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Every Poset Has a Central Element

NATHAN LINIAL

Institute of Mathematics and Computer Science,
Hebrew University,
Jerusalem 91904, Israel

AND

MICHAEL SAKS*

Department of Mathematics,
Rutgers University,
New Brunswick, New Jersey 08903

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It is proved that there exists a constant δ, $\frac{1}{2} > \delta > 0$, such that in every finite partially ordered set there is an element such that the fraction of order ideals containing that element is between δ and $1-\delta$. It is shown that δ can be taken to be at least $(3\text{-}\log_2 5)/4 \cong 0.17$. This settles a question asked independently by Colburn and Rival, and Rosenthal. The result implies that the information-theoretic lower bound for a certain class of search problems on partially ordered sets is tight up to a multiplicative constant. © 1985 Academic Press, Inc.

I. INTRODUCTION

The problem considered in this paper can be roughly stated as, "Does every finite partially ordered set have an element that belongs to approximately half of the order ideals?" This question, which arises out of a problem in computational complexity (see below) was raised independently by Colbourn and Rival, and Rosenthal (see [Sa]) and the authors [LS]. To formulate the problem precisely, say that an element x belonging to a finite partially ordered set $(P, \leqslant)$ is *δ-central* for some positive real number δ if the fraction of order ideals of P containing x is between δ and $1-\delta$. We are then asking whether there exists a constant $0 < \delta < \frac{1}{2}$ such

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that every finite poset has a δ-central element. Sands [Sa] provided a partial solution by proving that for each integer $h > 0$, there is a constant $\delta_h > 0$ such that every poset of height h (having largest chain of cardinality $h+1$) has a δ_h-central element; however, the values δ_h obtained approach 0 as h gets large. In this paper we settle the question affirmatively by proving

THEOREM 1.1. *In any finite partially ordered set* $(P, \leqslant)$ *there is an element* $x \in P$ *such that the fraction of order ideals of* P *that contain* x *is between* δ_0 *and* $1-\delta_0$, *where* $\delta_0 = (3 - \log_2 5)/4 \cong 0.17$.

Shearer [Sh] has recently shown that there exist posets which have no δ-central element if $\delta \gtrsim 0.197$, so the bound in the above theorem is not far from best possible.

Our interest in this question arose from the problem of searching partially ordered data structures, which we considered in [LS]. A *partially ordered data structure* is a poset P together with an order preserving injection $s: P \to R (x <_P y$ implies $s(x) < s(y))$. We think of the value $s(x)$ as being *stored* at location x in P. The *data location problem* for such a structure is: given a real number r determine whether r is stored in P and, if so, at which location (element). The basic step is a comparison of r to the value stored at some element x of P. Of interest is the minimum number of comparisons which are sufficient to solve the location problem in the worst case.

The hardest case of the location problem occurs when r is not stored in P, and establishing that r is not stored in P requires the identification of the ideal $I(r)$ of elements $x \in P$ such that $s(x) < r$. Thus the location problem is essentially the same as the *ideal identification problem*: given a poset P and an unknown ideal $I \subseteq P$, determining I using queries of the form "is $x \in I$?"

The fundamental lower bound on the number of queries required for this problem is the *information-theoretic bound* which is obtained as follows. The query "is $x \in I$?" partitions the set of possible ideals into two sets, those containing x and those not containing x. The response to the query eliminates from consideration all ideals in one of these two sets. It is possible that the set eliminated is the smaller of the two and the set of possible ideals is reduced by at most half. Thus, in worst case, at least $\log_2 i(P)$ queries are necessary to identify I (where $i(P)$ is the number of ideals of P).

The natural question is, of course, how close can we come to this bound? First, note that after each successive question we are reduced to a smaller problem of the same type since an answer of "yes" to "is $x \in I$?" effectively reduces the problem to the ideal identification problem on $P \backslash I(x)$, where $I(x)$ is the set of elements $\leqslant x$, and an answer of "no" reduces the problem to the same problem on $P\backslash F(x)$, where $F(x)$ is the set of elements $\geqslant x$. To attain the information-theoretic bound or close to it, we would need, at each stage, to ask about an element x that is in roughly half of the ideals in the residual poset. Whether this is always possible is the problem addressed by Theorem 1.1. More precisely, testing a δ-central element $(0<\delta<\frac{1}{2})$ at each stage, ensures that the number of possible ideals is reduced by a factor of at least δ, thereby guaranteeing the identification of the ideal I using at most $\log_{1/1-\delta} i(P)$ queries. Thus, as a consequence of Theorem 1.1, we have

THEOREM 1.2. *The number of queries required to solve either the ideal identification problem or the data location problem in a poset P is at most* $K_0 \log_2 i(P)$ *where* $K_0 = 1/(2-\log_2(1+\log_2 5)) \cong 3.73$. *Thus the information-theoretic bound is tight up to a multiplicative constant.*

We mention also that Theorem 1.1 can be restated via the Birkhoff representation theorem for distributive lattices [Bi] as

THEOREM 1.3. *Every finite distributive lattice L contains a prime ideal I (an ideal generated by a single meet irreducible element) such that*

$$\delta_0 \leqslant |I|/|L| \leqslant 1-\delta_0,$$

where $\delta_0 = (3-\log_2 5)/4$, *which settles a question of Colbourn and Rival* [Sa].

