phi(\partial \sigma) = \sum_{i=0}^{d} (-1)^{i} \Phi(\sigma^i) = (-1)^j \Phi(\sigma^j) + (-1)^k \Phi(\sigma^k) .
\]

However, by definition, \( \Phi(\sigma^j) \) and \( \Phi(\sigma^k) \) are both the same \((d-1)\)-simplex obtained from \( \sigma \) by removing \( v \). Moreover, it is easily verified that \( \text{sign}(\Phi(\sigma^j)) = (-1)^{k-j-1} \text{sign}(\Phi(\sigma^k)) \). Therefore,

\[
(-1)^j \Phi(\sigma^j) + (-1)^k \Phi(\sigma^k) = (-1)^j (-1)^{k-j-1} \Phi(\sigma^k) + (-1)^k \Phi(\sigma^k) = 0 .
\]

2. If \( u \notin \sigma, v \in \sigma \), then \( \Phi(\sigma^i) = \Phi(\sigma^i) \) for every \( 0 \leq i \leq d \), and thus

\[
\Phi(\partial \sigma) = \Phi \left( \sum_{i=0}^{d} (-1)^i \sigma^i \right) = \sum_{i=0}^{d} (-1)^i \Phi(\sigma^i) = \sum_{i=0}^{d} (-1)^i \Phi(\sigma^i) = \partial \Phi(\sigma) .
\]

3. Finally, if \( v \notin \sigma \), then \( \Phi(\sigma) = \sigma \). Since \( v \notin \sigma^i \), \( \Phi(\sigma^i) = \Phi(\sigma^i) = \sigma^i \) for every \( 0 \leq i \leq d \), and the argument of case 2 applies.
Let us consider next the restriction of $\Phi$ to $Z_d(Y)$, the vector space of $d$-cycles of $Y$. This restriction is a linear map that we call $\phi$. Clearly, $\text{dim}(\ker \phi) + \text{dim}(\text{Im} \phi) = \text{dim}(Z_d(Y))$ and the proof of the lemma will be completed by showing that
\[ \ker \phi \oplus \text{Im} \phi \subseteq Z_d(S_{uv}(Y)). \]

1. $\text{Im} \phi$: For every $\sigma \in Y$, either $\Phi(\sigma) = 0$ or it is in $S_{uv}(Y)$. But $\phi$ maps cycles to cycles, so $\text{Im} \phi \subseteq Z_d(S_{uv}(Y))$.

2. $\ker \phi$: Let $Z$ be a $d$-cycle of the form $Z = \sum_{\sigma \in T} \alpha_{\sigma} \cdot \sigma$ with $T \subseteq Y$, where $\alpha_{\sigma} \neq 0$ for $\sigma \in T$ and $\phi(Z) = 0$. We need to show that $\sigma \in S_{uv}(Y)$ for every $\sigma \in T$. This is clearly the case if $v \notin \sigma$ or if $u \in \sigma$. We only need to show that $\sigma' = \sigma \setminus \{v\} \cup \{u\} \in T$ when $v \in \sigma, u \notin \sigma$. But the coefficient of $\sigma'$ in the expansion of $\phi(Z)$ is $\alpha_{\sigma} \pm \alpha_{\sigma'}$, which is zero, since $\phi(Z) = 0$. Consequently, $\alpha_{\sigma'} \neq 0$ and $\sigma' \in T$, as claimed.

3. Finally, since $\Phi^2 = \Phi$, $\ker \phi \cap \text{Im} \phi = \emptyset$, and therefore the sum is direct.

This concludes the proof of Theorem 6.2.

Lovász’ version of the of the Kruskal-Katona Theorem yields the following appealing corollary:

**Corollary 6.6.** Let $|Y| = \binom{x}{d+1}$, where $x \geq d + 1$ is real. Then, $\text{rank}(Y) \geq \binom{x-1}{d}$. For $x \in \mathbb{N}$, the equality is attained if and only if $Y$ is the $d$-skeleton of a $k$-dimensional simplex.

