Simplicial complexes - Much more than a trick for distributed computing lower bounds

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- Partners in an economic transaction.
- Humans in some social context.

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- Proteins come, more often than not, in complexes that involve several proteins at once.
- Human social networks tend to include several individuals.
- Economics transactions often involve several parties at once.
- Most relevant to us here: Distributed systems are many-sided by their very nature.

Hypergraphs, anyone?

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These are the good news. The bad news are that the theory of hypergraphs is not nearly as well developed as graph theory.

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We only need to make a small modification to the notion of hypergraph to arrive at simplicial complexes. This way we make contact with a rich body of powerful mathematics in topology and geometry that can help us.

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We only need to make a small modification to the notion of hypergraph to arrive at simplicial complexes. This way we make contact with a rich body of powerful mathematics in topology and geometry that can help us.
What's more - many fascinating new connections and perspectives suggest themselves.

Definition

Let V be a finite set of *vertices*. A collection of subsets $X \subseteq 2^V$ is called a *simplicial complex* if it satisfies the following condition:

$$A \in X \quad \text{and} \quad B \subseteq A \Rightarrow B \in X.$$

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The dimension of X is the largest dimension of a face in X.

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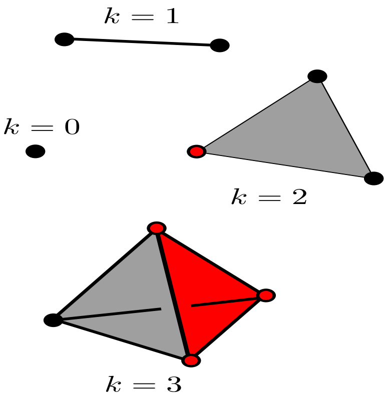
- Higher dimensional complexes offer a wonderful mix of combinatorics with geometric (mostly topological) ideas.
- The challenge - to develop a combinatorial perspective of higher dimensional complexes.

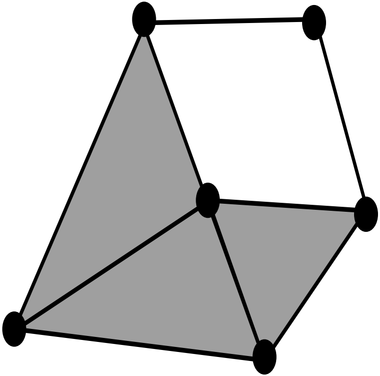
Simplicial complexes as geometric objects

Assign to $A \in X$ with $|A| = k+1$ a k-dim. simplex



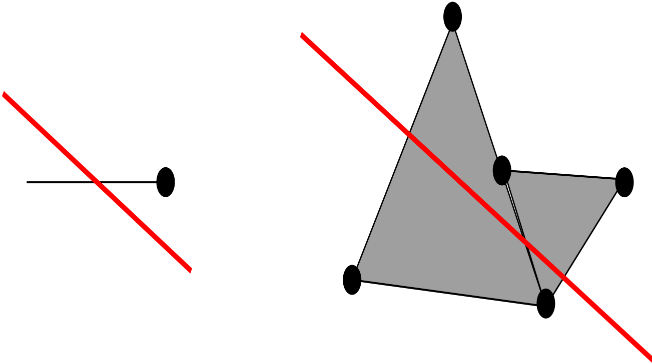
Putting simplices together properly

The intersection of every two simplices in X is a common face.



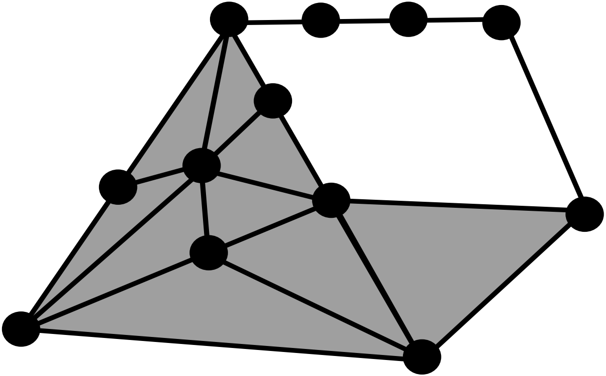
How NOT to do it

Not every collection of simplices in $\mathbb{R}^d$ is a simplicial complex



Geometric equivalence

Combinatorially different complexes may correspond to the same geometric object (e.g. via subdivision)



