Discrete Mathematics 23 (1978) 257–272.

COVERING DIGRAPHS BY PATHS

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Received 10 August 1977

The problems of minimum edge and minimum vertex covers by paths are discussed. The results relate to papers by Gallai–Milgram, Meyniel, Alspach–Pullman and others. One of the main results is concerned with partially ordered sets.

Notations. The terminology is rather standard, generally following Berge [3]. In any case where ambiguity may arise we give definitions. Graphs and digraphs are finite, except in Theorem 6, and have no loops or multiple edges. If multiple edges are allowed we use the term *multigraph*. Let $G=(V, E)$ be a graph. An *orientation* of G is a digraph whose underlying graph is G. If S is a set, $|S|$ denotes its cardinality. If a set is said to be maximum (minimum) it means that it is cardinality maximum (minimum). The *order* of G, that is, $|V(G)|$, is denoted by n. Also $|E(G)|=e$.

The *vertex independence number* of G is the maximum cardinality of an independent set of vertices. We denote it by $\beta_0(G)$. Let G be a digraph. DL(G) its *dilinegraph* is defined as follows: it is a digraph whose vertex set is $E(G)$. There is an edge from a vertex x to a vertex y if the terminal vertex of x and the initial vertex of y coincide.

We use the word path to mean a directed simple path. The *length* of a path is the number of edges it contains. A path of length two is called a *couple*. The two edges $[x, y]$, $[y, x]$ are also called the *two-way edge* joining x and y. The digraph obtained by deleting all two-way edges in DL(G), is denoted by $\widetilde{\mathrm{DL}}(G)$. *The edge independence number of* G is defined by $\beta_1(G)=\beta_0(\mathrm{DL}(G))$.

Given a set A of vertices in G, denote by $\Gamma^-(A)$ (resp. $\Gamma^+(A)$) the set of those vertices that are joined to (resp. from) some vertex in A. Also $\Gamma(A)=\Gamma^+(A)\cup\Gamma^-(A)$. If A is a singleton $A=\{x\}$, we define $d^+(x)=|\Gamma^+(\{x\})|$, $d^-(x)=|\Gamma^-(\{x\})|$, $d(x)=d^+(x)+d^-(x)$. Let A, B be two disjoint sets of vertices in G. We denote by $E(A)$ the set of edges in G both of whose endvertices belong to A. $E(A, B)$ is the set of edges joining a vertex in A and a vertex in B. If $B=V\backslash A$ we call $E(A, B)$ the *cut* associated with A. By $e(A)$, $e(A, B)$ we denote the cardinalities of $E(A)$, $E(A, B)$ respectively. If G is a digraph, A, B as above we denote by $E^+(A, B)$ the set of edges joining a vertex in A to a vertex in B. $e^+(A, B)$ is the cardinality of this set. In the sequel we sometimes add a subscript to Γ, d, E, e to clarify in which graph they are taken.

Let G be a digraph, $\{H_i\}$ a set of its subgraphs. $\{H_i\}$ is said to constitute a vertex (edge) *cover* of G if the H_i are vertex (edge) disjoint and the union of their vertex (edge) sets equals $V(G)$ ($E(G)$ resp.). In the sequel we shall refer to this as vertex (edge) cover or simply as cover when no confusion may arise. We wish to stress the fact *all covers are obtained by disjoint sets*. All the above definitions will be used freely for multigraphs, too.

1. Vertex covering by paths

A theorem due to Gallai–Milgram [5] (also [3, p. 298]) states:

Theorem GM. *A digraph G may be vertex covered by $\beta_0(G)$ paths.*

We establish a necessary condition for the minimality of a given cover. Let $M=\{\mu_i\}$ be a vertex cover of G by paths. A set of vertices $\{x_i\}$ with $x_i \in \mu_i$ is called a *transversal* of M. Also, let $A(M)=\{a_i\}$, $a_i \in \mu_i$, and $B(M)$ denote the initial vertices and terminal vertices, respectively, in M.

Theorem 1 and its proof are similar to Theorem GM and its proof, as presented in [3, p. 298].

Theorem 1. *Let G be a digraph and $M=\{\mu_i\}$ a vertex cover of G by paths. If M has no independent transversal, there exists a vertex cover N satisfying $A(N) \subsetneq A(M)$.*

Proof. The proof is by induction on $n=|V|$. The theorem is obviously true for $n \leqslant 2$. Let $A(M)=\{a_1, \ldots, a_t\}$ with $a_i \in \mu_i$. By assumption $A(M)$ is not independent so we may assume $[a_1, a_2] \in E$. If $\mu_1=\{a_1\}$, attach a_1 to μ_2 and the theorem follows. We assume therefore that a_1 has a successor in μ_1 which we call b_1. We apply the induction hypothesis to $G'=G\backslash a_1$. It is covered by $M'=\{\mu_1\backslash a_1, \mu_2, \ldots, \mu_t\}$ which does not have an independent transversal. Construct N' covering G' which satisfies $A(N') \subsetneq A(M')=\{b_1, a_2, \ldots, a_t\}$.

There are now three cases to be considered. If $b_1 \in A(N')$ attach a_1 to the path starting at b_1. If $b_1 \notin A(N')$, but $a_2 \in A(N')$ attach a_1 to the path starting at a_2. If $b_1, a_2 \notin A(N')$ add the path $\{a_1\}$ to N' to obtain N. In all three cases N has been obtained, fulfilling the assertion of the theorem.

We wish to sharpen Theorem 1. Let us make another two definitions. A transversal $\{x_i\}$ of a cover M is called *U-independent* if $[y_j, x_i] \in E$ for some $y_j \in \mu_j$, $i \neq j$, implies that y_j precedes x_j in μ_j. It is called *L-independent* if $[x_i, y_j] \in E$ for some $y_j \in \mu_j$, $i \neq j$, implies that x_j precedes y_j in μ_j.

By slightly altering the proof of Theorem 1 we achieve the following two results.

