

ON THE DENSITIES OF CLIQUES AND INDEPENDENT SETS IN GRAPHS

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Let $r, s \geq 2$ be integers. Suppose that the number of blue r -cliques in a red/blue coloring of the edges of the complete graph K_n is known and fixed. What is the largest possible number of red s -cliques under this assumption? The well known Kruskal-Katona theorem answers this question for $r = 2$ or $s = 2$. Using the shifting technique from extremal set theory together with some analytical arguments, we resolve this problem in general and prove that in the extremal coloring either the blue edges or the red edges form a clique.

1. Introduction

As usual we denote by K_s the complete graph on s vertices and by \overline{K}_s its complement, the edgeless graph on s vertices. By the celebrated Ramsey's theorem, for every two integers r, s every sufficiently large graph must contain K_r or \overline{K}_s . Turán's theorem can be viewed as a quantitative version of the case $s = 2$. Namely, it shows that among all \overline{K}_r -free n -vertex graphs, the graph with the least number of K_2 (edges) is a disjoint union of $r - 1$ cliques of nearly equal size. More generally, one can ask the following question. Fix two graphs H_1 and H_2 , and suppose that we know the number of induced copies of H_1 in an n -vertex graph G . What is the maximum (or minimum) number of induced copies of H_2 in G ? In its full generality, this problem

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seems currently out of reach, but some special cases already have important implications in combinatorics, as well as other branches of mathematics and computer science.

To state these classical results, we introduce some notation. Adjacency between vertices u and v is denoted by $u \sim v$, and the neighbor set of v is denoted by $N(v)$. If necessary, we add a subscript G to indicate the relevant graph. The collection of induced copies of a k -vertex graph H in an n -vertex graph G is denoted by $\text{Ind}(H; G)$, i.e.

$$\text{Ind}(H; G) := \{X \subseteq V(G) : G[X] \simeq H\}$$

and the *induced H -density* is defined as

$$d(H; G) := \frac{|\text{Ind}(H; G)|}{\binom{n}{k}}.$$

In this language, Turán's theorem says that if $d(K_r; G) = 0$ then $d(K_2; G) \leq 1 - \frac{1}{r-1}$ and this bound is tight. For a general graph H , Erdős and Stone [5] determined $\max d(K_2; G)$ when $d(H; G) = 0$ and showed that the answer depends only on the chromatic number of H . Zykov [18] extended Turán's theorem in a different direction. Given integers $2 \leq r < s$, he proved that if $d(K_s; G) = 0$ then $d(K_r; G) \leq \frac{(s-1) \cdots (s-r)}{(s-1)^r}$. The balanced complete $(s-1)$ -partite graphs show that this bound is also tight.

For fixed integers $r < s$, the Kruskal-Katona theorem [9,11] states that if $d(K_r; G) = \alpha$ then $d(K_s; G) \leq \alpha^{s/r}$. Again, the bound is tight and is attained when G is a clique on some subset of the vertices. On the other hand, the problem of *minimizing* $d(K_s; G)$ under the same assumption is much more difficult. Even the case $r=2$ and $s=3$ has remained unsolved for many years until it was recently answered by Razborov [14] using his newly-developed flag algebra method. Subsequently, Nikiforov [12] and Reiher [15] applied complicated analytical techniques to solve the cases $(r, s) = (2, 4)$, and $(r=2, \text{arbitrary } s)$, respectively.

In this paper, we study the following natural analogue of the Kruskal-Katona theorem. Given $d(\overline{K}_r; G)$, how large can $d(K_s; G)$ be? For integers $a \geq b > 0$ we let $Q_{a,b}$ be the a -vertex graph whose edge set is a clique on some b vertices. The complement of this graph is denoted by $\overline{Q}_{a,b}$. Let \mathcal{Q}_a denote the family of all graphs $Q_{a,b}$ and its complement $\overline{\mathcal{Q}}_{a,b}$ for $0 < b \leq a$. Note that for $r=2$ or $s=2$, the Kruskal-Katona theorem implies that the extremal graph comes from \mathcal{Q}_n . Our first theorem shows that a similar statement holds for all r and s .

Theorem 1.1. *Let $r, s \geq 2$ be integers and suppose that $d(\overline{K}_r; G) \geq p$ where G is an n -vertex graph and $0 \leq p \leq 1$. Let q be the unique root of $q^r + rq^{r-1}(1-q) = p$ in $[0, 1]$. Then $d(K_s; G) \leq M_{r,s,p} + o(1)$, where*

$$M_{r,s,p} := \max\{(1 - p^{1/r})^s + sp^{1/r}(1 - p^{1/r})^{s-1}, (1 - q)^s\}.$$

Namely, given $d(\overline{K}_r; G)$, the maximum of $d(K_s; G)$ (up to $\pm o(1)$) is attained in one of two graphs, (or both), one of the form $Q_{n,t}$ and another $\overline{Q}_{n,t'}$.

We obtain as well a *stability version* of Theorem 1.1. Two n -vertex graphs H and G are ϵ -close if it is possible to obtain H from G by adding or deleting at most ϵn^2 edges. As the next theorem shows, every near-extremal graph G for Theorem 1.1 is ϵ -close to a specific member of \mathcal{Q}_n .

Theorem 1.2. *Let $r, s \geq 2$ be integers and let $p \in [0, 1]$. For every $\epsilon > 0$, there exists $\delta > 0$ and an integer N such that every n -vertex graph G with $n > N$ satisfying $d(\overline{K}_r; G) \geq p$ and $|d(K_s; G) - M_{r,s,p}| \leq \delta$, is ϵ -close to some graph in \mathcal{Q}_n .*

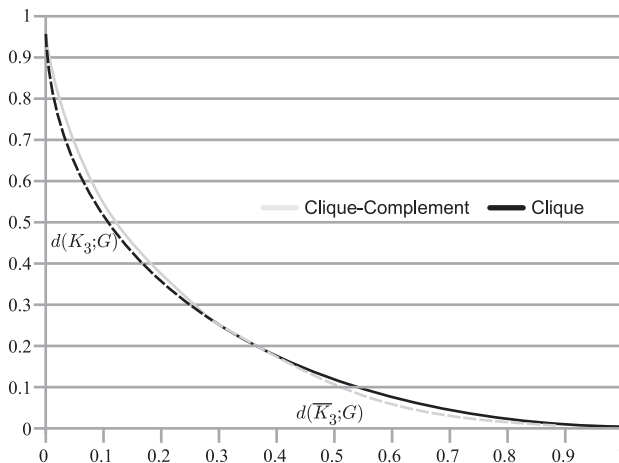


Figure 1. Illustration for the case $r = s = 3$. The green curve is $(d(\overline{K}_3; Q_{n,\theta n}), d(K_3; Q_{n,\theta n}))$ for $\theta \in [0, 1]$, and the red curve defined in the same way with $\overline{Q}_{n,\theta n}$. The maximum between the curves is the extremal function in Theorem 1.1. The intersection of the curves represents the solution of the max-min problem in Theorem 1.3