After a few preliminaries in Section 2, we present the proof of Theorem 1.1 in Sections 3–7. Sections 8–10 are devoted to some related results and questions.

II. Preliminaries

The notation for partially ordered sets and graphs is standard; see, for example, [Bi] and [Be]. An *order ideal* of a poset $(P, \leqslant)$ is a subset I such that if $y \in I$ and $x \leqslant y$ then $x \in I$. A *filter* is a subset that is the complement of an ideal. A bipartite graph Γ with bipartition A, B and edge set E is denoted (A, B, E). For $X \subseteq A$, $\Gamma(X)$ is the set of vertices in B which are adjacent to some vertex of X. A *stable set* is a set of mutually unrelated vertices.

All logarithms are to base 2.

We will be dealing extensively with real set functions; the domain of such a function is the set of subsets of some finite set S. A set function $f: 2^S \to R$ is additive if $f(\varnothing)=0$ and for $A \subseteq S$, $f(A)=\sum_{a \in A} f(a)$. The function f is multiplicative if $\log f$ is additive, that is, $f(0)=1$ and $f(A)=\prod_{a \in A} f(a)$ for each $A \subseteq S$. f is *log supermodular* (LSM) if $f(A \cup B)f(A \cap B) \geqslant f(A)f(B)$ for all $A, B \subseteq S$. f is *normalized log supermodular* (NLSM) if in addition $f(0) = 1$. Clearly all multiplicative functions are NLSM.

For a set function g on S we denote by $\hat{g}$ the set function defined by $\hat{g}(A) = \sum_{A' \subseteq A} g(A')$. The following proposition is one version of the principle of inclusion–exclusion (see, e.g., [Ai]).

PROPOSITION 2.1. *Let h and g be set functions on S. Then $h = \hat{g}$ if and only if for all $A \subseteq S$,*

$$g(A) = \sum_{A' \subseteq A} (-1)^{|A| - |A'|} h(A').$$

We state for reference the following simple

PROPOSITION 2.2. *Let h and g be set functions of S such that $h = \hat{g}$. Then h is multiplicative if and only if g is. Furthermore if both are multiplicative we have for all $A \subseteq S$,*

(i) $h(A) = \prod_{a \in A} (g(a) + 1)$,

(ii) $g(A) = \prod_{a \in A} (h(a) - 1)$.

III. PROOF OF THEOREM 1.1

Given a poset $(P, \leqslant)$ let N be the number of order ideals of P. For $x \in P$, let $p(x)$ be the proportion of order ideals that contain x. Our aim is to show the existence of a constant $\frac{1}{2} > \delta > 0$ such that every poset has an element x for which $1 - \delta \geqslant p(x) \geqslant \delta$. (The value $\delta_0 = \frac{1}{4}(3 - \log 5)$ will come out of the proof; for the moment we leave δ unspecified.)

Let J be the set of elements x such that $p(x) \geqslant \frac{1}{2}$ and $K = P \backslash J$. Since $p(x)$ is a decreasing function on P, J is an order ideal and K is a filter. We need to show that one of the followng holds:

(i) there is an element $x \in J$ such that $p(x) \leqslant 1 - \delta$,

(ii) there is an element $x \in K$ such that $p(x) \geqslant \delta$.

Clearly the search for such an element can be restricted to the sets A of maximal elements of J, and B of minimal elements of K. We will view the subposet induced on $A \cup B$ as a bipartite graph $\Gamma = (A, B, E)$.

It is natural to first consider (as Sands [Sa] did) the case where P is height one. In this case, J is the set of minimal elements of P and $A = J$ and $B = K$ and thus the vertex set of Γ is all of P. A set $X \cup Y$ with $X \subseteq A$ and $Y \subseteq B$ is an ideal if and only if $(A \backslash X) \cup Y$ is a stable set of Γ. Thus the stable sets of Γ are in one to one correspondence with the ideals of P.