Theorem 1.1. *If M does not have a U-independent transversal, G may be vertex covered by N satisfying $A(N) \subsetneq A(M)$.*

Theorem 1.2. *If M does not have an L-independent transversal, G may be covered by N satisfying $B(N) \subsetneq B(M)$.*

Theorem 1 may be used to obtain an interesting corollary. Let $G_1, \ldots, G_m$ be a family of disjoint digraphs. A digraph G is said to be a *join digraph* of this family if:

1. $V(G) = \bigcup_{i=1}^{m} V(G_i)$.
2. Let $x, y \in V(G_i)$ then $[x, y] \in E(G)$ iff $[x, y] \in E(G_i)$.
3. Let $x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$, then exactly one of the edges $[x, y]$, $[y, x]$ belongs to $E[G]$.

A digraph which contains a Hamiltonian path is called an *H-digraph* in the remainder of the paper.

Corollary 1.1. *Let $G_1, \ldots, G_m$ be pairwise disjoint H-digraphs. Each of their join digraphs is an H-digraph.*

Proof. A simple inductive argument reduces the proof to the case $m = 2$. Let G_1, G_2 be H-digraphs. Their H-paths vertex-cover each of their join digraphs. However, there are no $x_1 \in V(G_1)$, $x_2 \in V(G_2)$ that are nonadjacent. By Theorem 1 the join is an H-digraph too.

Theorem GM and Theorem 1 make no use of two way edges. We may take advantage of these edges by means of Meyniel's theorem [7].

Theorem M. *Let G be a digraph in which $d(x) + d(y) \geq 2n - 1$ for every two nonadjacent vertices x, y. If G is strongly connected then G contains a Hamiltonian cycle.*

Theorem 2. *Let G be a digraph in which $d(x) + d(y) \geq l$ for every two nonadjacent vertices x, y. If $l \leq 2n - 2$, G may be vertex covered by $n - [(l+1)/2]$ paths.*

Proof. We first show how the assumption of Theorem M may be reduced to give sufficient conditions for the existence of an H-path in G.

Lemma 2.1. *Let G be a digraph such that $d(x) + d(y) \geq 2n - 3$ for every two nonadjacent vertices x, y. Then G is an H-digraph*

Proof. Add to G a new vertex t and join it by a two-way edge to each other vertex of G. Call the resulting digraph G'.

Evidently G' is strongly connected and if $x, y \in V(G)$ are nonadjacent, then

$$d_{G'}(x) + d_{G'}(y) = (d_G(x) + 2) + (d_G(y) + 2) \geq 2n + 1 = 2(n+1) - 1.$$

By Theorem M, G' contains a Hamiltonian cycle. Deleting t from the cycle results in an H-path in G.

We now return to the proof of Theorem 2. Let $m=[n-(l/2)]-1\geq 0$. Add to G, m new vertices and join each new vertex to all other vertices (old or new) by a two way edge. Call the resulting digraph G'. We show that G' satisfies the assumptions of Lemma 2.1. Let x, y be nonadjacent vertices in $V(G)$.

$$\begin{aligned} d_{G'}(x) &= d_G(x)+2m \\ d_{G'}(y) &= d_G(y)+2m \end{aligned} \tag{2.1}$$

So $d_{G'}(x)+d_{G'}(y)=d_G(x)+d_G(y)+4m\geq l+4m\geq 2(n+m)-3$.

By Lemma 2.1 G' contains a H-path. Deleting the m new vertices G is vertex covered by at most $m+1$ paths.

2. Edge covering by paths

This problem has been discussed already in [2, 8]. It is rather easy to prove that $[n^2/4]$ paths are always sufficient to edge-cover a digraph without two-way edges. It has been proved recently [8] *that any digraph on $n\geq 4$ vertices may be edge covered by $[n^2/4]$ paths.*

When one studies the problem of how to cover a digraph by $[n^2/4]$ paths, an interesting fact reveals itself. If G does not contain two-way edges (it is *asymmetric*) it may be edge covered by $[n^2/4]$ paths whose length does not exceed two. It is rather easy to see that in order to prove the last assertion we need only prove it for tournaments. So we have to prove: *A tournament on n vertices may be edge covered by $[n^2/4]$ subgraphs, each being either a couple or an edge.*

Let us examine why $[n^2/4]$ may not be reduced in the last statement. Choose $A\subseteq V$ containing $[n/2]$ vertices. Orient each edge in $E(A,\bar{A})$ from A to $\bar{A}$. Orient $E(A)\cup E(\bar{A})$ arbitrarily. Such a tournament needs $|A|\cdot|\bar{A}|=[n^2/4]$ paths to be covered. (Here $\bar{A}=V\setminus A$). Let us push this idea a little further. Let G be any graph; let $A\subseteq V$. Orient $E(A,\bar{A})$ from A to $\bar{A}$. Orient all other edges arbitrarily. There is no cover of this orientation by less than $e(A,\bar{A})$ paths. Let us denote

$$p=p(G)=\max\{e(A,\bar{A}): A\subseteq V\}.$$

Evidently the following statement, if true, is best possible:

Each orientation of G may be edge covered by p paths.

We now combine the two underlined unproved statements and we state:

Theorem .. *Let G be a directed multigraph. It may be edge covered by $p(G)$ paths whose length do not exceed two.*

It is evident that the theorem is equivalent to the statement: G contains $e-p$ edge disjoint couples. We remind the reader that a *matching* in a graph, or a digraph, is a set of edges, no two of which have a vertex in common. We translate Theorem 3 to an equivalent theorem on $\widetilde{\mathrm{DL}}(G)$, namely:

Theorem 3.1. *Let G be a directed multigraph, then $\widetilde{\mathrm{DL}}(G)$ contains a matching of $e(G)-p(G)$ edges.*

It is Theorem 3.1 which we prove directly. The proof relies on the following matching theorem of Berge. Given a graph G and a set of its vertices, S, let $t_S = t_S(G)$ denote the number of odd components in $G\backslash S$. (A component is odd or even, according to the number of vertices it contains.)