Geometric equivalence

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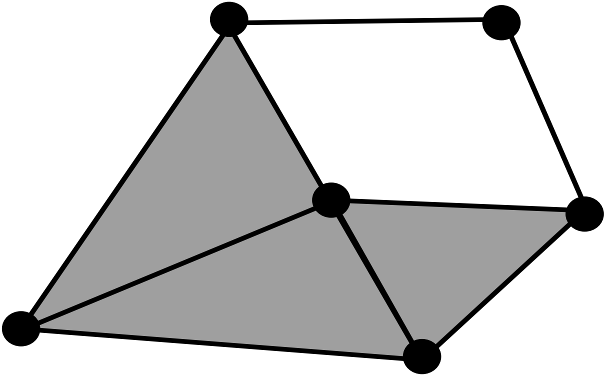

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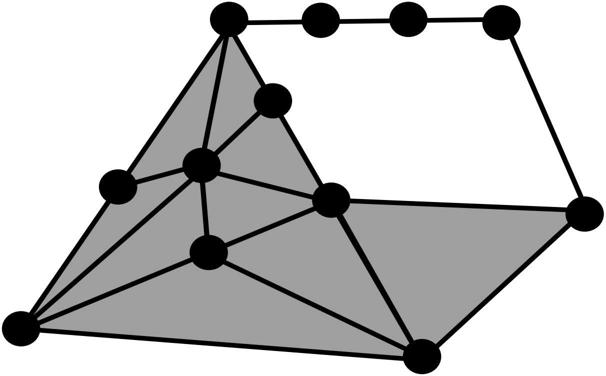
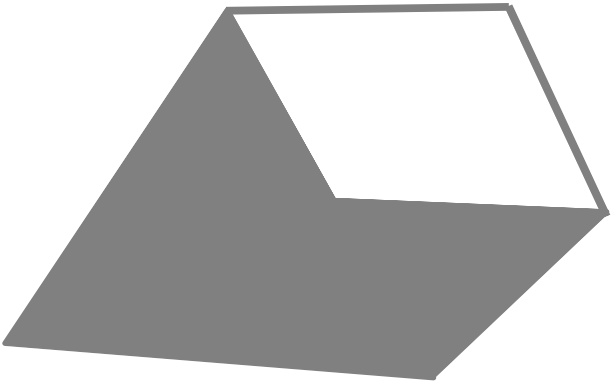


Geometric equivalence

are two different combinatorial descriptions of the same geometric object



Track record - SC's in theoretical computer science

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- Work on the evasiveness conjecture (See below).

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- Impossibility theorems in distributed asynchronous computation (Starting with [Borowsky, Gafni '93] [Herlihy, Shavit '93] and [Saks, Zaharoglou '93]).

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- Lower bounds on chromatic numbers: Kneser's graphs and hypergraphs. (Starting with [Lovász '78]).
- In the study of matching in hypergraphs (Starting with [Aharoni Haxell '00]).

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At each round Alice points at two vertices $x, y \in V$ and Bob answers whether they are adjacent in G, i.e. whether or not $xy \in E$.

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The game ends when Alice knows with certainty whether G has property $\mathcal{P}$.

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But how is this related to simplicial complexes, topology etc.?

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Careful: W is the set of vertices of the complex we consider.
The collection of all n-vertex graphs that have property $\mathcal{P}$ is a simplicial complex $=$ a down-closed family of subsets of W. (Since $\mathcal{P}$ is monotone).

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- Does a particular edge e belong to graph G ? $\Leftrightarrow$ Does a particular $x \in W$ belong to A ?

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- The game is played with a mutually known simplicial complex X on vertex set W.
- Alice's goal: to determine whether (an initially unknown) $A \subseteq W$ belongs to X.
- At each step: Alice points at some $x \in W$ and Bob responds whether or not x is in A.
- The simplicial complex X is said to be evasive if Bob has a strategy that forces Alice to query all elements in W.