Rather than talking about an n -vertex graph and its complement, we can consider a two-edge-coloring of K_n . A quantitative version of Ramsey Theorem asks for the minimum number of monochromatic s -cliques over

all such colorings. Goodman [8] showed that for $r = s = 3$, the optimal answer is essentially given by a random two-coloring of $E(K_n)$. In other words, $\min_G d(K_3; G) + d(\overline{K}_3; G) = 1/4 - o(1)$. Erdős [4] conjectured that the same random coloring also minimizes $d(K_r; G) + d(\overline{K}_r; G)$ for all r , but this was refuted by Thomason [16] for all $r \geq 4$. A simple consequence of Goodman's inequality is that $\min_G \max\{d(K_3; G), d(\overline{K}_3; G)\} = 1/8$. The following construction by Franek and Rödl [19] shows that the analogous statement for $r \geq 4$ is again false. Let H be a graph with vertex set $[2]^{13}$, the collection of all 8192 binary vectors of length 13. Two vertices are adjacent if the Hamming distance between the corresponding binary vectors is a number in $\{1, 4, 5, 8, 9, 11\}$. Let G be obtained from H by replacing each vertex with a clique of size n , and every edge with a complete bipartite graph. The number of K_4 and \overline{K}_4 in G can be easily expressed in terms of the parameters of H (see [19]), for large enough n one can show that $d(K_4; G) < 0.99 \cdot \frac{1}{64}$ and $d(\overline{K}_4; G) < 0.993 \cdot \frac{1}{64}$.

While the min-max question remains at present very poorly understood, we succeeded to completely answer the max-min version of this problem.

Theorem 1.3.

$$\max_G \min\{d(K_r; G), d(\overline{K}_r; G)\} = \rho^r + o(1),$$

where ρ is the unique root in $[0, 1]$ of the equation $\rho^r = (1 - \rho)^r + r\rho(1 - \rho)^{r-1}$.

This theorem follows easily from Theorem 1.1. Moreover, using Theorem 1.2, we can also show that for every $\epsilon > 0$ there is a $\delta > 0$ such that every n -vertex graph G with $\min\{d(K_r; G), d(\overline{K}_r; G)\} > \rho^r - \delta$ is ϵ -close to a clique of size ρn or to the complement of this graph.

Here we prove these theorems using the method of shifting. In the next section we describe this well-known and useful technique in extremal set theory. Using shifting, we show how to reduce the problem to *threshold graphs*. Section 3 contains the proof of our main result for threshold graphs and Section 4 contains the proof of the stability result. In Section 5 we sketch a second proof for the case $r = s$, based on a different representation of threshold graphs. We make a number of comments on the analogous problems for hypergraphs in Section 6. We finish this paper with some concluding remarks and open problems.

2. Shifting

Shifting is one of the most important and widely-used tools in extremal set theory. This method allows one to reduce many extremal problems to more

structured instances which are usually easier to analyze. Our treatment is rather shallow and we refer the reader to Frankl's survey article [6] for a fuller account.

Let \mathcal{F} be a family of subsets of a finite set V , and let u, v be two distinct elements of V . We define the (u, v) -shift map $S_{u \rightarrow v}$ as follows: for every $F \in \mathcal{F}$, let

$$S_{u \rightarrow v}(F, \mathcal{F}) := \begin{cases} (F \cup \{v\}) \setminus \{u\} & \text{if } u \in F, v \notin F \text{ and } (F \cup \{v\}) \setminus \{u\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

We define the (u, v) -shift of \mathcal{F} , to be the following family of subsets of V : $S_{u \rightarrow v}(\mathcal{F}) := \{S_{u \rightarrow v}(F, \mathcal{F}) : F \in \mathcal{F}\}$. We observe that $|S_{u \rightarrow v}(\mathcal{F})| = |\mathcal{F}|$. In this context, one may think of \mathcal{F} as a hypergraph over V . When all sets in \mathcal{F} have cardinality 2 this is a graph with vertex set V . As the next lemma shows, shifting of a graph does not reduce the number of l -cliques in it for every l . Recall that $\text{Ind}(K_l; G)$ denotes the collection of all cliques of size l in G .

Lemma 2.1. *For every integer $l > 0$, every graph G , and every $u \neq v \in V(G)$ there holds*

$$S_{u \rightarrow v}(\text{Ind}(K_l; G)) \subseteq \text{Ind}(K_l; S_{u \rightarrow v}(G)).$$

Proof. Let $A = S_{u \rightarrow v}(B, G)$, where B is an l -clique in G . First, consider the cases when $u \notin B$ or both $u, v \in B$ or $B \setminus \{u\} \cup \{v\}$ is also a clique in G . Then $A = B$ and we need to show that B remains a clique after shifting. Which edge in B can be lost by shifting? It must be some edge uw in B that gets replaced by the non-edge vw (otherwise we can not shift uw). Note that vw is not in B , since B is a clique. Hence, $u, w \in B$ and $v \notin B$. But then $B \setminus \{u\} \cup \{v\}$ is not a clique, contrary to our assumption.

In the remaining case when $u \in B$, $v \notin B$ and $B \setminus \{u\} \cup \{v\}$ is not a clique in G , we need to show that $A = B \setminus \{u\} \cup \{v\}$ is a clique after shifting $S_{u \rightarrow v}(G)$. Every pair of vertices in $A \setminus \{v\}$ belongs to B and the edge they span is not affected by the shifting. So consider $v \neq w \in A$. If $vw \in E(G)$, this edge remains after shifting. If, however, $vw \notin E(G)$, note that $uw \in E(G)$ since both vertices belong to the clique B . In this case $vw = S_{u \rightarrow v}(uw, G)$ and the claim is proved. ■

Since shifting edges from u to v is equivalent to shifting non-edges from v to u , it is immediate that $S_{u \rightarrow v}(\text{Ind}(\overline{K}_l; G)) \subseteq \text{Ind}(\overline{K}_l; S_{u \rightarrow v}(G))$. Therefore, we obtain the following corollary.

Corollary 2.2. *Let G be a graph, let $H = S_{u \rightarrow v}(G)$ and let l be a positive integer. Then*

$$d(K_l; H) \geq d(K_l; G) \quad \text{and} \quad d(\overline{K}_l; H) \geq d(\overline{K}_l; G).$$

We say that vertex u *dominates* vertex v if $S_{v \rightarrow u}(\mathcal{F}) = \mathcal{F}$. In the case when \mathcal{F} is a set of edges of G , this implies that every $w \neq u$ which is adjacent to v is also adjacent to u . If $V = [n]$, we say that a family \mathcal{F} is *shifted* if i dominates j for every $i < j$. Every family can be made shifted by repeated applications of shifting operations $S_{j \rightarrow i}$ with $i < j$. To see this note that a shifting operation that changes \mathcal{F} reduces the following non-negative potential function $\sum_{A \in \mathcal{F}} \sum_{i \in A} i$. As Corollary 2.2 shows, it suffices to prove Theorem 1.1 for shifted graphs.

In Section 3 we use the notion of *threshold graphs*. There are several equivalent ways to define threshold graph (see [2]), and we adopt the following definition.

Definition 2.3. We say that $G = (V, E)$ is a threshold graph if there is an ordering of V so that every vertex is adjacent to either all or none of the preceding vertices.