Theorem B [3, p. 159]. *A graph G contains a matching which does not meet k vertices or less iff*

$$t_S - |S| \leq k \tag{3.1}$$

for every $S \subseteq V$.

Note that G need not be connected as in [3, p. 159].

We prove in fact that $\widetilde{\mathrm{DL}}(G)$ has an $e-p$ matching. That is, a matching which does not meet $e-2(e-p)=2p-e$ vertices in $\widetilde{\mathrm{DL}}(G)$. We have to show that Berge's condition holds with $k=2p-e$. That is

$$t_S - |S| \leq 2p - e \tag{3.2}$$

for every $S \subseteq E(G)$. Here t_S is the number of odd components in $\widetilde{\mathrm{DL}}(G)\backslash S = \widetilde{\mathrm{DL}}(G\backslash S)$. If we set $H = G\backslash S$, we have to show that

$$t + e(H) \leq 2p(G) \tag{3.3}$$

where t is the number of odd components in $\widetilde{\mathrm{DL}}(H)$. To show that (3.3) holds for any subgraph H of G it is sufficient to prove the following lemma.

Lemma 3.1. *Let G be a directed multigraph. Let t be the number of odd components in* $\widetilde{\mathrm{DL}}(G)$. Then

$$t \leq 2p(G) - e(G). \tag{3.4}$$

A proof of Lemma 3.1 will provide us with a proof of Theorem 3.1. First we prove Lemma 3.1 under the extra assumption that G is asymmetric. So we prove, using the above notation:

Lemma 3.2. *Let G be a directed asymmetric multigraph. Then*

$$t \leq 2p(G) - e(G). \tag{3.4a}$$

Proof. We first note that $\mathrm{DL}(G)$ and $\widetilde{\mathrm{DL}}(G)$ coincide, since G is asymmetric. We prove (3.4a) by induction on n. For $n = 1$ it is trivially true. Let us assume that there is a vertex $x \in V(G)$ such that the edges incident with x in G (viewed as vertices of $\mathrm{DL}(G)$) span a connected subgraph of $\mathrm{DL}(G)$. Let (3.4a) hold for graphs of order smaller than n. Define $G' = G \setminus x$ and let e', p', t' stand for $e(G')$, $p(G')$, $t(G')$ respectively. We shall distinguish two cases:

Case I: $e(x, V\setminus x)$ is even. We show

$$t' \geq t \tag{3.5a}$$

$$2p - e \geq 2p' - e'. \tag{3.5b}$$

To prove (3.5a) remember that by assumption $E(x, V\setminus x)$ is contained in one component of $\mathrm{DL}(G)$. We have to show that when $E(x, V\setminus x) \subseteq V(\mathrm{DL}(G))$ is deleted from $\mathrm{DL}(G)$ the number of odd components in $\mathrm{DL}(G)$ does not decrease. If the component containing $E(x, V\setminus x)$ is even this is evident. Assume it is odd. If the deletion of $E(x, V\setminus x)$ does not disconnect the component we have $t' = t$. If it does, at least one of the new components is odd, and (3.5a) again holds.

To prove (3.5b) suppose that $E(B, C)$ is a maximum cut in G' (here $C = (V\setminus x)\setminus B$). Assume w.l.o.g. that $e(x, B) \geq e(x, C)$. Then:

$$\begin{aligned} 2p - e \geq 2e(B, C \cup x) - e &= 2e(B, C) + 2e(B, x) - (e' + e(B, x) + e(C, x)) \\ &= 2p' - e' + e(B, x) - e(C, x) \geq 2p' - 2e'. \end{aligned}$$

By the induction hypothesis we have

$$2p' - e' \geq t'. \tag{3.6}$$

From (3.5a), (3.5b) and (3.6) we obtain (3.4a).

Case II: $e(x, V\setminus x)$ is odd. Then

$$t' \geq t - 1 \tag{3.7a}$$

$$2p - e \geq 2p' - e' + 1. \tag{3.7b}$$

The proofs of (3.7a), (3.7b) follows along the same lines as above. Again from (3.7a), (3.7b) and (3.6) we obtain (3.4a).

We reconsider now the assumption made at the beginning of the proof. If $E(x, V\setminus x)$ spans a disconnected subgraph of $\mathrm{DL}(G)$, then $d^+(x) \cdot d^-(x) = 0$. If all vertices in G satisfy $d^+(x) \cdot d^-(x) = 0$ then G is a bipartite multigraph in which all the edges are oriented from one independent set of vertices to the other one. In such a directed multigraph $e = p = t$ holds so that (3.4) is valid. This completes the proof of Lemma 3.2.

The proof of Lemma 3.2 provides us with a proof of Theorems 3 and 3.1 under the extra assumption that G is asymmetric. To prove Lemma 3.1 we first follow the proof of Lemma 3.2. The only point in the proof which depends on the asymmetry of G is the assertion that if $E(x, V\setminus x)$ spans a disconnected subgraph of $\widetilde{\mathrm{DL}}(G)$ then $d^+(x) \cdot d^-(x) = 0$. To obtain a proof of Lemma 3.1 we investigate under what conditions does $E(x, V\setminus x)$ span a disconnected subgraph of $\widetilde{\mathrm{DL}}(G)$.

We say that x, y are *M-neighbours* (or *M-adjacent*) if $E(G)$ contains the two way edge joining x and y.