Another simple consequence of Theorem 6.2 is an upper bound on $|Y|$ in terms of its rank:

**Theorem 6.7.** The largest size of a $d$-complex with $\text{rank}(Y) = r$ is
\[ m(d, r) = \sum_{i=s}^{d} \left( \binom{r_i + 1}{i + 1} \right) \text{ where } r = \sum_{i=s}^{d} \binom{r_i}{i} \]
in cascade form. The bound is attained e.g., at the $d$-complex $K(d, r)$ whose $d$-faces are the first $m(d, r)$ $(d + 1)$-tuples over $\mathbb{N}$ in the co-lexicographic order.

It is a good place to comment that $K(d, r)$ is a subcomplex of $K_n^d$, provided that $\text{rank}(K_n^d) = \binom{n-1}{d} \geq r$. This follows from the fact that the size of the underlying set of $K(d, r)$ is $r_d + 1$ if $s = d$, and $r_d + 2$ otherwise, where $r_d$ is the parameter from the cascade form of $r$. Observe also that $K(d, r)$ is necessarily closed, since otherwise it could be augmented without increasing its the rank, contrary to its optimality. An alternative description of the $d$-faces of $K(d, r)$ is the largest prefix of $(d + 1)$-tuples in the co-lexicographic order in which the number 1 is contained in $r$ tuples.

Keeping in mind that a $(d, k)$-hycruct in $K_n^d$ is a complement of a closed set of rank $\binom{n}{d-1} - k$, Theorem 6.7 can be rephrased as a lower bound on the size of $(d, k)$-hycruct.

**Theorem 6.8.** Let $r = \binom{n-1}{d} - k$. Then, the complement of $K(d, r)$ in $K_n^d$ is a $(d, k)$-hycruct of minimal size.

We turn to derive concrete bounds on the size of the extremal $(d, k)$-hycruct. Since $K(d, r)$ is a prefix in the co-lexicographic order, its complement $H(d, k)$ is a suffix in this order. But a suffix in the co-lexicographic order is a prefix in the lexicographic order on a reversed alphabet. Consequently, by renaming the vertex set, $H(d, k)$ is the smallest prefix of the lexicographic order on $[n]_{d+1}$ in which the number $n$ is contained in $k$ tuples.

Equivalently, let $\tau_1, \ldots, \tau_k$ be the prefix of length $k$ of the lexicographic order on $[n]_{d+1}$. Let $H_i, \text{ for every } 1 \leq i \leq k$, be the $d$-hycruct consists of the $d$-faces $\sigma \supset \tau_i$. Then, $H(d, k) = \bigcup_{i=1}^{k} H_i$.

**Claim 6.9.** $k(n-d) \geq |H(k, d)| \geq \frac{1}{d+1} \cdot k(n-d)$.
Proof. On one hand,
\[ |H(k, d)| = \left| \bigcup_{i=1}^{k} H_i \right| \leq k \cdot |H_i| = k(n - d). \]

On the other hand, any \(d\)-simplex in \(H(k, d)\) may belong to at most \((d + 1)\) different \(H_i\)'s, hence \(H(k, d) \geq \frac{1}{d+1} \cdot k(n - d). \)

\[ \square \]

7 Open Problems

- There are several problems that we solved here for 2-dimensional complexes. It is clear that some completely new ideas will be required in order to answer these questions in higher dimensions. In particular it would be interesting to extend the construction based on arithmetic triples for \(d > 2\).
- An interesting aspect of the present work is that the behavior over \(\mathbb{F}_2\) and \(\mathbb{Q}\) differ, some times in a substantial way. It would be of interest to investigate the situation over other coefficient rings.
- How large can an acyclic closed set over \(\mathbb{F}_2\) be? Theorem 5.2 gives a bound, but we do not know the exact answer yet.
- We still do not even know how large a \(d\)-cycle can be. In particular, for which integers \(n, d\) and a field \(\mathbb{F}\) does there exist a set of \(\binom{n-1}{d} + 1\) \(d\)-faces on \(n\) vertices such that removing any face yields a \(d\)-hypertree over \(\mathbb{F}\)?
- Many basic (approximate) enumeration problems remain wide open. How many \(n\)-vertex \(d\)-hypertrees are there? What about \(d\)-collapsible complexes? A fundamental work of Kalai [10] provides some estimates for the former problem, but these bounds are not sharp. In one dimension there are exactly \(\frac{(n-1)!}{2}\) inclusion-minimal \(n\)-vertex cycles. We know very little about the higher-dimensional counterparts of this fact.