A nice feature of this frame of thought is this: Whether Bob responds that $x \in A$ or $x \notin A$, we now proceed to a new game with a new simplicial complex X' or X'' on vertex set $W \setminus \{x\}$.

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For a non-evasive X, the query Is $x \in W$? is a good step for Alice iff both X' and X'' are non-evasive.
This allows for an inductive approach.

and indeed

Non-evasiveness implies collapsibility

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We will later return to this notion.

Kahn Saks and Sturtevant

The additional ingredient is that $\mathcal{P}$ is a graph property. Namely, it does not depend on vertex labeling. This implies that the simplicial complex of all graphs with property $\mathcal{P}$ is highly symmetric. Using some facts from group theory they conclude:

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Theorem (KSS '83)

The evasiveness conjecture holds for all n-vertex graphs if n is prime.

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- A model of random simplical complexes (main issue for the rest of this talk).
- Study extremal problems on simplicial complexes.
- In even bigger terms: Develop High dimensional combinatorics.

Exporting the probabilistic method to topology?

We want to develop a theory of random simplicial complexes, in light of to random graph theory. Specifically we seek a higher-dimensional analogue to $G(n, p)$.

Recollections of $G(n, p)$

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Closely related model: the evolution of random graphs starts with n vertices and no edges. At each step add a random edge to the evolving graph.

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We denote by $X(n, p)$ this probability space of two-dimensional complexes.

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Theorem (ER '60)

The threshold for graph connectivity in $G(n, p)$ is

$$p = \frac{\ln n}{n}$$

A few more words on this theorem

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- If $p < (1-\epsilon)\frac{\ln n}{n}$, then a random graph in $G(n,p)$ is almost surely disconnected.
- If $p > (1+\epsilon)\frac{\ln n}{n}$, then a random graph in $G(n,p)$ is almost surely connected.

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If $p < (1 - \epsilon)\frac{\ln n}{n}$, then a random graph in $G(n, p)$ is not only almost surely disconnected.

In fact, in this range of p, the graph almost surely has some isolated vertices. This is an easy consequence of the coupon-collector principle from probability theory.

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- The vanishing of the $(d-1)$-st homology $=$ the matrix ∂_d has a nontrivial left kernel. (This is the higher dimensional analog of a graph's incidence matrix - more below).
- Being simply connected (vanishing of the fundamental group).

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- Likewise, if S is the vertex set of a connected component of G, then $\mathbf{1}_S M = 0$.
- It is not hard to see that G is connected iff the only nonzero vector x that satisfies $xM = 0$ is $x = \mathbf{1}$.

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... and how to view the easy part of the ER theorem from this perspective

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This is the row corresponding to an isolated vertex in the resulting graph.

... and how to view the easy part of the ER theorem from this perspective

If $p < (1-\epsilon)\frac{\ln n}{n}$, then the resulting matrix almost surely has an all-zeros row.
This is the row corresponding to an isolated vertex in the resulting graph.
A matrix with a row of zeros clearly has a non-trivial left kernel.

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- The transformations associated with A_1 resp. A_2 are called *the boundary operator* (of the appropriate dimension) and are denoted ∂ (perhaps with an indication of the dimension).

It is an easy exercise to verify that $A_1 A_2 = 0$ (the general form is $\partial\partial = 0$, a key fact in topology).

A natural question suggests itself - Baby homology theory

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Clearly, the right kernel of X contains the column space of Y. The question to ask is:
Is this a proper inclusion or an equality?

This is quantified by considering the quotient space

$$\text{right kernel}(X)/\text{column space}(Y).$$

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In our situation where X and Y are inclusion matrices of k vs. $(k+1)$-dimensional faces of a simplicial complex, these quotient spaces are the relevant homology and cohomology groups.

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Several things are clear: We now start from the $\binom{n}{2} \times \binom{n}{3}$ inclusion matrix and select a random subset of the columns where every column is selected independently and with probability p.

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That should be clear now: The row space of the $n \times \binom{n}{2}$ matrix.

A little terminology

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The process of selecting the columns yields a random two-dimensional complex with a full one-dimensional skeleton. We call this model of random complexes $X_2(n, p)$. (So, e.g. $X_1(n, p)$ is nothing but good old $G(n, p)$).