Lemma 3.3. *Let G be a directed multigraph and $x \in V(G)$. Then $E(x, V\backslash x)$ spans a disconnected subgraph of $\widetilde{\mathrm{DL}}(G)$ iff x satisfies one of the following conditions:*

A. $\Gamma(x) = \{y\}$; *x, y are M-adjacent.*
B. B_1 $d^+(x) = 0$, or
B_2 $d^-(x) = 0$
C. $|\Gamma(x)| = 2$; *both neighbours of x are M-neighbours.*
D. D_1 $|\Gamma(x)| \geq 2$, $\Gamma^+(x) = \{y\}$; *x, y are M-neighbours*, or
D_2 $|\Gamma(x)| \geq 2$, $\Gamma^-(x) = \{y\}$; *x, y are M-neighbours.*

Proof. Sufficiency is evident. To prove necessity assume that $E(x, V\backslash x)$ spans a disconnected subgraph of $\widetilde{\mathrm{DL}}(G)$. Because of conditions A and B we may assume that $|\Gamma(x)| \geq 2$. If none of the neighbours of x is an M-neighbour then condition B must hold. If exactly one neighbour is an M-neighbour, then condition D must be satisfied. If at least two neighbours are M-neighbours, then it is easily checked that $|\Gamma(x)| = 2$, so that condition C holds.

We shall now complete the proof of Lemma 3.1. This will be done by induction on n, the order of G. The lemma clearly holds if $n = 1$. Suppose that the lemma holds for multigraphs of order $<n$.

Denote by A, (B_1, B_2, C etc.) the set of vertices of G for which condition A, (B_1, B_2, C etc.) holds. Also let $B = B_1 \cup B_2$ and $D = D_1 \cup D_2$. If G has a vertex x not in $A \cup B \cup C \cup D$, then we can apply (at x) the proof of Lemma 3.2. Assume, therefore, that $V(G) = A \cup B \cup C \cup D$.

If G' is a multigraph, we use e', p', t' to denote $e(G')$, $p(G')$, $t(G')$ respectively.

Let $x \in A$ and let $G' = G\backslash x$. It is easy to verify that

$$e = e' + e(x, V\backslash x),$$
$$p = p' + e(x, V\backslash x),$$
$$t \leq t' + e(x, V\backslash x).$$

G' satisfies (3.4), hence so does G. We may assume therefore that $A = \emptyset$.

Let x, y be M-neighbours so that

$$E(x, V\backslash x) \cup E(y, V\backslash y) \text{ spans a subgraph of } \widetilde{\mathrm{DL}}(G) \text{ with at most two components.} \tag{3.8}$$

Let $G' = G\backslash\{x, y\}$, and let M, N satisfy $M \cup N = V\backslash\{x, y\}$, $M \cap N = \emptyset$, $e(M, N) = p'$. Clearly

$$p \geq p' + e(x, N) + e(y, M) + e(x, y)$$
$$p \geq p' + e(x, M) + e(y, N) + e(x, y).$$

The first inequality follows by adjoining x to M, and y to N. The second one follows by adjoining x to N, and y to M. Summing up we obtain

$$2p \geq 2p' + (e - e' + e(x, y))$$

Hence

$$2p - e \geq 2p' - e' + e(x, y) \geq t' + 2 \geq t.$$

The last inequality is an immediate consequence of (3.8). So we may assume that there are no M-adjacent x, y in G for which (3.8) holds. It follows that $C = \emptyset$, for if $x \in C$ and y is an M-neighbour of x, then x and y satisfy (3.8). The same argument shows that if $x, y \in D_1$ (or $x, y \in D_2$) then x, y are not M-adjacent. By definition of D_1, D_2, if two vertices in D_1 (or in D_2) are adjacent, then they are M-adjacent. Thus we conclude that $E(D_1) = E(D_2) = \emptyset$. It is also easily verified that $E(B_1, D_1) = E(B_2, D_2) = \emptyset$. Evidently $E(B_1) = E(B_2) = \emptyset$.

It follows that G is bipartite, $B_1 \cup D_1$ and $B_2 \cup D_2$ being independent sets. In a bipartite graph $e = p \geq t$ holds and the proof is complete.

3. Edge covering of a poset

The vertex covering problem of a partially ordered set (=poset) was remarkably solved by Dilworth [4]. Before we state and solve the edge covering problem, let us make two remarks on acyclic digraphs. It has been shown by Alspach and Pullman [2]:

Proposition AP. *A minimal edge cover of an acyclic digraph contains precisely* $\sum_{x \in V} \max\{d^+(x) - d^-(x), 0\}$ *paths.*

If G is acyclic, M a minimum edge cover of G by paths, then the initial vertex of one path in M cannot coincide with the terminal vertex of another path in M. That is why Proposition AP holds.

The following theorem is a direct consequence of Theorem GM.

Theorem 4. *Let G be an acyclic digraph. Then it may be edge covered by $\beta_1(G)$ paths.*

Proof. Apply Theorem GM to DL(G), As $\beta_0(\mathrm{DL}(G)) = \beta_1(G)$, DL($G$) is vertex covered by $\beta_1(G)$ paths. A path in DL(G) corresponds to a path in G which is not necessarily simple. However, if G is acyclic, every path in G is simple. This concludes the proof. Let us remark that it is also easy to derive Theorem 4 from Proposition AP.

A poset $(S, >)$ will be represented by a digraph $G = (V, E)$ where $V = S$ and $[x, y] \in E$ iff $x > y$. This representation enables us to use graph theoretic language for the poset. We shall interchangebly refer to S as both a poset and its representing digraph. We investigate minimal edge coverings of a poset by paths. Our main result in this section, Theorem 6F, may be formulated so that it resembles Dilworth's theorem (see Theorem 6.1).