Lemma 2.4. *A graph is shifted if and only if it is a threshold graph.*

Proof. Let G be a shifted graph. We may assume that $V = [n]$, and i dominates j in G for every $i < j$. Consider the following order of vertices,

$$\dots, 3, N_G(2) \setminus N_G(3), 2, N_G(1) \setminus N_G(2), 1, V \setminus N_G(1),$$

where the vertices inside the sets that appear here are ordered arbitrarily. We claim that this order satisfies Definition 2.3. First, every vertex $v \notin N_G(1)$ is isolated. Indeed, if $u \sim v$, then necessarily $v \sim 1$, since 1 dominates u . Therefore, vertex 1 and its non-neighbors satisfy the condition in the definition. The proof that G is threshold proceeds by induction applied to $G[N_G(1)]$.

Conversely, let G be a threshold graph. Let v_1, v_2, \dots, v_n be an ordering of V as in Definition 2.3. We say that a vertex is good (resp. bad) if it is adjacent to all (none) of its preceding vertices. Consider two vertices v_i and v_j . It is straightforward to show that v_i dominates v_j if either (1) v_i is good and v_j is bad, (2) they are both good and $i > j$ or (3) they are both bad and $i < j$. Therefore, we can reorder the vertices by first placing the good vertices in reverse order followed by the bad vertices in the regular order. This new ordering demonstrates that G is shifted. ■

3. Main result

In this section, we prove Theorem 1.1. It will be convenient to reformulate the theorem, in a way that is analogous to the Kruskal-Katona theorem.

Theorem 3.1. *Let $r, s \geq 3$ be integers and let $a, b > 0$ be real numbers. The maximum (up to $\pm o(1)$) of the function $f(G) := \min\{a \cdot d(K_s; G), b \cdot d(\overline{K}_r; G)\}$ over all n -vertex graphs is attained in one of two graphs, (or both), one of the form $Q_{n,t}$ and another $\overline{Q}_{n,t'}$. In particular, $f(G) \leq \max\{a \cdot \alpha^s, b \cdot \beta^r\} + o(1)$, where α is the unique root in $[0, 1]$ of $a \cdot \alpha^s = b \cdot [(1 - \alpha)^r + r\alpha(1 - \alpha)^{r-1}]$ and β is the unique root in $[0, 1]$ of $b \cdot \beta^r = a \cdot [(1 - \beta)^s + s\beta(1 - \beta)^{s-1}]$.*

We turn to show how to deduce Theorem 1.1 from Theorem 3.1. We assume that $r, s \geq 3$, since the other cases follow from Kruskal-Katona theorem.

Proof of Theorem 1.1. Let M be the maximum of $d(K_s; G)$ over all graphs G on n vertices with $d(\overline{K}_r; G) \geq p$. Fix such an extremal G with $d(\overline{K}_r; G) = p' \geq p$ and $d(K_s; G) = M$. Now apply Theorem 3.1 with $a = p$ and $b = M$ and the same n, r and s . The extremal graph G' that Theorem 3.1 yields, satisfies

$$f(G') \geq f(G) = \min\{a \cdot d(K_s; G), b \cdot d(\overline{K}_r; G)\} = p \cdot M,$$

hence $d(K_s; G') \geq M$ and $d(\overline{K}_r; G') \geq p$. Therefore, the same G' is extremal for Theorem 1.1 as well and we know that the maximum in this theorem is achieved asymptotically by a graph of \mathcal{Q}_n .

Note that we can always assume that in the extremal graph $d(\overline{K}_r; G') = p$ since otherwise we can add edges to G' without decreasing $d(K_s; G')$ until $d(\overline{K}_r; G') = p$ is obtained. Therefore, the maximum is attained either by a graph of the form $\overline{Q}_{n, p^{1/r}n}$ or by $Q_{n, (1-q)n}$, where $q^r + rq^{r-1}(1-q) = p$. This implies that asymptotically the maximum in Theorem 1.1 is indeed

$$M_{r,s,p} = \max\{(1 - p^{1/r})^s + sp^{1/r}(1 - p^{1/r})^{s-1}, (1 - q)^s\}. \quad \blacksquare$$

By Corollary 2.2 and Lemma 2.4, $f(G)$ is maximized by a threshold graph. We turn to prove Theorem 3.1 for threshold graphs. Let G be a threshold graph on an ordered vertex set V , as in Definition 2.3. There exists an integer $k > 0$, and a partition A_1, \dots, A_{2k} of V such that

1. if $v \in A_i$ and $u \in A_j$ for $i < j$, then $v < u$,
2. every vertex in A_{2i-1} (respectively A_{2i}) is adjacent to all (none) of its preceding vertices.

Let $x_i = \frac{|A_{2i-1}|}{|V|}$ and $y_i = \frac{|A_{2i}|}{|V|}$. Clearly $\sum_{i=1}^k (x_i + y_i) = 1$. Up to a negligible error-term,

$$d(K_s; G) = p(\mathbf{x}, \mathbf{y}) := \left(\sum_{i=1}^k x_i \right)^s + s \cdot \sum_{i=1}^{k-1} \left[y_i \cdot \left(\sum_{j=i+1}^k x_j \right)^{s-1} \right],$$

$$d(\overline{K}_r; G) = q(\mathbf{x}, \mathbf{y}) := \left(\sum_{i=1}^k y_i \right)^r + r \cdot \sum_{i=1}^k \left[x_i \cdot \left(\sum_{j=i}^k y_j \right)^{r-1} \right],$$

where $\mathbf{x} = (x_1, x_2, \dots, x_k)$ and $\mathbf{y} = (y_1, y_2, \dots, y_k)$. Occasionally, p will be denoted by p_s and q by q_r to specify the parameter of these functions.

Our problem can therefore be reformulated as follows. For given integers $k \geq 2$, $r, s \geq 3$ and real $a, b > 0$, let $W_k \subseteq \mathbb{R}^{2k}$ be the set

$$W_k := \left\{ (x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k) \in \mathbb{R}^{2k} : \right. \\ \left. x_i, y_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^k (x_i + y_i) = 1 \right\}.$$

Let $p, q: W_k \rightarrow \mathbb{R}$ be the two homogeneous polynomials defined above, we are interested in maximizing the real function

$$\varphi(\mathbf{x}, \mathbf{y}) := \min\{a \cdot p(\mathbf{x}, \mathbf{y}), b \cdot q(\mathbf{x}, \mathbf{y})\}.$$

This problem is well defined since W_k is compact and φ is continuous.