Moreover, for $b \in B$, the ideals of P containing b are in one to one correspondence with the stable sets of Γ containing b and for $a \in A$ the ideals not containing a are in one to one correspondence with the stable sets of Γ containing a. Thus if we define $p^*(x)$ to be the fraction of stable sets of Γ containing x, then $p^*(x) = p(x)$ if $x \in B$ and $p^*(x) = 1 - p(x)$ if $x \in A$. Thus the existence of a δ-central element in every height one poset is equivalent to:

In any bipartite graph $\Gamma = (A, B, E)$ some vertex x belongs to a fraction of at least δ of the stable sets, i.e., $p^*(x) \geqslant \delta$. (3.1)

This was proved by Sands for $\delta \cong 0.12$, but his argument does not seem applicable to posets of height greater than one. Our aim was to find a new proof which could be extended. To this end, we considered the stronger conjecture that the average of $p^*(x)$ over $x \in P$ is at least δ. Now this average is equal to

$$\frac{1}{|P|} \sum_{x \in P} p^*(x) = \frac{1}{|P| N} \sum_{x \in P} \sum_{\substack{S \text{stable} \\ x \in S}} 1$$
$$= \frac{1}{N} \sum_{S \text{stable}} \frac{|S|}{|P|}, \tag{3.2}$$

which is the average proportion of elements in a stable set of Γ. Thus (3.1) is a corollary of

THEOREM 3.1. *In any bipartite graph $\Gamma = (A, B, E)$ the average size of a stable set of Γ is at least $\delta_0 |A \cup B|$.*

We present a proof of this result in Section 4. Another proof has been found independently by Erdos and Sands [ES]. A very simple proof, which we present in Section 9, has also been given by N. Alon (for a smaller value of δ). Our proof, however, leads to an approach to the general problem for posets of arbitrary height.

In this general case, we first need to find the appropriate generalization of the above correspondence between ideals of P and stable sets of Γ. Say a subset T of P is *quasi-stable* if it is the symmetric difference of J and some ideal I of P, i.e., $T = (J \backslash I) \cup (I \backslash J)$. The following characterization is obvious.

PROPOSITION 3.2. *T is quasi-stable* iff *$T \cap K$ is an ideal in the subposet K, $T \cap J$ is a filter in the subposet J and every element of $T \cap K$ is incomparable to every element of $T \cap J$.*

The quasi-stable sets of P are in natural one to one correspondence with the ideals of P. For $x \in P$, let $p^*(x)$ be the proportion of quasi-stable sets containing x. If $x \in K$, then $p^*(x) = p(x)$, and if $x \in J$, then $p^*(x) = 1 - p(x)$. Thus, as in the bipartite case, the existence of a δ-central element in P is equivalent to the existence of an element for which $p^*(x) \geqslant \delta$.

Pursuing the analogy to the height one case, we tried to show that the average of $p^*(x)$ is at least δ_0. In general this is false. However, it is sufficient for our purposes to show that the elements of P can be assigned nonnegative weights $\lambda: P \to R^+$ such that the λ-weighted average of $p^*(x)$ over x in P is at least δ, i.e.,

$$\sum_{x \in P} \lambda(x)\, p^*(x) \geqslant \delta \sum_{x \in P} \lambda(x). \tag{3.3}$$

It will be convenient to view λ as an additive set function on P by defining $\lambda(T) = \sum_{x \in T} \lambda(x)$. Now we have

$$\begin{aligned}\sum_{x \in P} \lambda(x)\, p^*(x) &= \sum_{x \in P} \lambda(x) \frac{1}{N}\Bigg(\sum_{\substack{T\text{quasi-stable}\\ x \in T}} 1\Bigg)\\ &= \frac{1}{N} \sum_{T\text{quasi-stable}} \sum_{x \in T} \lambda(x)\\ &= \frac{1}{N} \sum_{T\text{quasi-stable}} \lambda(T).\end{aligned} \tag{3.4}$$

Thus, Theorem 1.1 follows from

THEOREM 3.3. *For any poset P there is an additive set function λ on P such that the average λ-weight of a quasi-stable set is at least $\delta_0\, \lambda(P)$*, i.e.,

$$\frac{1}{N} \sum_{T\text{quasi-stable}} \lambda(T) \geqslant \delta_0 \lambda(P).$$

Note that for the case of height 1 posets, this follows from Theorem 3.1 with the weight $\lambda(T) = |T|$.