We prepare some order theoretic terminology. A poset $(S, >)$ is said to be *locally finite* if $d_G(x) < \infty$ for every $x \in S$. Let S be a locally finite poset. An element $x \in S$ is called *large* provided $d_G^+(x) \geqslant d_G^-(x)$, *small* if $d_G^+(x) \leqslant d_G^-(x)$. We use S^+, S^- to denote the sets of large and small elements respectively. We also denote $M = S^+ \cap S^-$. A subset $A \subseteq S$ is an order *ideal* (*filter*) if $x < a$ ($x > a$) for some $a \in A$ implies $x \in A$. In particular S^+ is a filter and S^- is an ideal. Let $N \subseteq S$ be a filter, an edge $[x, y]$ ($x > y$) is called *N-mixed* if $x \in N$, $y \notin N$. A couple $[x, y]$, $[y, z] \in E$ (that is $x > y > z$) is called *N-mixed* if $x \in N$, $z \notin N$. An edge or a couple which is not N-mixed is said to be *N-unmixed.*

Lemma 5. *Let S be a finite poset. Then*

$$\max\{e^+(A, \bar{A}) : A \subseteq S\} = \beta_1(S) = e(S^+, S^- \backslash M) = e^+(S^+, S^- \backslash M). \tag{5.1}$$

where $\bar{A} = S \backslash A$.

Proof. The equality $e(S^+, S^- \backslash M) = e^+(S^+, S^- \backslash M)$ is evident. $E^+(A, \bar{A})$ does not contain a couple from which $\max e^+(A, \bar{A}) \leqslant \beta_1(S)$ follows. To prove the reverse inequality suppose $F \subseteq E$ does not contain a couple. Let $P = P(F)$, $Q = Q(F)$ be the sets of initial, resp. terminal vertices of the edges in F. By assumption P, Q are disjoint so $F \subseteq E^+(P, Q) \subseteq E^+(P, \bar{P})$ which proves $\max e^+(A, \bar{A}) \geqslant \beta_1(S)$.

All that remains to prove is that $\max e^+(A, \bar{A}) = e^+(S^+, S^- \backslash M)$. Let $B \subseteq S$ satisfy $e^+(B, \bar{B}) = \max e^+(A, \bar{A})$. We show that B is a filter and $\bar{B}$ is an ideal. If there is some $x \in B$, $y \in \bar{B}$ for which $y > x$ let $B' = B \cup y \backslash x$. One can easily check that

$$e^+(B', \bar{B}) > e^+(B, \bar{B})$$

holds. So in searching for a set A to maximize $e^+(A, \bar{A})$ we have to consider only filters. Let B be a maximizing subset. Let x be a minimal element in B. From $e^+(B \backslash x, \bar{B} \cup x) \leqslant e^+(B, \bar{B})$ it follows easily that $x \in S^+$. Hence, $y \in S^+$ for all $y \in B$ so that $S^+ \supseteq B$. By the same reasoning $S^- \supseteq \bar{B}$. In other words $B \supseteq S^+ \backslash M$, $\bar{B} \supseteq S^- \backslash M$. If $x \in M$ then $\Gamma^-(x) \subseteq S^+ \backslash M$, $\Gamma^+(x) \subseteq S^- \backslash M$, and $d^+(x) = d^-(x)$. Therefore we can move x from B to $\bar{B}$ and vice versa without affecting $e^+(B, \bar{B})$. This concludes the proof of the lemma.

Theorem 6F. *A finite poset S may be edge covered by edge disjoint S^+-mixed couples and S^+-mixed edges.*

The "edges" of a poset are of course, the edges of the representing graph i.e. the pairs $[x, y]$ where $x > y$.

Proof. The proof is based on the following theorem of Hall [6]. We use the formulation given in [3, p. 136]. If (X, Y, E) is a bipartite graph we say that X *may be matched* into Y if there is a matching meeting each vertex in X.

Theorem H. *Let* (X, Y, E) *be a bipartite graph. X may be matched into Y iff for every* $A \subseteq Y$

$$|Y \backslash A| \geq |X \backslash \Gamma(A)|. \tag{6.1}$$

We now return to the proof of Theorem 6F. Let $(S, >)$ be a finite poset. Let $G = (S, E)$ be the representing digraph.

We assume that for all $x \in S$, $d_G(x) > 0$. Otherwise replace S by the set $S' = \{x \in S : d_G(x) > 0\}$. The truth of Theorem 6F for S' clearly implies its validity for S.

Note that an edge $[[x, y], [y, z]]$ in $\mathrm{DL}(G)$ represents an ordered triple $x > y > z$ in S. Delete from $\mathrm{DL}(G)$ those edges which correspond to S^+ unmixed couples in G and denote the resulting subgraph of $\mathrm{DL}(G)$ by H.

Let X, Y be the sets of S^+ unmixed and S^+ mixed edges of G, respectively. Note that H is a bipartite digraph, with independent sets X, Y. It is easily checked that Theorem 6F is equivalent to the assertion that X can be matched into Y within H.

We show that condition (6.1) holds for H. Assume on the contrary that for some $A \subseteq Y$

$$|Y \backslash A| < |X \backslash \Gamma_H(A)|.$$

Or equivalently

$$|Y| - |X| < |A| - |\Gamma_H(A)|. \tag{6.2}$$

We wish to apply two reductions to (6.2). Let $P = P(A)$, $Q = Q(A)$ be the sets of initial, resp. terminal vertices in G of the edges in A. If we replace A by $E_G(P, Q)$ in (6.2) the inequality still holds, as $A \subseteq E_G(P, Q)$, but $\Gamma_H(A) = \Gamma_H(E_G(P, Q))$. P and Q are disjoint and we have

$$|Y| - |X| < e_G^+(P, Q) - |\Gamma_H(E_G(P, Q))|. \tag{6.3a}$$

Note also that P and Q satisfy the following conditions:

$$\text{For all } x \in P,\ e^+(x, Q) \neq 0, \quad \text{and for all } y \in Q,\ e^+(P, y) \neq 0. \tag{6.3b}$$

So to refute (6.2) it suffices to show that if $P \subseteq S^+$ and $Q \subseteq S^- \backslash M$, then they cannot satisfy (6.3a) and (6.3b) simultaneously.