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We have asked for the critical p where there a non-trivial left kernel exists.
In topological language: What is the critical p at which the first homology with $\mathbb{F}_2$ coefficients of a random $X \in X_2(n, p)$ vanishes?

...and the answer is...

Theorem (L. + Meshulam '06)

The threshold for the vanishing of the first homology of $X_2(n,p)$ with $\mathbb{F}_2$ coefficients is

$$p = \frac{2 \ln n}{n}$$

Again, one part is easy

For the very same reason, when $p < (1-\epsilon)\frac{2 \ln n}{n}$

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For the very same reason, when $p < (1 - \epsilon)\frac{2 \ln n}{n}$ the resulting matrix contains an all zeros row and consequently it has a nontrivial left kernel.

Such a row corresponds to an edge that is not contained in any of the randomly chosen 2-dimensional faces.

More generally

Likewise define $X_d(n, p)$, the random d-dimensional simplicial complexes with a full $(d-1)$-st dimensional skeleton.

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Theorem

In d dimensions the critical probability for the vanishing of the $(d-1)$-st homology with an arbitrary finite group of coefficients is

$$\frac{d \ln n}{n}.$$

The vanishing of the fundamental group

Theorem (Babson, Hoffman, Kahle '11)

The threshold for the vanishing of the fundamental group in $X(n, p)$ is near

$$p = n^{-1/2}.$$

Comment: When the field is not $\mathbb{F}_2$

We have to select an (arbitrary but fixed) orientation to the triples and pairs. The entries of the inclusion matrix are ± 1 depending on whether the orientation of the edge and the 2-face containing it are consistent or not.

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The d-dimensional case is similar (with an appropriate adaptation).

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If A is the incidence matrix of the graph and if $Ax = 0$, then x is the indicator vector of a set of edges that picks an even number of 1's in every row.

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Such subgraphs form a linear subspace. This subspace is generated by the simple cycles in G.

To sum up, A has a nonzero right kernel iff G contains cycles.

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So the relevant 1-dimensional theorem is:

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Theorem (Erdős-Rényi)

The critical probability for almost sure existence of a cycle in $G(n, p)$ is

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Theorem (Erdős-Rényi)

The critical probability for almost sure existence of a cycle in $G(n, p)$ is

$$p = \frac{1}{n}.$$

$=$ *the critical p for $G \sim G(n, p)$ to be a forest.*

The best show in town - Watch it unfold

The Erdős-Rényi papers on $G(n, p)$ is a monumental piece of science which had taught us many important and unexpected things. However, the most dramatic chapter in this fascinating story is the phase transition in the evolution of random graphs.

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Start with n isolated vertices and sequentially add a new random edge, one at a time. Observe the connected components of the evolving graph.

Prelude - The early stages

At the very beginning we see only isolated edges (a matching).

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As we proceed, more complex connected components start to appear, but they are all small and simple.

- small $=$ cardinality $O(\log n)$.
- simple $=$ a tree.
- Possibly a constant number of exceptions which are a small tree plus one edge $=$ unicylic graphs.

Crescendo - The phase transition

Around step $\frac{n}{2}$ and over a very short period of time

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Note: Time $\frac{n}{2}$ corresponds to $p = \frac{1}{n}$.

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Around step $\frac{n}{2}$ many other parameters are undergoing an abrupt change.

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However, for $p > \frac{1+\epsilon}{n}$, the graph almost surely contains a cycle.

Let's talk business again

Having a cycle means a nonzero right kernel to the graph's adjacency matrix.