We say that $(\mathbf{x}, \mathbf{y}) \in W_k$ is *non-degenerate* if the set of zeros in the sequence $(y_1, x_2, y_2, \dots, x_k, y_k)$, with x_1 omitted, forms a suffix. If $(\mathbf{x}, \mathbf{y}) \in W_k$ is degenerate, then there is a non-degenerate $(\mathbf{x}', \mathbf{y}') \in W_k$ with $\varphi(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}', \mathbf{y}')$. Indeed, if $y_i = 0$ and $x_{i+1} \neq 0$ for some $1 \leq i < k$, let $(\mathbf{x}', \mathbf{y}') \in W_{k-1}$ be defined by

$$\mathbf{x}' = (x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_k)$$

$$\mathbf{y}' = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_k)$$

It is easy to verify that $p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}', \mathbf{y}')$ and $q(\mathbf{x}, \mathbf{y}) = q(\mathbf{x}', \mathbf{y}')$. By induction on k , we assume that $(\mathbf{x}', \mathbf{y}')$ is non-degenerate, and by padding \mathbf{x}' and \mathbf{y}' with a zero, the claim is proved. The case $x_i = 0$ and $y_i \neq 0$ is proved similarly. In particular, φ has a non-degenerate maximum in W_k .

Our purpose is to show that the original problem is optimized by graphs from \mathcal{Q}_n . This translates to the claim that a non-degenerate (\mathbf{x}, \mathbf{y}) that maximizes φ is supported only on either x_1, y_1 or y_1, x_2 , which corresponds to either a clique $Q_{n,t}$ or a complement of a clique $Q_{n,t}$, respectively.

Lemma 3.2. *Let $(\mathbf{x}, \mathbf{y}) \in W_k$ be a non-degenerate maximum of φ . If $x_1 > 0$, then for every $i \geq 2$, $x_i = y_i = 0$. On the other hand, if $x_1 = 0$ then $y_i = 0$ for every $i \geq 2$, and $x_i = 0$ for every $i \geq 3$.*

Proof. We note first that the second part of the lemma is implied by the first part. Define \mathbf{x}' by

$$x'_i := \begin{cases} x_{i+1} & \text{if } i < k, \\ 0 & \text{if } i = k. \end{cases}$$

Clearly, if $x_1 = 0$, then $p_s(\mathbf{x}, \mathbf{y}) = q_s(\mathbf{y}, \mathbf{x}')$, $q_r(\mathbf{x}, \mathbf{y}) = p_r(\mathbf{y}, \mathbf{x}')$, and

$$\varphi'(\mathbf{y}, \mathbf{x}') := \min\{b \cdot p_r(\mathbf{y}, \mathbf{x}'), a \cdot q_s(\mathbf{y}, \mathbf{x}')\} = \varphi(\mathbf{x}, \mathbf{y}).$$

Since φ attains its maximum when $x_1 = 0$, maximizing it is equivalent to maximizing $\varphi'(\mathbf{y}, \mathbf{x}')$. Since (\mathbf{x}, \mathbf{y}) is non-degenerate, $y_1 > 0$, and applying the first part of Lemma 3.2 for $\varphi'(\mathbf{y}, \mathbf{x}')$ finishes the proof, by obtaining that for every $i \geq 2$, $y_i = x'_i = 0$.

The first part of Lemma 3.2 is proved in the following lemmas. We successively show that $x_3 = 0$, then $y_2 = 0$ and finally $x_2 = 0$. ■

Here is a local condition that maximum points of φ satisfy.

Lemma 3.3. *If φ takes its maximum at a non-degenerate $(\mathbf{x}, \mathbf{y}) \in W_k$, then $a \cdot p(\mathbf{x}, \mathbf{y}) = b \cdot q(\mathbf{x}, \mathbf{y})$.*

Proof. Note that $0 < y_1 < 1$, since $(\mathbf{x}, \mathbf{y}) \in W$ is non-degenerate. We consider two perturbations of the input, one of which increases $p(\mathbf{x}, \mathbf{y})$, and the other increases $q(\mathbf{x}, \mathbf{y})$. Consequently, if $a \cdot p(\mathbf{x}, \mathbf{y}) \neq b \cdot q(\mathbf{x}, \mathbf{y})$, by applying the appropriate perturbation, we increase the smaller between $a \cdot p(\mathbf{x}, \mathbf{y})$ and $b \cdot q(\mathbf{x}, \mathbf{y})$, thus increasing $\min\{a \cdot p(\mathbf{x}, \mathbf{y}), b \cdot q(\mathbf{x}, \mathbf{y})\}$, contrary to the maximality assumption.

To define the perturbation that increases p , let $\mathbf{x}' = \mathbf{x} + t\mathbf{e}_1$ and $\mathbf{y}' = \mathbf{y} - t\mathbf{e}_1$, where $0 < t < y_1$, and \mathbf{e}_1 is the first unit vector in \mathbb{R}^k . Then, $(\mathbf{x}', \mathbf{y}') \in W$ and

$$\frac{\partial p(\mathbf{x}', \mathbf{y}')}{\partial t} = s \left(t + \sum_{i=1}^k x_i \right)^{s-1} - s \cdot \left(\sum_{j=2}^k x_j \right)^{s-1} > 0$$

as claimed.

In order to increase q , consider two cases. If $x_1 = 0$, let $\mathbf{x}' = \mathbf{x} - t\mathbf{e}_2$ and $\mathbf{y}' = \mathbf{y} + t\mathbf{e}_1$, where $0 < t < x_2$. Then, $(\mathbf{x}', \mathbf{y}') \in W$ and

$$\frac{\partial q(\mathbf{x}', \mathbf{y}')}{\partial t} = r \left(t + \sum_{i=1}^k y_i \right)^{r-1} - r \cdot \left(\sum_{j=2}^k y_j \right)^{r-1} > 0.$$

If $x_1 > 0$, we let $\mathbf{x}' = \mathbf{x} - t\mathbf{e}_1$ and $\mathbf{y}' = \mathbf{y} + t\mathbf{e}_1$, where $0 < t < x_1$. Then,

$$\frac{\partial q(\mathbf{x}', \mathbf{y}')}{\partial t} = r(x_1 - t)(r - 1) \left(t + \sum_{i=1}^k y_i \right)^{r-2} > 0. \quad \blacksquare$$

Lemma 3.4. *If $(\mathbf{x}, \mathbf{y}) \in W_k$ is a non-degenerate maximum of φ with $x_1 > 0$, then $x_3 = 0$.*

Proof. Suppose, that $x_3 > 0$ and let $1 \leq l \leq m \leq k$. Then

$$\begin{aligned} \frac{\partial p}{\partial x_l} &= s \cdot \left(\sum_{i=1}^k x_i \right)^{s-1} + s(s-1) \cdot \sum_{i=1}^{l-1} \left[y_i \cdot \left(\sum_{j=i+1}^k x_j \right)^{s-2} \right], \\ \frac{\partial q}{\partial x_l} &= r \cdot \left(\sum_{j=l}^k y_j \right)^{r-1}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 p}{\partial x_l \partial x_m} &= s(s-1) \cdot \left(\sum_{i=1}^k x_i \right)^{s-2} \\ &\quad + s(s-1)(s-2) \cdot \sum_{i=1}^{l-1} \left[y_i \cdot \left(\sum_{j=i+1}^k x_j \right)^{s-3} \right], \\ \frac{\partial^2 q}{\partial x_l \partial x_m} &\equiv 0. \end{aligned}$$

