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Asymptotically almost every 2r-regular graph has an internal partition

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תקציר

יהי G גרף פשוט, סופי ולא-מכוון ותהי V קבוצת הקדקדים של G. "חלוקה פנימית" של גרף היא חלוקה של V לשתי קבוצות זרות ("צדדים") כך שלכל קדקד ב V, רוב שכניו נמצאים באותה קבוצה בה הוא נמצא. הפתרון הטריוואלי בו אחד הצדדים הוא הקבוצה הריקה קיים תמיד ואנו מתעניינים בפתרון לא טריוואלי.

"חלוקה פנימית" כזו לא קיימת עבור כל גרף. דוגמאות טריואליות הן קליקות, או גרפים דו-צדדיים כשאחד הצדדים בגודל אי-זוגי.

אנו מוכיחים שלכל r טבעי, אסימפטוטית עבור גרפים גדולים, כמעט לכל גרף $2r$-רגולרי, קיימת חלוקה פנימית. למעשה, אנו מוכיחים טענה כללית יותר, כדלהלן.

עבור $U \subseteq V$ נסמן ב- $d_U(x)$ את דרגת קדקד x. תהיינה $a, b \geq 2$ פונקציות שלמות כלשהן על קדקדי G. נאמר שלגרף G יש חלוקה (a,b)-פנימית, אם קיימת חלוקה של הקדקדים (A,B), כך שלכל קדקד ב A, $d_A(x) \geq a(x)$, ולכל קדקד ב- B, $d_B(x) \geq b(x)$, .

יהי G גרף כך שכל ארבעה קדקדים פורשים לכל היותר ארבע צלעות. אנו מוכיחים את המשפט הבא: אם לכל קדקד מתקיים $d(x) \geq a(x) + b(x)$ אזי ל – G יש חלוקה (a,b)-פנימית.

כמובן, עבר גרף $2r$-רגולרי ובחירה $a(x) = b(x) = r$ המשפט גורר קיום חלוקה פנימית.

מכיוון שבגרפים רגולריים מקריים, ההסתברות לקיום ארבעה קדקדים עם יותר מארבע צלעות שואפת לאפס, אנו מקבלים את המשפט שבכותרת: כמעט לכל גרף 2r-רגולרי יש חלוקה פנימית.

Asymptotically Almost Every $2r$-regular Graph has an Internal Partition

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Abstract

An *internal partition* of a graph is a partitioning of the vertex set into two parts such that for every vertex, at least half of its neighbors are on its side. We prove that for every positive integer r, asymptotically almost every $2r$-regular graph has an internal partition.

1 Introduction

1.1 Notations

We denote the neighbor set of a vertex x in a graph $G = (V, E)$ by $N(x)$ and its degree by $d(x)$. The number of neighbors that x has in a subset $A \subseteq V$ is denoted by $d_A(x) = |N(x) \cap A|$. We denote by $\mathbb{N}_{\geq k}$ the set $\{k, k+1, \ldots\}$. For a vertex x and a set A we use the shorthand notation $A \cup x$ and $A \backslash x$ rather than $A \cup \{x\}$ and $A \backslash \{x\}$ respectively.

Our main concern is with *partitions* of V into two parts $\langle A, B \rangle$. We denote by $e(A, B)$ the *cut size* of this partition, i.e., the number of edges $e = (x, y)$ with $x \in A$ and $y \in B$.

1.2 Internal Partitions

Let $G = (V, E)$ be a simple undirected graph. A partition $\langle A, B \rangle$ of V is called *external* if every vertex has at least as many neighbors on the other side as it has on its own side. Clearly, every graph has an external partition, e.g., one that maximizes the cut size $e(A, B)$. Likewise, in an *internal partition* every vertex has at least as many neighbors on its own side as on the other side. This requirement is clearly satisfied by the trivial partition $\langle \emptyset, V \rangle$, but we insist on a *non-trivial* internal partition where both parts are non-empty.