Before proceeding with the proofs of these results, we define some additional notation.

$$\mathscr{K} \equiv \text{set of ideals of the subposet } K,$$
$$\mathscr{J} \equiv \text{set of filters of the subposet } J.$$

For $L \subseteq K$,

$$\mathscr{K}(L) \equiv \{I \in \mathscr{K} \mid I \subseteq L\},$$
$$k(L) \equiv |\mathscr{K}(L)|.$$

For $H \subseteq J$,

$$C(H) \equiv \{ y \in K \mid y \text{ is incomparable to every element of } H \}.$$

For $X \subseteq A$,

$$f(X) \equiv |\{F \in \mathscr{J} \mid F \cap A = X\}|,$$
$$\hat{f}(X) \equiv |\{F \in \mathscr{J} \mid F \cap A \subseteq X\}| = \sum_{X' \subseteq X} f(X').$$

For $Y \subseteq B$,

$$g(Y) \equiv |\{I \in \mathscr{K} \mid I \cap B = Y\}|,$$
$$\hat{g}(Y) \equiv |\{I \in \mathscr{K} \mid I \cap B \subseteq Y\}| = \sum_{Y' \subseteq Y} g(Y').$$

IV. Proof of Theorem 3.1

For $X \subseteq A$, let $\Gamma(X) = \{ y \in B \mid [x, y] \in E \text{ for some } x \in X\}$. For $Y \subseteq B$, the set $X \cup Y$ is stable if and only if $Y \subseteq B \backslash \Gamma(X)$; thus the average size of a stable set of Γ is equal to

$$\begin{aligned}
&\frac{1}{N} \sum_{X \subseteq A} \sum_{Y \subseteq B \backslash \Gamma(X)} |X| + |Y| \\
&\quad = \frac{1}{N} \sum_{X \subseteq A} |X| \, 2^{|B \backslash \Gamma(X)|} + \frac{1}{2} |B \backslash \Gamma(X)| \, 2^{|B \backslash \Gamma(X)|} \\
&\quad = \sum_{X \subseteq A} \frac{2^{|B \backslash \Gamma(X)|}}{N} \left(|X| + \frac{1}{2} |B \backslash \Gamma(X)| \right).
\end{aligned} \tag{4.1}$$

To prove the theorem we must show that this quantity is greater than or equal to $\delta_0(|A| + |B|)$.

For $X \subseteq A$, let $q(X) = 2^{|B \backslash \Gamma(X)|}/N$. Since $N = \sum_{X \subseteq A} 2^{|B \backslash \Gamma(X)|}$ we have $\sum q(X) = 1$; $q(X)$ can be interpreted as the probability that the intersection of A with a randomly chosen stable set S is X. Rewriting (4.1) in terms of $q(X)$ and simplifying yields the following expression:

$$\begin{aligned}
&\frac{1}{2} \sum_{X \subseteq A} q(X)(2|X| + \log(q(X) N)) \\
&\quad = \frac{\log N}{2} + \frac{1}{2} \sum_{X \subseteq A} q(X)(2|X| + \log q(X)).
\end{aligned}$$

Thus we must show that

$$\log N + \sum_{X \subseteq A} q(X)(2|X| + \log q(X)) \geqslant 2\delta_0(|A| + |B|). \tag{4.2}$$

We need the following simple

LEMMA 4.1. *Let $q: S \to [0, 1]$ be a discrete probability mass function on the finite set S, and let $f: S \to R^+$ be any function. Then*

$$\sum_{X \in S} (f(X) + \log q(X))\, q(X) \geqslant -\log \sum_{X \in S} 2^{-f(X)}.$$

Proof. A routine exercise using Lagrange multipliers. ∎

From (4.2) and Lemma 4.1 with $f(X) = 2|X|$ it now suffices to show that

$$\log N - \log \sum_{X \subseteq A} 4^{-|X|} \geqslant 2\delta_0(|A| + |B|). \tag{4.3}$$

Now

$$\sum_{X \subseteq A} 4^{-|X|} = \sum_{k=0}^{|A|} \binom{|A|}{k} 4^{-k} = \left(\frac{5}{4}\right)^{|A|}$$

so

$$\log \sum_{X \subseteq A} 4^{-|X|} = |A| \log \frac{5}{4}.$$

Without loss of generality we may assume $|A| \geqslant |B|$. Also, since each subset of A is stable, $N \geqslant 2^{|A|}$ so (4.3) follows from

$$|A| - |A| \log \tfrac{5}{4} \geqslant 4\,|A|\,\delta_0$$

which holds provided that $\delta_0 \leqslant (3 - \log 5)/4$.

V. PROOF OF THEOREM 3.3

The strategy for proving Theorem 3.3 is to mimic the proof of Theorem 3.1 as closely as possible, modifying the details when necessary. The key difference is that here we must find an additive set function $\lambda: 2^P \to R^+$ that validates the theorem. We will restrict our search to those functions λ with support on $A \cup B$, i.e., those satisfying

$$\lambda(x) = 0 \qquad \text{for } x \notin A \cup B. \tag{5.1}$$

For the moment we postpone the choice of λ, and, using a derivation analogous to (but more complicated than) that of (3.1), reduce the theorem to another inequality which is more convenient to work with.