The problem may be further reduced to the case where P is a filter, Q is an ideal, P and Q are disjoint and they satisfy (6.3a) and (6.3b). Suppose $x \notin P$, $y \in P$ satisfy $x > y$. Replacing P by $P \cup x \backslash y$ the r.h.s of (6.3a) increases, while (6.3b) remains valid. Since $P \subseteq S^+$, $Q \subseteq S^- \backslash M$, before the replacement and after it we have $P \cap Q = \emptyset$. We repeat the replacement procedure until we finally replace P

by a filter. When the same is performed on Q for $x \in Q$, $y \notin Q$, it is replaced by an ideal. P and Q remain disjoint and (6.3b) still holds.

If $P \cap Q = \emptyset$, $P \cup Q = S$ hold for some filter P and some ideal Q which satisfy (6.3a) and (6.3b), then

$$\Gamma_H(E_G(P, Q)) = E_G(P) \cup E_G(Q)$$

So (6.3a) becomes

$$|Y| - |X| < e_G^+(P, Q) - e_G(P) - e_G(Q). \tag{6.4a}$$

But evidently

$$e(G) = |Y| + |X| = e_G^+(P, Q) + e_G(P) + e_G(Q). \tag{6.4b}$$

Adding (6.4a) to (6.4b) we obtain

$$e^+(S^+, S^- \backslash M) = |Y| < e_G^+(P, Q),$$

which contradicts Lemma 5. Therefore $P \cup Q \neq S$. Next we choose a filter P and an ideal Q that are disjoint and satisfy (6.3a), (6.3b), such that $|P \cup Q|$ is maximal under these conditions. Every $x \notin P \cup Q$ satisfies $e_G^+(x, P) = e_G^+(Q, x) = 0$ as P is a filter and Q is an ideal.

Note that the quantity $e^+(x, Q)$ defined for $x \in S$, is a nondecreasing function of x. Similarly, $e^+(P, x)$ is nonincreasing in x. Therefore if for some $x \in S \backslash (P \cup Q)$ $e_G^+(x, Q) \geq e_G^+(P, x)$ and $e_G^+(x, Q) > 0$, then there is a maximal element x^* of $S \backslash (P \cup Q)$ for which $e_G^+(x^*, Q) \geq e_G^+(P, x^*)$ and $e^+(x^*, Q) > 0$. Replace P by $P \cup x^*$. This does not injure (6.3b) nor the disjointness of P and Q. The r.h.s. of (6.3a) increases by $e_G^+(x^*, Q) - e_G^+(P, x^*) \geq 0$. $P \cup x^*$ is a filter and this contradicts the maximality of $P \cup Q$. If for some $x \in S \backslash (P \cup Q)$, $e_G^+(x, Q) \leq e_G^+(P, x)$ and $e_G^+(P, x) > 0$ a similar construction is applied to obtain a contradiction.

We may assume then, that if $x \in S \backslash (P \cup Q)$, then $e_G^+(P, x) = e_G^+(x, Q) = 0$. That means, no element of $S \backslash (P \cup Q)$ is comparable to any element of $P \cup Q$. If $x \in S \backslash (P \cup Q)$ then x is comparable to some element of $S \backslash (P \cup Q)$ since, by assumption, $d_G(x) > 0$. Therefore there exists a pair x, y of elements in $S \backslash (P \cup Q)$, such that $x > y$, x is maximal in S and y is minimal in S. Replace P by the filter $P \cup x$ and Q by the ideal $Q \cup y$. These two sets are disjoint and satisfy (6.3b). The r.h.s. of (6.3a) has increased by one. This contradicts the maximality of $|P \cup Q|$ and concludes the proof of Theorem 6F.

We now give two reformulations of Theorem 6F. The first one exhibits the similarity of Theorem 6F and Dilworth's Theorem. We say that a set of edges is *independent* if it does not contain a couple. In Dilworth's theorem the minimal vertex cover is discussed. The maximal cardinality of an independent set of vertices (which is a trivial lower bound) is shown to be equal to this minimum. Quite the same is done in

Theorem 6.1. *In a minimum edge cover of a poset the number of paths equals the maximal cardinality of an independent set of edges. Moreover the cover may consist of paths whose length does not exceed two.*

To derive Theorem 6.1 from Theorem 6F we only have to observe that no path contains more than one edge in $E^+(S^+, S^-\setminus M)$. It is also worth noting that if we omit the restriction on the length of the paths, Theorem 6.1 is a special case of Theorem 4. This part of the theorem is also easily derivable from Proposition AP.

Another reformulation is:

Theorem 6.2. *A finite poset S contains $e(S^+)+e(S^-\setminus M)$ edge disjoint couples but no more.*

We extend Theorem 6F to infinite posets which satisfy a restrictive finitary condition. Let $(S, >)$ be an infinite poset, again we represent it by the (infinite) digraph $G=(S, E)$. If $u, v \in E$ form a couple, in this order, we denote this couple by (u, v). We use C to denote the set of all couples in G.

Theorem 6. *Let S be a locally finite poset, then it may be edge-covered by edge-disjoint S^+-mixed couples and S^+-mixed edges.*

Proof. The proof depends on Theorem 6F and on the following theorem of Rado [9]:

Theorem R. *Let Y, Z be sets, $\phi=\phi_Y$ the collection of all finite subsets of Y. Associate with every $A \in \phi$ a function $f_A : A \to Z$. Assume that for every $y \in Y$ the set $\{f_A(y) \mid y \in A \in \phi\}$ is finite. Then there exists a function $f : Y \to Z$ so that*

$$\text{for every } A \in \phi \text{ there exists a set } B \text{ satisfying } A \subseteq B \in \phi \text{ and}$$
$$f|_A = f_B|_A. \tag{6.5}$$

To prove Theorem 6 proceed as follows: Define $Y=E(=\{[x, y]: x \in S, y \in S, x > y\})$, and let $Z=E \cup \{0\}$, where 0 is an element not in E. For $A \in \phi_E$ let $\langle A\rangle$ be the set of all endpoints of edges in A. $\langle A\rangle$ is a finite subposet of S. Choose a fixed cover C_A of $\langle A\rangle$, which satisfies the conditions of Theorem 6F (relative to $\langle A\rangle$, of course). Now define f_A as follows. If $u \in A$, then either u itself belongs to C_A, or u is covered by a couple (u, v) or (v, u) in C_A. In the first case let $f_A(u)=0$, and in the second case let $f_A(u)=v$.