Clearly $\frac{\partial^2 p}{\partial x_l \partial x_m} = \frac{\partial^2 p}{\partial x_l^2}$, for $l \leq m$. We define two matrices \mathbf{A} and \mathbf{B} as following.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\partial p}{\partial x_1} & \frac{\partial p}{\partial x_2} & \frac{\partial p}{\partial x_3} \\ \frac{\partial q}{\partial x_1} & \frac{\partial q}{\partial x_2} & \frac{\partial q}{\partial x_3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1 \partial x_2} & \frac{\partial^2 p}{\partial x_1 \partial x_3} \\ \frac{\partial^2 p}{\partial x_1 \partial x_2} & \frac{\partial^2 p}{\partial x_2^2} & \frac{\partial^2 p}{\partial x_2 \partial x_3} \\ \frac{\partial^2 p}{\partial x_1 \partial x_3} & \frac{\partial^2 p}{\partial x_2 \partial x_3} & \frac{\partial^2 p}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_1^2} \\ \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_2^2} & \frac{\partial^2 p}{\partial x_2^2} \\ \frac{\partial^2 p}{\partial x_1^2} & \frac{\partial^2 p}{\partial x_2^2} & \frac{\partial^2 p}{\partial x_3^2} \end{bmatrix}.$$

It is easy to see that if (\mathbf{x}, \mathbf{y}) is non-degenerate with $x_3 > 0$, then $\frac{\partial^2 p}{\partial x_3^2} > \frac{\partial^2 p}{\partial x_2^2} > \frac{\partial^2 p}{\partial x_1^2} > 0$. This implies that \mathbf{B} is positive definite.

For a vector $\mathbf{v} \in \mathbb{R}^3$ and $\epsilon > 0$, we define \mathbf{x}' by

$$x'_i = \begin{cases} x_i + \epsilon v_i & \text{if } i \leq 3, \\ x_i & \text{if } i > 3. \end{cases}$$

If \mathbf{A} is invertible, let \mathbf{v} be the (unique) vector for which

$$\mathbf{A} \cdot \mathbf{v}^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

In particular $\sum_i x'_i = \sum_i x_i$. For ϵ sufficiently small,

$$p(\mathbf{x}', \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) + \epsilon + O(\epsilon^2) > p(\mathbf{x}, \mathbf{y})$$

$$q(\mathbf{x}', \mathbf{y}) = q(\mathbf{x}, \mathbf{y}) + \epsilon > q(\mathbf{x}, \mathbf{y})$$

contrary to the maximality of (\mathbf{x}, \mathbf{y}) .

If \mathbf{A} is singular, pick some $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{A} \cdot \mathbf{v}^T = \mathbf{0}$. Again $\sum_i x'_i = \sum_i x_i$. Since \mathbf{B} is positive definite, for a sufficiently small ϵ ,

$$p(\mathbf{x}', \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) + \frac{\epsilon^2}{2} \cdot \mathbf{v} \cdot \mathbf{B} \cdot \mathbf{v}^T + O(\epsilon^3) > p(\mathbf{x}, \mathbf{y})$$

$$q(\mathbf{x}', \mathbf{y}) = q(\mathbf{x}, \mathbf{y}),$$

contradicting Lemma 3.3. ■

Lemma 3.5. *If $(\mathbf{x}, \mathbf{y}) \in W_k$ is a non-degenerate maximum of φ with $x_1 > 0$, then $y_2 = 0$.*

Proof. By Lemma 3.4 we may assume that $x_i = y_i = 0$ for all $i \geq 3$. Suppose, towards contradiction, that $y_2 \neq 0$. Let

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

where

$$\begin{aligned} a_1 &= \frac{\partial p}{\partial x_1} - \frac{\partial p}{\partial x_2} = -s(s-1) \cdot y_1 \cdot x_2^{s-2}, \\ a_2 &= \frac{\partial p}{\partial y_1} - \frac{\partial p}{\partial y_2} = s \cdot x_2^{s-1}, \\ b_1 &= \frac{\partial q}{\partial x_1} - \frac{\partial q}{\partial x_2} = r \cdot ((y_1 + y_2)^{r-1} - y_2^{r-1}), \\ b_2 &= \frac{\partial q}{\partial y_1} - \frac{\partial q}{\partial y_2} = -r(r-1) \cdot x_2 \cdot y_2^{r-2}. \end{aligned}$$

If $\text{rank}(\mathbf{M})=2$, then there is a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that $\mathbf{M} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Define $x'_1 = x_1 + \epsilon v_1$, $x'_2 = x_2 - \epsilon v_1$ and $y'_1 = y_1 + \epsilon v_2$, $y'_2 = y_2 - \epsilon v_2$. Then $x'_1 + x'_2 + y'_1 + y'_2 = 1$ and for sufficiently small $\epsilon > 0$

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}') &= p(\mathbf{x}, \mathbf{y}) + \epsilon \left(\frac{\partial p}{\partial x_1} v_1 - \frac{\partial p}{\partial x_2} v_1 + \frac{\partial p}{\partial y_1} v_2 - \frac{\partial p}{\partial y_2} v_2 \right) + O(\epsilon^2) \\ &= p(\mathbf{x}, \mathbf{y}) + \epsilon(a_1 v_1 + a_2 v_2) + O(\epsilon^2) = p(\mathbf{x}, \mathbf{y}) + \epsilon + O(\epsilon^2) > p(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Similarly $q(\mathbf{x}', \mathbf{y}') = q(\mathbf{x}, \mathbf{y}) + \epsilon + O(\epsilon^2) > q(\mathbf{x}, \mathbf{y})$. Thus (\mathbf{x}, \mathbf{y}) cannot be a maximum of φ . Hence, $\text{rank}(\mathbf{M}) \leq 1$, and in particular

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0,$$

which implies that

$$0 = x_2^{s-1} y_2^{r-1} \left((r-1)(s-1) \frac{y_1}{y_2} - \left(\frac{y_1}{y_2} + 1 \right)^{r-1} + 1 \right).$$

The function

$$g(\alpha) = (r-1)(s-1)\alpha - (\alpha+1)^{r-1} + 1$$

is strictly concave for $\alpha > 0$ and vanishes at 0. Since $\alpha=0$ is not a maximum of g , the equation $g\left(\frac{y_1}{y_2}\right)=0$ determines $\frac{y_1}{y_2}$ uniquely.