By Proposition 3.2, the left-hand side of (3.5) can be rewritten as

$$\begin{aligned}\frac{1}{N}&\sum_{T\text{quasi-stable}}\lambda(T)\\&=\frac{1}{N}\sum_{H\in\mathscr{I}}\sum_{L\in\mathscr{K}(C(H))}[\lambda(H)+\lambda(L)]\\&=\frac{1}{N}\sum_{H\in\mathscr{I}}\left[k(C(H))\,\lambda(H)+\sum_{L\in\mathscr{K}(C(H))}\lambda(L)\right].\end{aligned}\tag{5.2}$$

Now by the additivity of λ and the assumption (5.1),

$$\begin{aligned}\sum_{L\in\mathscr{K}(C(H))}\lambda(L)&=\sum_{L\in\mathscr{K}(C(H))}\sum_{y\in L\cap B}\lambda(y)\\&=\sum_{y\in C(H)\cap B}\lambda(y)(k(C(H))-k(C(H)\setminus y))\\&\geqslant\sum_{y\in C(H)\cap B}\lambda(y)\,\tfrac{1}{2}k(C(H))=\tfrac{1}{2}\lambda(C(H)\cap B)\,k(C(H)),\end{aligned}\tag{5.3}$$

where the inequality follows from the observation that if I is an ideal in $\mathscr{K}(C(H)\setminus y)$, then I and $I\cup\{y\}$ are ideals in $\mathscr{K}(C(H))$.

Substituting (5.3) into (5.2), we obtain

$$\frac{1}{N}\sum_{T\text{quasi-stable}}\lambda(T)\geqslant\frac{1}{N}\sum_{H\in\mathscr{I}}k(C(H))\left[\lambda(H\cap A)+\frac{1}{2}\lambda(C(H)\cap B)\right].$$

Thus to prove Theorem 3.3, it is enough to exhibit a nonnegative additive set function λ on $A\cup B$ such that

$$\begin{aligned}\frac{1}{N}&\sum_{H\in\mathscr{I}}k(C(H))[2\lambda(H\cap A)+\lambda(C(H)\cap B)]\\&\geqslant 2\delta_0(\lambda(A)+\lambda(B)).\end{aligned}\tag{5.4}$$

We now need to construct an appropriate additive set function λ on $A\cup B$. The idea is to continue along the lines of Theorem 3.1 and see what properties λ must satisfy in order that the proof generalizes. These considerations lead us to

LEMMA 5.1. *If λ is an additive set function on $A \cup B$ satisfying*

$$\begin{aligned} &\lambda(X) \geqslant \log \hat{f}(X), \quad \forall X \subseteq A \quad \text{and} \quad \lambda(A) = \log \hat{f}(A), \\ &\lambda(Y) \geqslant \log \hat{g}(Y), \quad \forall Y \subseteq B \quad \text{and} \quad \lambda(B) = \log \hat{g}(B); \end{aligned} \tag{5.5}$$

then λ satisfies (5.4)

Once this is proved, Theorem 3.3 will follow from the existence of an additive set function λ satisfying (5.5), which is a consequence of the following two lemmas.

LEMMA 5.2. *The functions $\hat{f}$: $2^A \to R$ and $\hat{g}$: $2^B \to R$ are* NLSM *functions*

LEMMA 5.3. *Let h: $2^S \to R^+$ be a* NLSM *function. Then there exists a multiplicative function H: $2^S \to R^+$ satisfying $H(T) \geqslant h(T)$ for all $T \subseteq S$ and $H(S) = h(S)$.*

Applying this lemma to $\hat{f}$ and $\hat{g}$ produces multiplicative functions α and β. The function λ defined by $\lambda(X) = \log \alpha(X)$ for $X \subseteq A$ and $\lambda(Y) = \log \beta(Y)$ for $Y \subseteq B$ then satisfies the hypotheses of Lemma 5.1.

VI. PROOF OF LEMMA 5.1

We continue the parallel with the proof of Theorem 3.1 by defining for $H \in \mathscr{J}$,

$$q(H) = k(C(H))/N.$$

Note $\sum q(H) = 1$; $q(H)$ is the proportion of quasi-stable sets whose intersection with J is H.