The finiteness condition in Theorem R is a direct consequence of the local finiteness of S. Let $f : E \to E \cup \{0\}$ be a function which satisfies (6.5), (as in Theorem R).

By the definition of f_A we have

$$\forall A \in \phi_E \qquad \forall u, v \in A \qquad f_A(u)=v \Leftrightarrow f_A(v)=u. \tag{6.6a}$$

Therefore

$$\forall u, v \in E \qquad f(u)=v \Leftrightarrow f(v)=u. \tag{6.6b}$$

For let $A=\{u, v\}$ and assume that $f(u)=v$. Consider B as in (6.5). We have

$f_B(u)=v$ so by (6.6a) $f_B(v)=u$. By (6.5) $f(v)=u$. This proves (6.6b). It is also evident that $f(u)=v$ implies that (u, v) or (v, u) is a couple in G.

Define

$$F_A=\{(u, v)\in C \mid u, v\in A, \quad f_A(u)=v\}$$
$$F=\{(u, v)\in C \mid u, v\in E, \quad f(u)=v\}.$$

By the above remarks, F is a collection of edge disjoint couples. Let S_1, S_2 be the sets of those vertices in S that are initial, resp. terminal, in some couple of F. We show that $S_1\cap S_2=\emptyset$. Assume on the contrary that (u, v), $(g, h)\in F$ and that the terminal vertex of v coincides with the initial vertex of g. Let $A=\{u, v, g, h\}$, then there exists a set B, $A\subseteq B\in\phi_E$ for which (u, v), $(g, h)\in F_B$. Thus, the couples (u, v) and (g, h) belong to the edge cover C_B of $\langle B\rangle$ and C_B does not satisfy the conditions of Theorem 6F, contrary to the choice of C_B.

We now denote $M=S\backslash(S_1\cup S_2)$. Let J be the set of edges that are not covered by F. By definition $J=f^{-1}(0)$. We let M_1 (resp. M_2) be the set of those vertices in M that are initial (resp. terminal) vertices in some edge of J. We show that $M_1\cap M_2=\emptyset$. If on the contrary there exist $u, v\in J$ so that $(u, v)\in C$, we choose $A=\{u, v\}$ and consider B as in (6.5). As $f(u)=f(v)=0$, also $f_B(u)=f_B(v)=0$. This means that u and v belong to C_B, which again contradicts the conditions of Theorem 6F.

We further show that if $(u, v)\in F$ and $t\in J$ it is impossible that $(v, t)\in C$. If such u, v, t exist in E, let $A=\{u, v, t\}$ and let B be as in (6.5). By assumption $f_B(t)=0$, $(u, v)\in F_B$ which again contradicts the choice of C_B. Similarly we find that if $(u, v)\in F$, $t\in J$ it is impossible that $(t, u)\in C$.

By definition of J it is clear that $F\cup J$ cover all edges of S. We show that $S^+\backslash M=S_1\cup M_1$ and the proof is thus complete. We have already shown that no vertex in S is both the head of an element in $F\cup J$ and a tail of another element in $F\cup J$. Therefore, $x\in S^+\backslash M$ iff x is the head of some element of $F\cup J$. In other words $S^+\backslash M=S_1\cup M_1$.

4. Edge-covering by couples and edges

Let us remind the reader that for a digraph G, we define $\beta_1(G)$ as $\beta_0(\widetilde{\mathrm{DL}}(G))$. We also note that

$$\beta_1(G)=\max\{e^+(A, V\backslash A) \mid A\subseteq V\}.$$

A proof of this fact is contained in the first paragraph of the proof of Lemma 5. Evidently, a digraph G cannot be edge-covered by fewer than $\beta_1(G)$ couples and edges. The purpose of this section is to study digraphs G which have an edge-cover of $\beta_1(G)$ couples and edges. Digraphs which represent partial orders enjoy this property as is shown in Theorem 6.1. Not every acyclic digraph shares this property. This is shown by the digraph

$$D_1=(V_1,E_1),\quad \text{where}\quad V_1=\{1,2,3,4,5,\},$$
$$E_1=\{[1,2],[2,3],[3,4],[4,5],[2,4]\}.$$

It is easily verified that $\beta_1(D_1)=2$ whereas D has 5 edges.

Let $(S,>)$ be a poset. It is quite customary to deal with $(S,>)$ by means of its *reduced representing digraph* $G=(S,E)$ where $[x,y]\in E$ iff $x>y$ and no $z\in S$ satisfies $x>z>y$. Unlike digraphs which represent partial orders in the sense of Theorem 6.1, reduced representing digraphs G do not necessarily have an edge-cover of $\beta_1(G)$ couples and edges. This is demonstrated by the digraph $D_2=(V_2,E_2)$, where $V_2=\{1,\ldots,7\}$, $E_2=\{[i,i+1]\mid 1\le i\le 5\}\cup\{[2,7],[7,5]\}$. We omit the details of this example.

Now we find some conditions on a digraph G which ensure that G can be edge-covered by $\beta_1(G)$ couples and edges. The digraph G enjoys this property iff it contains $e(G)-\beta_1(G)$ edge-disjoint couples, or, equivalently, if $\widetilde{\mathrm{DL}}(G)$ contains a matching of $e(G)-\beta_1(G)$ edges.

By definition we have $\beta_1(G)=\beta_0(\widetilde{\mathrm{DL}}(G))$, and the problem is: for which digraphs G does $\widetilde{\mathrm{DL}}(G)$ contain a matching of $v(\widetilde{\mathrm{DL}}(G))-\beta_0(\widetilde{\mathrm{DL}}(G))$ edges. It follows from one of the consequences of Hall's Theorem (see [3, p. 133]) that this is the case when $\widetilde{\mathrm{DL}}(G)$ is (an orientation of) a bipartite graph.