Denote $\alpha = \frac{y_1}{y_2}$, and consider the following change of variables.

$$\begin{aligned} x'_1 &= x_1 + \frac{1}{1 + (r-1)(s-1)\alpha} \cdot x_2, & x'_2 &= \frac{(r-1)(s-1)\alpha}{1 + (r-1)(s-1)\alpha} \cdot x_2, \\ y'_1 &= y_1 + y_2 = (\alpha+1)y_2, & y'_2 &= 0. \end{aligned}$$

Clearly, $x'_1 + x'_2 = x_1 + x_2$ and $y'_1 + y'_2 = y_1 + y_2$. Moreover,

$$\begin{aligned} q(\mathbf{x}', \mathbf{y}') &= (y'_1)^r + r \cdot x'_1 \cdot (y'_1)^{r-1} \\ &= (y_1 + y_2)^r + r \cdot x_1 \cdot (y_1 + y_2)^{r-1} + \frac{r \cdot x_2 \cdot (y_1 + y_2)^{r-1}}{1 + (r-1)(s-1)\alpha} \\ &= (y_1 + y_2)^r + r \cdot x_1 \cdot (y_1 + y_2)^{r-1} + \frac{r \cdot (1+\alpha)^{r-1} \cdot x_2 \cdot y_2^{r-1}}{(1+\alpha)^{r-1}} = q(\mathbf{x}, \mathbf{y}), \\ p(\mathbf{x}', \mathbf{y}') &= (x'_1 + x'_2)^s + s \cdot y'_1 \cdot (x'_2)^{s-1} \\ &= (x_1 + x_2)^s + s \cdot (\alpha+1) \cdot \left(\frac{(r-1)(s-1)\alpha}{1 + (r-1)(s-1)\alpha} \right)^{s-1} \cdot y_2 \cdot x_2^{s-1} \\ &> (x_1 + x_2)^s + s \cdot \alpha \cdot y_2 \cdot x_2^{s-1} = p(\mathbf{x}, \mathbf{y}), \end{aligned}$$

where the last inequality is a consequence of Lemma 3.6 below. This contradicts Lemma 3.3. ■

Lemma 3.6. *Let $r, s \geq 3$ be integers. Let $\alpha > 0$ be the unique positive root of*

$$(\alpha + 1)^{r-1} - 1 = (r - 1)(s - 1)\alpha.$$

Then

$$\left(1 + \frac{1}{(r-1)(s-1)\alpha}\right)^{s-1} < 1 + \frac{1}{\alpha}.$$

Proof. First, we show that $(r-1)\alpha > 1$. Let $t = (r-1)\alpha$ and assume, by contradiction, that $t \leq 1$. For $0 < t \leq 1$, we have $e^t < 1 + 2t$. On the other hand, $e \geq (1 + \alpha)^{1/\alpha}$, implying $e^t \geq (1 + \alpha)^{t/\alpha} = (1 + \alpha)^{r-1}$. Thus we have $2t > (1 + \alpha)^{r-1} - 1 = (r-1)(s-1)\alpha = (s-1)t$, which implies $2 > s-1$, a contradiction. Therefore $(r-1)\alpha > 1$. Also, since $1+x < e^x$ for all $x > 0$, we have that $\left(1 + \frac{1}{(r-1)(s-1)\alpha}\right)^{s-1} < e^{\frac{1}{(r-1)\alpha}}$. So it suffices to show that $e^{\frac{1}{(r-1)\alpha}} \leq 1 + \frac{1}{\alpha}$. But since $(r-1)\alpha > 1$, we have

$$\left(1 + \frac{1}{\alpha}\right)^{(r-1)\alpha} > 1 + \frac{(r-1)\alpha}{\alpha} = r \geq 3 > e,$$

which finishes the proof of the lemma. ■

Lemma 3.7. *If $(\mathbf{x}, \mathbf{y}) \in W_k$ is a non-degenerate maximum of φ with $x_1 > 0$, then $x_2 = 0$.*

Proof. This proof is very similar to the proof of Lemma 3.5. Now $x_1, x_2, y_1 > 0$ and $x_1 + x_2 + y_1 = 1$. Also

$$\begin{aligned} p(\mathbf{x}, \mathbf{y}) &= (x_1 + x_2)^s + s \cdot y_1 \cdot x_2^{s-1}, \\ q(\mathbf{x}, \mathbf{y}) &= y_1^r + r \cdot x_1 \cdot y_1^{r-1}. \end{aligned}$$

Let

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix},$$

where

$$\begin{aligned} a_1 &= \frac{\partial p}{\partial x_1} - \frac{\partial p}{\partial x_2} = -s(s-1) \cdot y_1 \cdot x_2^{s-2}, \\ a_2 &= \frac{\partial p}{\partial y_1} - \frac{\partial p}{\partial x_1} = -s \cdot ((x_1 + x_2)^{s-1} - x_2^{s-1}), \\ b_1 &= \frac{\partial q}{\partial x_1} - \frac{\partial q}{\partial x_2} = r \cdot y_1^{r-1}, \\ b_2 &= \frac{\partial q}{\partial y_1} - \frac{\partial q}{\partial x_1} = r(r-1) \cdot x_1 \cdot y_1^{r-2}. \end{aligned}$$

If \mathbf{M} is nonsingular, then there is a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that $\mathbf{M} \cdot \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Define $x'_1 = x_1 + \epsilon(v_1 - v_2)$, $x'_2 = x_2 - \epsilon v_1$ and $y'_1 = y_1 + \epsilon v_2$. Then $x'_1 + x'_2 + y'_1 = 1$ and for sufficiently small $\epsilon > 0$

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}') &= p(\mathbf{x}, \mathbf{y}) + \epsilon \left(\frac{\partial p}{\partial x_1}(v_1 - v_2) - \frac{\partial p}{\partial x_2}v_1 + \frac{\partial p}{\partial y_1}v_2 \right) + O(\epsilon^2) \\ &= p(\mathbf{x}, \mathbf{y}) + \epsilon(a_1v_1 + a_2v_2) + O(\epsilon^2) = p(\mathbf{x}, \mathbf{y}) + \epsilon + O(\epsilon^2) > p(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Similarly $q(\mathbf{x}', \mathbf{y}') = q(\mathbf{x}, \mathbf{y}) + \epsilon + O(\epsilon^2) > q(\mathbf{x}, \mathbf{y})$ and therefore, (\mathbf{x}, \mathbf{y}) cannot be a maximum of φ . Hence,

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0,$$

which implies

$$0 = y_1^{r-1} x_2^{s-1} \left((r-1) \cdot (s-1) \cdot \frac{x_1}{x_2} - \left(\frac{x_1}{x_2} + 1 \right)^{s-1} + 1 \right).$$

Let $\gamma = \frac{x_1}{x_2} > 0$. Then $1 + (r-1)(s-1)\gamma - (1+\gamma)^{s-1} = 0$ and concavity of the left hand side shows that γ is determined uniquely by this equation. Now make the following substitution:

$$\begin{aligned} x'_1 &= 0 \\ x'_2 &= x_1 + x_2 = (1 + \gamma) \cdot x_2 \\ y'_1 &= \frac{1}{1 + (r-1)(s-1)\gamma} \cdot y_1 \\ y'_2 &= \frac{(r-1)(s-1)\gamma}{1 + (r-1)(s-1)\gamma} \cdot y_1 \end{aligned}$$

Clearly $x'_1 + x'_2 = x_1 + x_2$ and $y'_1 + y'_2 = y_1$. Since $(1+\gamma)^{s-1} = 1 + (r-1)(s-1)\gamma$, we have

$$\begin{aligned} p(\mathbf{x}', \mathbf{y}') &= (x'_2)^s + s \cdot y'_1 \cdot (x'_2)^{s-1} \\ &= (x_1 + x_2)^s + s \cdot y_1 \cdot x_2^{s-1} = p(\mathbf{x}, \mathbf{y}) \\ q(\mathbf{x}', \mathbf{y}') &= (y'_1 + y'_2)^r + r \cdot x'_2 \cdot (y'_2)^{r-1} \\ &= y_1^r + r \cdot \frac{(1+\gamma)}{\gamma} \cdot \left(\frac{(r-1)(s-1)\gamma}{1 + (r-1)(s-1)\gamma} \right)^{r-1} \cdot x_1 \cdot y_1^{r-1} \\ &> y_1^r + r \cdot x_1 \cdot y_1^{r-1} = q(\mathbf{x}, \mathbf{y}), \end{aligned}$$