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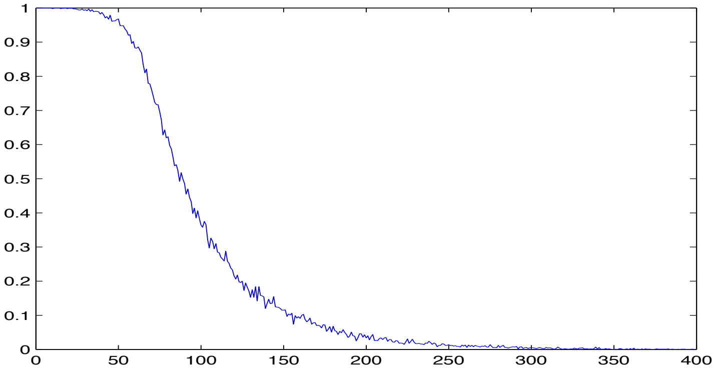
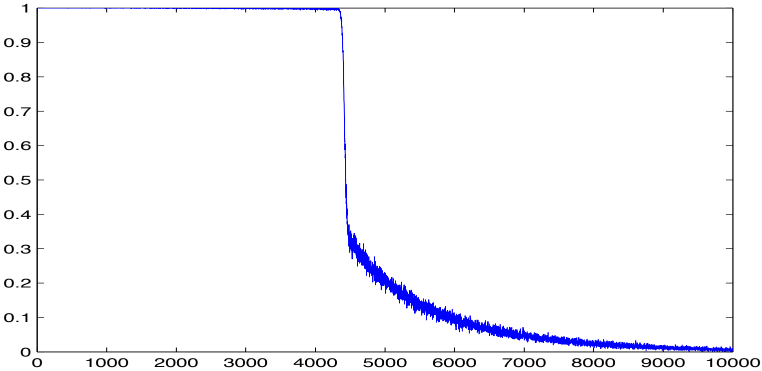
But before we turn to do that

You're calling this a phase transition????
A view of phase transition in $G(n, p)$



THIS is a phase transition - Phase transition in random 2-dim complexes



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- What are high-dimensional trees and forests?

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As usual, in higher dimensions the plot is thicker......

A well known theorem - and a twist

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OK, OK, you mean that G is a tree, but what is this collapsible thing???

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Theorem

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- *G is connected.*
- *G is acyclic.*
- *G is collapsible.*

OK, OK, you mean that G is a tree, but what is this collapsible thing??? Never heard this term before.

This is not such a great mystery

An elementary collapse is a step where you remove a vertex of degree one and the single edge that contains it.

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An elementary collapse is a step where you remove a vertex of degree one and the single edge that contains it.

A graph G is collapsible if by repeated application of elementary collapses you can eliminate all of the edges in G.

Collapsing - a linear algebra perspective

Let M be a matrix. In an elementary collapse we erase row i and column j of M provided that M_{ij} is the only nonzero entry in the i-th row.

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For an incidence matrix of a graph, this coincides with the graph-theoretic definition: Remove a vertex of degree 1 and the edge incident with it.

Some questions to answer before we can go high-dimensional

Q_1 : In dimension 1 (graphs) we speak of n vertices and $n-1$ edges.

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A_1 : $\binom{n-1}{d}$. See below.

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Q_4 : What is the analog of acyclic?

A_4 : No right kernel.

High-dimensional analogues of trees/forests

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Q : Which graphs are bases?

High-dimensional trees and forests

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But doesn't the answer depend on the underlying field?

No.

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This is a combinatorial condition and so it holds over any base field.

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2. In particular, is it still equivalent to collapsibility? (It's easy to see that in every dimension collapsibility is a sufficient condition).

A little surprise

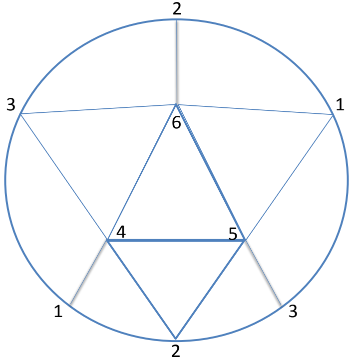

Figure: A triangulation of the projective plane

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The example we just saw is a column basis for $\mathbb{Q}$, but not for $\mathbb{F}_2$.

Open Problem

Consider a random column basis for the $\binom{n}{2} \times \binom{n}{3}$ *inclusion matrix have over some fixed field (most interestingly over* $\mathbb{F}_2$ *or over* $\mathbb{Q}$*).*

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As we'll see, there is strong evidence (but still no proof) that the answer should be $o(1)$.

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$$p_{non/vanishing\ of\ H_d} \leq (1 - o_d(1))\frac{d}{n}.$$

That's all folks