Now we have the following chain of inequalities.

$$\lambda(C(H) \cap B) \geqslant \log(\hat{g}(C(H) \cap B)) \geqslant \log(k(C(H))) = \log(Nq(H)). \tag{6.1}$$

The first inequality follows by hypothesis (5.5) and the second inequality and the equality are immediate from the definition. Thus (5.4) follows from

$$\begin{aligned} &\log N + \sum_{H \in \mathscr{J}} q(H)(2\lambda(H \cap A) + \log q(H)) \\ &\geqslant 2\delta_0(\log \hat{f}(A) + \log \hat{g}(B)). \end{aligned} \tag{6.2}$$

Without loss of generality, assume $\hat{f}(A) \geqslant \hat{g}(B)$ (if not we can redo the preceding argument on the dual poset). Also, note that since every filter of J is quasi-stable, $N \geqslant \hat{f}(A)$. Thus (6.2) follows from

$$-\left(\sum_{H \in \mathcal{J}} q(H)(2\lambda(H \cap A) + \log q(H))\right) \leqslant (1 - 4\delta_0) \log \hat{f}(A). \tag{6.3}$$

Note $\hat{f}(A) \geqslant 2^{|A|}$. Also by Lemma 4.1, the left-hand side of (6.3) is less than or equal to

$$\log \sum_{H \in \mathcal{J}} 2^{-2\lambda(H \cap A)} = \log \sum_{X \subseteq A} f(X)\, 2^{-2\lambda(X)}.$$

Setting $\alpha(X) = 2^{\lambda(X)}$, (6.3) follows from

$$\sum_{X \subseteq A} \frac{f(X)}{(\alpha(X))^2} \leqslant \left(\frac{5}{4}\right)^{|A|} \tag{6.4}$$

which we now prove. By Proposition 2.1,

$$\begin{aligned}\sum_{X \subseteq A} \frac{f(X)}{\alpha(X)^2} &= \sum_{X \subseteq A} \frac{1}{\alpha(X)^2} \sum_{S \subseteq X} (-1)^{|X| - |S|} \hat{f}(S) \\ &= \sum_{S \subseteq A} \hat{f}(S) \sum_{X \supseteq S} \frac{(-1)^{|X| - |S|}}{\alpha(X)^2}.\end{aligned} \tag{6.5}$$

Using the multiplicativity of α and Proposition 2.2, we have for any S

$$\begin{aligned}\sum_{X \supseteq S} \frac{(-1)^{|X| - |S|}}{\alpha(X)^2} &= \frac{1}{\alpha(S)^2} \sum_{Z \subseteq A \setminus S} \frac{(-1)^{|Z|}}{\alpha(Z)^2} \\ &= \frac{1}{\alpha(S)^2} \prod_{a \in A \setminus S} \left(1 - \frac{1}{\alpha(a)^2}\right)\end{aligned}$$

which is nonnegative for all $S \in A$ since $\alpha(a) \geqslant 1$ for all $a \in A$. It follows that the last expression in (6.5) is a linear combination of the $\hat{f}(S)$ with non-negative coefficients. Since $\alpha(S) \geqslant \hat{f}(S)$ for all $S \subseteq A$ we may replace $\hat{f}(S)$ by $\alpha(S)$ to obtain a larger sum. That is,

$$\begin{aligned}\sum_{X \subseteq A} \frac{f(X)}{\alpha(A)^2} &\leqslant \sum_{S \subseteq A} \alpha(S) \sum_{X \supseteq S} \frac{(-1)^{|X| - |S|}}{\alpha(X)^2} \\ &= \sum_{X \subseteq A} \frac{1}{\alpha(X)^2} \left(\sum_{S \subseteq X} (-1)^{|X| - |S|} \alpha(S)\right).\end{aligned} \tag{6.6}$$

According to Propositions 2.1 and 2.2, the function $\gamma: 2^A \to R$ defined by

$\gamma(X)=\sum_{S\subseteq X}(-1)^{|X|-|S|}\alpha(S)$ is a multiplicative function with $\gamma(a)=\alpha(a)-1$ for $a\in A$. We deduce from (6.6) that

$$\sum_{X\subseteq A}\frac{f(X)}{\alpha(X)^2}\leqslant\sum_{X\subseteq A}\frac{\gamma(X)}{\alpha(X)^2}=\sum_{X\subseteq A}\prod_{a\in X}\frac{\alpha(a)-1}{\alpha(a)^2}.$$

Using the inequality $(t-1)/t^2\leqslant\frac{1}{4}$ for real t, we obtain (6.4) which completes the proof of Theorem 3.3.

VII. Proofs of Lemmas 5.2, 5.3 and 5.4

Lemma 5.2 is a consequence of the following inequality originally due to Daykin [D] which is a special case of an inequality proved later by Ahlswede and Daykin [AD] and is closely related to the well-known FKG inequality [FKG].