This observation leads to the following:

Theorem 7. *Let $G=(V,E)$ be a digraph, and assume that*

$$E=E^+(A,V\backslash A)\cup E^+(B,V\backslash B),$$

for some $A,B\subseteq V$. Then it is possible to cover the edges of G by $\beta_1(G)$ (but not fewer) couples and edges

To prove Theorem 7 observe that under these assumptions $\widetilde{\mathrm{DL}}(G)$ is bipartite with independent sets: $E^+(A,V\backslash A)$ and $E\backslash E^+(A,V\backslash A)$. The proof now follows from the above mentioned corollary of Hall's Theorem. The following is an interesting special case of the above.

Corollary 7.1. *If G is an orientation of a bipartite graph, then it is possible to cover the edges of G by $\beta_1(G)$ couples and edges*

If (A,B) is a partition of $V(G)$ into two independent sets, then the assumption of Theorem 7 holds and the assertion follows.

As a consequence of Theorem 7 we obtain:

Theorem 7.1 *Let G be a digraph which contains no cyclic triangle and no directed path of length 4. Then G can be edge-covered by $\beta_1(G)$ couples and edges.*

Proof. If G contains a directed cycle, then it must have length 4. This follows directly from the assumptions of the theorem. Moreover, it is easily seen that such a cycle is a component of G. Therefore, we may assume that G is acyclic.

Define a height function h on $V(G)$ as follows: For $x \in V = V(G)$, let $h(x)$ be the maximum length of a directed path in G which terminates at x. By our assumptions on G, $0 \leq h(x) \leq 3$ for every $x \in V$. Since G is assumed to be acyclic, it is easy to verify that $[x, y] \in E$ implies $h(x) < h(y)$. For $i = 0, 1, 2, 3$ let

$$V_i = \{x \in V \mid h(x) = i\}.$$

By the preceeding remark

$$E = \cup\{E^+(V_i, V_j) \mid 0 \leq i < j \leq 3\}.$$

Thus E is the union of $E^+(V_0 \cup V_1, V_2 \cup V_3)$ and $E^+(V_0 \cup V_2, V_1 \cup V_3)$. The conditions of Theorem 7 hold and the proof is thus complete.

Since components of dilinegraphs play a major role in the above discussions, we now consider the following problem: Let G be a digraph, what are the components of $\mathrm{DL}(G)$? We may assume w.l.o.g. that for every $x \in V(G)$, $d(x) > 0$ for an isolated vertex in G has no influence on $\mathrm{DL}(G)$. Denote

$$M = \{x \in V \mid d^+(x) = 0\}, \; P = \{x \in V \mid d^-(x) = 0\}.$$

We prove:

Theorem 8. *Let G be a digraph, let $C_1, \ldots, C_l$ be the components of $G \backslash (P \cup M)$ and let V_k be the vertex set of C_k. Then the components of* $\mathrm{DL}(G)$ are spanned by the following elements or sets:

1. *Each element of $E^+(P, M)$*
2. $E^+(P, V_k) \cup E(V_k) \cup E^+(V_k, M) \qquad (k = 1, \ldots, l)$.

Proof. First we show that each of these sets spans a connected subgraph of $\mathrm{DL}(G)$. Evidently each element of $E^+(P, M)$ is an isolated vertex in $\mathrm{DL}(G)$. Let us show that $E_k = E^+(P, V_k) \cup E(V_k) \cup E^+(V_k, M)$ spans a connected subgraph of $\mathrm{DL}(G)$. To show this we have to show that for every $u, v \in E_k$ there exists a sequence $u = u_0, \ldots, u_m = v$ in $E = E(G)$ such that u_{i-1} and u_i form a couple in G for $i = 1, \ldots, m$. Each of the edges u, v has an endvertex which does not belong to $P \cup M$. Let x be an endvertex of u, y an endvertex of v, $x, y \notin P \cup M$.

Since, x, y belong to the same component C_k of $G \backslash (P \cup M)$, we can find a path (not necessarily directed) connecting them in C_k, say $x = z_0, z_1, \ldots, z_m = y$. For $1 \leq j \leq m$, let $w_j \in E_k$ be the directed edge which contains z_{j-1} and z_j as endvertices, i.e. $w_j = [z_{j-1}, z_j]$ or $w_j = [z_j, z_{j-1}]$. Let also $w_0 = u$, $w_{m+1} = v$. We only need to show that for $1 \leq j \leq m+1$ either w_{j-1} and w_j form a couple in G or there is an edge w^* in E_k which forms a couple with w_{j-1} and also with w_j. If w_{j-1} and w_j do not form a couple, then z_{j-1} is the head or the tail of both w_{j-1} and w_j. Assume, w.l.o.g., that z_{j-1} is the head of w_{j-1} and of w_j. Since $z_{j-1} \in V_k \subseteq V(G) \setminus (P \cup M)$, z_{j-1} is the tail of an edge $w^* = [z_{j-1}, z^*]$. If $z^* \in M$, then $w^* \in E^+(V_k, M)$, otherwise $z^* \in V_k$. In any case, $w^* \in E_k$, and (w^*, w_{j-1}), (w^*, w_j) are couples in G.

We have shown that E_k spans a connected subgraph of DL(G) for $k = 1, \ldots, l$. We still have to show that this connected subgraph is indeed a component of DL(G). Since $E^+(P, M) \cup \bigcup_{k=1}^{l} E_k = E(G) = V(\mathrm{DL}(G))$, it suffices to show that no edge in E_k forms a couple with an edge in E_j for $j \neq k$. This is evident since no couple in G has a vertex in $P \cup M$ as its medial vertex. This concludes the proof of Theorem 8.

Acknowledgement

This paper contains part of author's Ph.D thesis. The author wishes to convey his gratitude to Professor M. Lewin from the Technion, Haifa, and to Professor M. A. Perles of the Hebrew University, Jerusalem for their guidance.

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