References

Let $h(A) = \frac{1}{2} - x$. To minimize $h$, we need to show that $h \geq 0$ under the conditions of the optimization problem. This involves some case analysis.

First note $\gamma - \frac{1}{2} - x \geq 0$ by condition 3, so that for fixed $\gamma, x$ we have that $h$ is a decreasing function of $y$. Thus, to minimize $h$, we need to determine the largest possible value of $y$.

1. We first consider the range $\gamma \leq 2$. Here condition 4 is redundant, and $y \leq \min\{x, \gamma - \frac{1}{2} - x\}$.

   (a) We further restrict to the range $x \leq \frac{\gamma}{2} - \frac{3}{4}$, where $x \leq \gamma - \frac{1}{2} - x$, so the largest feasible value of $y$ is $y = x$. Note that $h|_{y=x} = \gamma - \frac{3}{4} - x(\gamma - \frac{1}{2})$. But $\gamma - \frac{1}{2} \geq 0$ by condition 1, so $h$ is minimized by maximizing $x$, namely taking $x = \frac{\gamma}{2} - \frac{3}{4}$. This yields $h = \frac{1}{8} - \frac{1}{2}(\gamma - 1)(\gamma - 2)$ which is positive in the relevant range $2 \geq \gamma \geq 1$.

   (b) In the complementary range $\frac{\gamma}{2} - \frac{3}{4} \leq x$ the largest value for $y$ is $y = \gamma - \frac{1}{2} - x$ which yields $h = \gamma - x^2 - \frac{3}{4} - (\gamma - \frac{1}{2} - x)^2 = -2(x - \frac{\gamma}{2})(x - \frac{\gamma - 1}{2}) - \frac{(\gamma - 1)(\gamma - 2)}{2}$. It suffices to check that $h \geq 0$ at both extreme value of $x$, namely $\frac{\gamma}{2} - \frac{1}{4}$ and $\gamma/2$. Also $h = 0$ only at $x = \gamma/2$ with $\gamma = 1$ or $2$.

2. In the complementary range $\gamma \geq 2$, condition 3 is redundant and condition 4 takes over.

   (a) Assume first that $x \leq \frac{1+\gamma}{4}$, then $x \leq \frac{1+\gamma}{2} - x$ and the extreme value for $y$ is $y = x$. Again $h|_{y=x} = \gamma - \frac{3}{4} - x(\gamma - \frac{1}{2})$ and now the largest possible value of $x$ is $x = \frac{1+\gamma}{4}$ which yields $h = \frac{5-2\gamma(\gamma-1)}{8}$. This is positive at the range $\frac{9}{4} \geq \gamma \geq 2$.

   (b) When $x \geq \frac{1+\gamma}{4}$ the minimum $h$ is attained at $y = \frac{1+\gamma}{2} - x$, so that

   $h = \gamma - x^2 - \frac{3}{4} - \frac{1+\gamma}{2} - x \cdot (\gamma - \frac{1}{2} - x) = -2(x - 1)(x + 1 - \frac{3\gamma}{4}) - \frac{1}{2}(\gamma - 2)(\gamma - \frac{5}{2})$.

For fixed $\gamma$ it suffices to check that $h \geq 0$ at the two ends of the range $1 \geq x \geq \frac{1+\gamma}{4}$. At $x = 1$ we get $h = -\frac{1}{2}(\gamma - 2)(\gamma - \frac{5}{2})$ which is nonnegative when $\frac{9}{4} \geq \gamma \geq 2$ with $h = 0$ only at $\gamma = 2$. When $x = \frac{1+\gamma}{4}$, we get $h = \frac{5-2\gamma(\gamma-1)}{8}$ which is positive for $\frac{9}{4} \geq \gamma \geq 2$. 

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To sum up, $h \geq 0$ throughout the relevant range with two points where $h = 0$, namely $\gamma = 2, x = 1, y = \frac{1}{2}$ and $\gamma = 1, x = \frac{1}{2}, y = 0$. 