Lemma 7.1. *Let $\mathscr{A}$, $\mathscr{B}$ be sets of subsets of the finite set S. Then*

$$|\mathscr{A}|\,|\mathscr{B}|\leqslant|\mathscr{A}\vee\mathscr{B}|\,|\mathscr{A}\wedge\mathscr{B}|,$$

where $\mathscr{A}\vee\mathscr{B}=\{A\cup B\mid A\in\mathscr{A},\ B\in\mathscr{B}\}$ *and* $\mathscr{A}\wedge\mathscr{B}=\{A\cap B\mid A\in\mathscr{A},\ B\in\mathscr{B}\}$.

Proof of Lemma 5.2. By symmetry, it is enough to show that $\hat{g}$ is NLSM. Clearly $\hat{g}(\phi)=1$. For $Y\subseteq B$ let $\mathscr{B}(Y)$ be the set of ideals $L\in\mathscr{K}$ such that $L\cap B\subseteq Y$. By Lemma 7.1, for Y_1, $Y_2\subseteq B$, we have

$$|\mathscr{B}(Y_1)\vee\mathscr{B}(Y_2)|\cdot|\mathscr{B}(Y_1)\wedge\mathscr{B}(Y_2)|\geqslant|\mathscr{B}(Y_1)|\cdot|\mathscr{B}(Y_2)|.$$

Note that $\mathscr{B}(Y_1)\vee\mathscr{B}(Y_2)\subseteq\mathscr{B}(Y_1\cup Y_2)$ and $\mathscr{B}(Y_1)\wedge\mathscr{B}(Y_2)\subseteq\mathscr{B}(Y_1\cap Y_2)$ so we have

$$|\mathscr{B}(Y_1\cup Y_2)|\cdot|\mathscr{B}(Y_1\cap Y_2)|\geqslant|\mathscr{B}(Y_1)|\cdot|\mathscr{B}(Y_2)|$$

or

$$\hat{g}(Y_1\cup Y_2)\,\hat{g}(Y_1\cup Y_2)\geqslant\hat{g}(Y_1)\,\hat{g}(Y_2).\quad\blacksquare$$

Proof of Lemma 5.3. Let $S=\{1,\ldots,s_n\}$ and define the multiplicative function H by $H(s_j)=h(s_1,\ldots,s_j)/h(s_1,\ldots,s_{j-1})$ for $1\leqslant j\leqslant n$, extending to arbitrary subsets by multiplicativity. Clearly $H(S)=h(S)$. We show $H(T)\geqslant h(T)$ for all $T\subseteq S$ by induction on $|T|$. We have $H(\varnothing)=h(\varnothing)=1$ for the basis. For $T\neq 0$, let s_k be the element of largest index in T and let

$U=\{s_1, s_2,..., s_k\}$. Since $T \subseteq U$, the LSM inequality applied to T and $U \backslash s_k$ yields

$$h(U)\, h(T \backslash s_k) \geqslant h(U \backslash s_k)\, h(T).$$

By the induction hypothesis, $H(T \backslash s_k) \geqslant h(T \backslash s_k)$ so

$$\frac{h(U)}{h(U \backslash s_k)} H(T \backslash s_k) \geqslant h(T),$$

but, the left-hand side is, by definition, $H(T)$, proving the lemma, and Theorem 3.3.

VIII. A Weighted Version of Theorem 3.1

Let $\Gamma=(A, B, E)$ be a bipartite graph and W a nonnegative set functon (weight function) on $A \cup B$. We now ask: is there a vertex v of Γ such that the total weight of the stable sets containing v is a nontrivial fraction of the total weight of all stable sets. Obviously this depends on W. The following theorem provides a sufficient condition for the existence of such a vertex.

Theorem 8.1. *Let $\Gamma=(A, B, E)$ be a bipartite graph and suppose f and g are normalized nondecreasing supermodular set functions on A and B, respectively. Let W be the set function on $A \cup B$ defined by $W(Z)=f(Z \cap A)\, g(Z \cap B)$. Then there is an additive set function λ on $A \cup B$ such that*

$$\sum_{\substack{S \subseteq A \cup B \\ S \text{Stable}}} W(S)\, \lambda(S) \geqslant \delta_0 \sum_{\substack{S \subseteq A \cup B \\ S \text{Stable}}} W(S)\, \lambda(A \cup B).$$

The proof of Theorem 8.1 follows along the same lines as the proofs of Theorems 3.1 and 3.3. The reader can fill in the details.

IX. On the Information-Theoretic Bound

The information-theoretic argument given in Section 1 to give a lower bound on the ideal identification problem is a standard one, and has been used to give lower bounds for a wide variety of problems. The natural question "how good is this bound?" has been addressed, either directly or indirectly by many researchers (e.g., [Fr, GYY, see [S] for a survey). Theorem 1.2 answers this question for the ideal identification problem.