Where the last inequality follows from Lemma 3.6, with r and s switched. Again, this contradicts Lemma 3.3. ■

By combining Lemmas 3.3 – 3.7, we obtain a proof of Lemma 3.2, which states that the maximum of φ is attained by a non-degenerate (\mathbf{x}, \mathbf{y}) supported only on either x_1, y_1 or y_1, x_2 . In the first case, let $x_1 = \alpha$ and $y_1 = 1 - \alpha$. Then by Lemma 3.3, $a \cdot p(\mathbf{x}, \mathbf{y}) = a \cdot \alpha^s = b \cdot q(\mathbf{x}, \mathbf{y}) = b[(1 - \alpha)^r + r\alpha(1 - \alpha)^{r-1}]$ and $\varphi(\mathbf{x}, \mathbf{y}) = a \cdot \alpha^s$. In the second case, let $y_1 = \beta$ and $x_2 = 1 - \beta$. Then $b \cdot q(\mathbf{x}, \mathbf{y}) = b \cdot \beta^r = a \cdot p(\mathbf{x}, \mathbf{y}) = a[(1 - \beta)^s + s(1 - \beta)^{s-1}]$ and $\varphi(\mathbf{x}, \mathbf{y}) = b \cdot \beta^r$. This shows that the maximum of φ is $\max\{a \cdot \alpha^s, b \cdot \beta^r\}$ with α, β satisfying the above equations. In terms of the original graph, this proves that φ is maximized by a graph of the form $Q_{n,t}$ or $\bar{Q}_{n,t}$, respectively. In particular, our problem has at most two extremal configurations (in some cases a clique and the complement of a clique can give the same value of φ).

4. Stability analysis

In this section we discuss the proof of Theorem 1.2. In essentially the same way that Theorem 3.1 implies Theorem 1.1, this theorem follows from a stability version of Theorem 3.1:

Theorem 4.1. *Let $r, s \geq 3$ be integers and let $a, b > 0$ be real. For every $\epsilon > 0$, there exists $\delta > 0$ and an integer N such that every n -vertex G with $n > N$ for which*

$$f(G) \geq \max\{a \cdot \alpha^s, b \cdot \beta^r\} - \delta$$

is ϵ -close to some graph in \mathcal{Q}_n . Here f, α and β are as in Theorem 3.1.

Proof. If G is a threshold graph, the claim follows easily from Lemma 3.2. Since G is a threshold graph, $f(G) = \varphi(\mathbf{x}, \mathbf{y}) + o(1)$ for some $(\mathbf{x}, \mathbf{y}) \in W_k$ and some integer k . As this lemma shows, the continuous function φ attains its maximum on the compact set W_k at most twice, and this in points that correspond to graphs from \mathcal{Q}_n . Since $f(G)$ is δ -close to the maximum, it follows that (\mathbf{x}, \mathbf{y}) must be ϵ' -close to at least one of the two optimal points in W_k . This, in turn implies ϵ -proximity of the corresponding graphs.

For the general case, we use the stability version of the Kruskal-Katona theorem due to Keevash [10]. Suppose G is a large graph such that $f(G) \geq \max\{a \cdot \alpha^s, b \cdot \beta^r\} - \delta$. Let G_1 be the shifted graph obtained from G . Thus G_1 is a threshold graph with the same edge density as G , and $f(G_1) \geq f(G)$ by Corollary 2.2. Pick a small $\epsilon' > 0$. We just saw that for δ sufficiently small, G_1 is ϵ' -close to $G_{\max} \in \mathcal{Q}_n$. As we know, either $G_{\max} = Q_{n,t}$ or $G_{\max} = \bar{Q}_{n,t}$ for some $0 < t \leq n$. We deal with the former case, and the second case can be done similarly. Now $|d(K_2; G) - d(K_2; G_{\max})| \leq \epsilon'$, since G and G_1 have the same edge density. Moreover, $d(K_s; G) \geq d(K_s; G_{\max}) - \delta/a$, because

$f(G) \geq f(G_{max}) - \delta$. Since G_{max} is a clique, it satisfies the Kruskal-Katona inequality with equality. Consequently G has nearly the maximum possible K_s -density for a given number of edges. By choosing ϵ' and δ small enough and applying Keevash's stability version of Kruskal-Katona inequality, we conclude that G and G_{max} are ϵ -close. \blacksquare

5. Second proof

In this section we briefly present the main ingredients for an alternative approach to Theorem 1.1. We restrict ourselves to the case $r = s$. This proof reduces the problem to a question in the calculus of variations. Such calculations occur often in the context of shifted graphs.

Let G be a shifted graph with vertex set $[n]$ with the standard order. Then, there is some $n \geq i \geq 1$ such that $A = \{1, \dots, i\}$ spans a clique, whereas $B = \{i+1, \dots, n\}$ spans an independent set. In addition, there is some non-increasing function $F: A \rightarrow B$ such that for every $j \in A$ the highest index neighbor of j in B is $F(j)$, and all vertices of B up to index $F(j)$ are connected to j . Let x be the relative size of A and $1-x$ the relative size of B . In this case we can express (up to a negligible error term)

$$\begin{aligned} d(\overline{K}_k; G) &= \binom{n}{k}^{-1} \left[\binom{(1-x)n}{k} + \sum_{1 \leq j \leq xn} \binom{n - F(j)}{k-1} \right] \\ &= (1-x)^k + \frac{k}{n} \sum_{1 \leq j \leq xn} \left(\frac{n - F(j)}{n} \right)^{k-1} \\ &= (1-x)^k + kx(1-x)^{k-1} \sum_{1 \leq j \leq xn} \frac{1}{nx} \left(1 - \frac{F(j) - xn}{(1-x)n} \right)^{k-1}. \end{aligned}$$

Let f be a non-increasing function $f: [0, 1] \rightarrow [0, 1]$ such that $f(t) = \frac{F(j) - xn}{(1-x)n}$ for every $\frac{j-1}{xn} \leq t \leq \frac{j}{xn}$ (Think of f as a relative version of F both on its domain with respect to A and its codomain with respect to B). Then we can express $d(\overline{K}_k; G)$ in terms of x and f

$$d(\overline{K}_k; G) = (1-x)^k + kx(1-x)^{k-1} \int_0^1 (1-f(t))^{k-1} dt = d(\overline{K}_k; G_{x,f}).$$

Similarly one can show that

$$d(K_k; G) = x^k + kx^{k-1}(1-x) \int_0^1 (k-1)t^{k-2}f(t)dt = d(K_k; G_{x,f}).$$

Note that in this notation, $x=\theta$, $f=0$ (resp. $x=1-\theta$, $f=1$) corresponds to $Q_{n,\theta,n}$, (resp. $Q_{n,\theta,n}$).