To place these questions in a general setting, we define the *edge iden-*

tification problem for hypergraphs: given a hypergraph $H=(V,\mathscr{E})$ and an unknown edge $E\in\mathscr{E}$, determine E using queries of the form "is $x\in\mathscr{E}$?" (The ideal identification problem is the case when $\mathscr{E}$ is the set of ideals of some finite poset.)

After a response of "no" to the question "is $x\in\mathscr{E}$?", we are left with the edge identification problem for the hypergraph Hx (the *deletion* of x) on $V\backslash x$ obtained by discarding all edges not containing x. A "yes" response leaves us with the same problem on the hypergraph $H:x$ (the *localization* of x) on $V\backslash x$ obtained by discarding all edges not containing x and deleting x from all the remaining edges.

Thus, as before, the information-theoretic bound implies that a least $\log|\mathscr{E}|$ questions are required. In general this bound may be very weak, for example, if consists of singletons, $|\mathscr{E}|-1$ questions are needed. We are interested in identifying classes of hypergraphs for which the information theoretic bound is good.

Say that x is δ-central in H if the fraction of edges containing x is between δ and $1-\delta$. Arguing as for the ideal identification problem, we see that the information-theoretic bound is tight (up to a multiplicative constant) for the edge identification problem of H if there is a constant $\delta>0$ such that every hypergraph H' obtained from H by (repeated) deletions or localizations has a δ-central element. This suggests that we consider classes of hypergraphs that are closed under deletions and localizations and such that every hypergraph in the class has a δ-central element. We call such a class a *δ-central class of hypergraphs*. Summarizing the above discussion we have:

PROPOSITION 9.1. *Let Ω be a δ-central class of hypergraphs. Then for any $H=(V,\mathscr{E})\in\Omega$ the edge identification problem requires at least* $\log|\mathscr{E}|$ *questions and can be solved in* $C\log|\mathscr{E}|$ *questions, where* $C=-1/\log(1-\delta)$.

Which leads us to the following nebulous

PROBLEM 9.2. Find nice classes of δ-central hypergraphs.

In this context, the following (slightly stronger) restatement of Theorem 1.1 is appealing.

THEOREM 9.3. *The class Ω consisting of hypergraphs whose edge sets are closed under unions and intersections is δ_0-central for* $\delta_0=(3-\log 5)/4$.

One intriguing problem that can be formulated along these lines is the following conjecture of Fredman, which appears in [Li]: in every partially ordered set $(P,\leqslant)$ there exists a pair of elements, x, y such that the proportion of linear extensions of P (i.e., total orders compatible with P) in which $x > y$ is between $\frac{1}{3}$ and $\frac{2}{3}$. To fit this in the framework of Problem 9.2, associate to every finite poset P a hypergraph H_P as follows: the vertex set consists of all ordered pairs of incomparable elements. For each linear extension σ there is an edge E_σ which contains the ordered pair (x, y) iff x precedes y in σ. Then the above conjecture is equivalent to the conjecture that the class Ω of all such hypergraphs is $\frac{1}{3}$-central.

The conjecture is true for posets of width 2 [Li]. Kahn and Saks [KS] have recently proven the result with $\frac{1}{3}$ replaced by $\frac{3}{11}$.

X. The Average Size of a Stable Set in a Graph

Theorem 3.1 asserts the existence of a constant (δ_0) which is a lower bound for the ratio of the average size of a stable set to the number of vertices in any bipartite graph. The existence of such a constant is actually quite trivial, as was pointed out to us by N. Alon. He observed that for an appropriately chosen δ ($\delta = 0.1$ will do), the number of stable sets of size less than δv, indeed the *total number* of vertex subsets of size less than δv, is less than the number of subsets of the larger set in the bipartition (which have average size at least $v/4 > 2\delta v$) and thus the overall average is at least δv.

What if Γ is not bipartite? For any graph we have

$$\alpha(\Gamma) \geqslant v(\Gamma)/\chi(\Gamma),$$

where $\alpha(\Gamma)$ is the size of the largest stable set, $\chi(\Gamma)$ is its chromatic number, and $v(\Gamma)$ is the number of vertices. Let $\bar{\alpha}(\Gamma)$ be the average size of a stable set of Γ.

We ask

Question 10.1. *Does there exist a universal constant $C > 0$ such that for every graph Γ, $\bar{\alpha}(\Gamma) \geqslant Cv(\Gamma)/x(\Gamma)$?*

Using Alon's argument, one can obtain

Proposition 10.2. *There is a universal constant C such that*

$$\bar{\alpha}(\Gamma) \geqslant C \frac{v(\Gamma)}{\chi(\Gamma) \log \chi(\Gamma)}.$$

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