To prove Theorem 1.1 for the case $r=s=k$, we show that assuming $d(K_k; G_{x,f}) \geq \alpha$, the maximum of $d(\overline{K}_k; G_{x,f})$ is attained by either $f=0$ or $f=1$. For this purpose, we prove upper bounds on the integrals.

Lemma 5.1. *If $f: [0,1] \rightarrow [0,1]$ is a non-increasing function, then*

$$\begin{aligned} & \int_0^1 (1-f(t))^{k-1} dt \\ & \leq \max \left\{ 1 - \left(\int_0^1 (k-1)t^{k-2}f(t)dt \right)^{\frac{1}{k-1}}, \left(1 - \int_0^1 (k-1)t^{k-2}f(t)dt \right)^{k-1} \right\}. \end{aligned}$$

The bounds in Lemma 5.1 are tight. Equality with the first term holds for f that takes only the values 1 and 0, and equality with the second term occurs for f a constant function. Proving Theorem 1.1 for such functions is done using rather standard (if somehow tedious) calculations. Lemma 5.1 itself is reduced to the following lemma through a simple affine transformation and normalization.

What non-decreasing function in $[0,1]$ minimizes the inner product with a given monomial?

Lemma 5.2. *Let $g: [0,1] \rightarrow [0,B]$ be a non-decreasing function with $B \geq 1$ and $\|g\|_{k-1} = 1$. Then*

$$\begin{aligned} \langle (k-1)t^{k-2}, g \rangle &= \int_0^1 (k-1)t^{k-2}g(t)dt \\ &\geq \min \left\{ B \left(1 - \left(1 - \frac{1}{B^{k-1}} \right)^{k-1} \right), 1 \right\}. \end{aligned}$$

Equality with the first term holds for

$$g(t) = \begin{cases} 0 & t < 1 - \frac{1}{B^{k-1}} \\ B & t \geq 1 - \frac{1}{B^{k-1}} \end{cases}$$

The second equality holds for $g=1$.

We omit the proof which is based on standard calculations and convexity arguments.

6. Shifting in hypergraphs

In this section, we will discuss a possible extension of Lemma 2.1 to hypergraphs. Consider two set systems \mathcal{F}_1 and \mathcal{F}_2 with vertex sets V_1 and V_2 , respectively. A (not necessarily induced) *labeled copy of \mathcal{F}_1 in \mathcal{F}_2* is an injection $I: V_1 \rightarrow V_2$ such that $I(F) \in \mathcal{F}_2$ for every $F \in \mathcal{F}_1$. We denote by $\text{Cop}(\mathcal{F}_1; \mathcal{F}_2)$ the set of all labeled copies of \mathcal{F}_1 in \mathcal{F}_2 and let

$$t(\mathcal{F}_1; \mathcal{F}_2) := |\text{Cop}(\mathcal{F}_1; \mathcal{F}_2)|.$$

Recall that a vertex u *dominates* vertex v if $S_{v \rightarrow u}(\mathcal{F}) = \mathcal{F}$. If either u dominates v or v dominates u in a family \mathcal{F} , we call the pair $\{u, v\}$ *stable* in \mathcal{F} . If every pair is stable in \mathcal{F} , then we call \mathcal{F} a *stable set system*.

Theorem 6.1. *Let \mathcal{H} be a stable set system and let \mathcal{F} be a set system. For every two vertices u, v of \mathcal{F} there holds*

$$t(\mathcal{H}; S_{u \rightarrow v}(\mathcal{F})) \geq t(\mathcal{H}; \mathcal{F}).$$

Corollary 6.2. *Let G be an arbitrary graph and let H be a threshold graph H . Then*

$$t(H; S_{u \rightarrow v}(G)) \geq t(H; G),$$

for every two vertices u, v of G .

Proof of Theorem 6.1 (sketch). We define a new shifting operator $\tilde{S}_{u \rightarrow v}$ for sets of labeled copies. First, for every $u, v \in V$, and a labeled copy $I: U \rightarrow V$, define $I_{u \leftrightarrow v}: U \rightarrow V$ by

$$I_{u \leftrightarrow v}(w) = \begin{cases} I(w) & \text{if } I(w) \neq u, v, \\ v & \text{if } I(w) = u, \\ u & \text{if } I(w) = v \end{cases}$$

For \mathcal{I} a set of labeled copies, $I \in \mathcal{I}$, we let

$$\tilde{S}_{u \rightarrow v}(I, \mathcal{I}) = \begin{cases} I_{u \leftrightarrow v} & \text{if } I_{u \leftrightarrow v} \notin \mathcal{I} \text{ and } \text{Im}(I) \cap \{u, v\} = \{u\}, \\ I_{u \leftrightarrow v} & \text{if } I_{u \leftrightarrow v} \notin \mathcal{I}, \{u, v\} \subset \text{Im}(I), \\ & \text{and } I^{-1}(u) \text{ dominates } I^{-1}(v) \text{ in } \mathcal{H}, \\ I & \text{otherwise.} \end{cases}$$

Finally, let $\tilde{S}_{u \rightarrow v}(\mathcal{I}) := \{\tilde{S}_{u \rightarrow v}(I, \mathcal{I}) : I \in \mathcal{I}\}$. Clearly, $|\tilde{S}_{u \rightarrow v}(\mathcal{I})| = |\mathcal{I}|$, and we prove that

$$\tilde{S}_{u \rightarrow v}(\text{Cop}(\mathcal{H}; \mathcal{F})) \subseteq \text{Cop}(\mathcal{H}; S_{u \rightarrow v}(\mathcal{F}))$$

thereby proving that $t(\mathcal{H}; S_{u \rightarrow v}(\mathcal{F})) \geq t(\mathcal{H}; \mathcal{F})$. As often in shifting, the proof is done by careful case analysis which is omitted. \blacksquare

7. Concluding remarks

In this paper, we studied the relation between the densities of cliques and independent sets in a graph. We showed that if the density of independent sets of size r is fixed, the maximum density of s -cliques is achieved when the graph itself is either a clique on a subset of the vertices, or a complement of a clique. On the other hand, the problem of minimizing the clique density seems much harder and has quite different extremal graphs for various values of r and s (at least when $\alpha=0$, see [3,13]).

Question 7.1. Given that $d(\overline{K}_r; G) = \alpha$ for some integer $r \geq 2$ and real $\alpha \in [0, 1]$, which graphs minimize $d(K_s; G)$?

In particular, when $\alpha=0$ we ask for the least possible density of s -cliques in graphs with independence number $r-1$. This is a fifty-year-old question of Erdős, which is still widely open. Das et al [3], and independently Pikhurko [13], solved this problem for certain values of r and s . It would be interesting if one could describe how the extremal graph changes as α goes from 0 to 1 in these cases. As mentioned in the introduction, the problem of minimizing $d(K_s; G)$ in graphs with fixed density of r -cliques for $r < s$ is also open and so far solved only when $r=2$.

Note added in proof. After writing this paper, we learned that P. Frankl, M. Kato, G. Katona and N. Tokushige [7] independently considered the same problem and obtained similar results when $r=s$.

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