Yet, it is worthwhile to consider the following more general definition:

Definition 1.1. *Let $G = (V, E)$ be a graph and let $a, b : V \to \mathbb{N}$ be two functions. We say that a partition $\langle A, B \rangle$ of V is* (a, b)-internal*, if:*

1. *$d_A(x) \geq a(x)$ for every $x \in A$, and*
2. *$d_B(x) \geq b(x)$ for every $x \in B$.*

In these terms, an *internal partition* as mentioned above is the same as a $\left(\lceil \frac{d(x)}{2} \rceil, \lceil \frac{d(x)}{2} \rceil\right)$-internal partition.

A vertex in A (resp. B) for which condition (1) (resp. (2)) holds, is said to be *satisfied*. Otherwise, we say that it is *unsatisfied*. Clearly, a partition is internal iff all vertices are satisfied.

The problem of the existence of, or the efficient finding of, internal partitions appears in the literature under various names: *Decomposition under Degree Constraints* [T83], *Cohesive Subsets* [M00], *q-internal Partition* [BL16] and *Satisfactory Graph Partition* [GK00]. A related subject is *Defensive Alliance Partition Number* of a graph. A short survey about *alliance* and *friendly* partitions, and the link to the internal partitions, can be found in [GE08]. A generalization (and a survey) of *Alliance Partitions* can be found in [FR14].

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It is also interesting to reformulate, saying that $\langle A, B\rangle$ is an (a, b)-internal partition iff $d_B(x) \leq d(x) - a(x)$ for every $x \in A$ and $d_A(y) \leq d(y) - b(y)$ for every $y \in B$. From this perspective, the existence of an (a, b)-internal partition can be viewed as a strong *isoperimetric inequality* for G. Recall that the isoperimetric number (or Cheeger constant) of a graph is min $\frac{e(A,B)}{|A|}$ over all partitions $\langle A, B\rangle$ and where $|A| \leq |B|$. That is, it is defined by a set $A \subset V$ which minimizes the *average* $\frac{1}{|A|}\sum_{x\in A} d_B(x)$, whereas in considering (a, b)-internal partitions, we bound $d_B(x)$ for *each* vertex in A. We return to this perspective in the open problems section.

While every graph has an external partition, there are simple examples of graphs with no *internal* partitions, e.g., cliques or complete bipartite graphs in which at least one part has odd cardinality. On the other hand, it is not easy to find large sparse graphs that have no internal partition. Also, as some of the theorems next mentioned show, *nearly* internal partitions (the exact meaning of this is clarified below) always exist.
Stiebitz [S96], responding to a problem of Thomassen [T83], made a breakthrough in this area. His results and later work by others in a similar vein are summarized in the following theorem.

Theorem 1.2. *Let $G = (V, E)$ be a graph and let $a, b : V(G) \to \mathbb{N}$. Each of the following conditions implies the existence of an (a, b)-internal partition:*

1. *[S96] $d(x) \geq a(x) + b(x) + 1$ for every $x \in V$.*
2. *[K98] $d(x) \geq a(x) + b(x)$ for every $x \in V$ and G is triangle-free.*
3. *[D00][GK04] $d(x) \geq a(x) + b(x) - 1$ for every $x \in V$ and girth$(G) \geq 5$.*

A very recent manuscript by Ma and Yang [MY17] shows that in the last statement of the theorem it suffices to assume that G is C_4-free.

Further work in this area falls into several main categories:

- **How hard is it to decide whether a given graph has an (a, b)-internal partition?** This question is investigated in a series of papers by Bazgan et al., surveyed in [BTV10]. Each existence statement in theorem 1.2 comes with a polynomial time algorithm to find the promised partition. For large a and b the problem seems to become difficult. For example, a theorem of Chvátal [C84] says that the case $a = b = d - 1$ is NP-hard for graphs in which all vertex degrees are 3 and 4.
- **Generalizations and variations:** Gerber and Kobler [GK00] introduced vertex- and edge-weighted versions of the problem and showed that these are NP-complete. Recent works by Ban [B16] and by Schweser and Stiebitz [SS17] extend theorem 1.2 to edge-weighted graphs. It is NP-hard to decide the existence of an internal *bisection* i.e., an internal partition with $|A| = |B|$ [BTV06]. There is literature concerning approximate internal partitions, partitions into more than two parts etc. See [GK03], [GK04] , [BTV05], [BTV10].
- **Sufficient conditions:** Shafique and Dutton [S02] and [GK04] provide several sufficient conditions for the existence of an internal partition in general graphs, and more specific conditions for line-graphs and triangle-free graphs.
- **Necessary conditions:** It is proved in [S02] that there is no forbidden subgraph characterization for the existence or non-existence of an internal partition. Given a graph's edge-density it is possible to bound the cardinality of the parts of an internal partition, if it exists [GK00].
- **Regular graphs:** For $d = 3, 4$ the only d-regular graphs with no internal partition are $K_{3,3}, K_4, K_5$ [S02]. As shown by Ban and Linial [BL16], every 6-regular graph with 14 or more vertices has an internal partition. The case of 5-regular graphs remains open.

The most comprehensive survey of the subject of which we know is [BTV10].

2 The Theorem : 2r-regular Graphs

As mentioned, the repertoire of graphs with no internal partitions seems rather limited, and for $d \in \{3, 4, 6\}$ there are only finitely many d-regular graphs with no internal partition. This has led to the following conjecture [BL16]:

Conjecture 2.1. *For every d there are only finitely many d-regular graphs with no internal partitions.*

The main theorem of this paper is a weaker version of this conjecture, namely, an asymptotic result for an even d.

We say that a graph G is 4-*sparse* if every set of four vertices spans at most four edges (i.e., has no K_4 and no diamond-graph as subgraph). We prove:

Theorem 2.2. *Let G be a graph and let $a, b : V(G) \to \mathbb{N}_{\geq 2}$ be such that $d(x) \geq a(x) + b(x)$ for every vertex $x \in V$. If G is 4-sparse then it has an (a, b)-internal partition.*

Corollary 2.3. *If G is a 4-sparse graph and all its vertices have even degrees, then G has an internal partition.*

We note the following simple fact about random regular graphs:

Proposition 2.4. *For every $d \geq 3$ asymptotically almost every d-regular graph is 4-sparse.*

Proof. We work with the configuration model of random n-vertex d-regular graphs. Let X be the random variable that counts the number of sets of four vertices that span five or six edges. Then $\mathbb{E}(X) \leq O(n^4) \cdot \frac{(dn-11)!!}{(dn-1)!!} = O(\frac{1}{n})$. $\square$

We can now conclude the theorem in the title of this paper, namely:

Corollary 2.5. *Asymptotically almost every $2r$-regular graph has an internal partition.*

Before we prove theorem 2.2, we need to introduce several definitions.

Definition 2.6. *Let $G = (V, E)$ be a graph, and let $f : V \to \mathbb{N}$.*

- *We say that $A \subseteq V$ is f-internal if $d_A(x) \geq f(x)$ for every $x \in A$.*
- *We say that $A \subseteq V$ is f-degenerate if every non-empty subset $K \subseteq A$ has a vertex $x \in K$ such that $d_K(x) \leq f(x)$.*

Remark 2.7. *Clearly, a non-empty set is not $(a-1)$-degenerate if and only if it contains a non-empty a-internal subset.*

Definition 2.8. *Let $A, B \subseteq V$ be non-empty disjoint sets, and let $f, g : A \to \mathbb{N}$. The pair $\langle A, B\rangle$ is said to be*

- *(f, g)-internal if A is f-internal and B is g-internal.*
- *(f, g)-degenerate if A is f-degenerate and B is g-degenerate.*

It is an (f, g)-degenerate partition if it is an (f, g)-degenerate pair and, in addition, $B = V \setminus A$.

We claim next that for every (a, b)-degenerate partition $\langle A, B\rangle$ it holds that $|A| \geq 2$ (and, by symmetry, also $|B| \geq 2$). Otherwise, every vertex $x \in B$ satisfies $d_B(x) = d(x) - d_A(x) \geq d(x) - 1 > d(x) - a(x) = b(x)$, so that B is not b-degenerate. The last inequality uses the assumption that $a(v) \geq 2$ for all $v \in V$.

Proof of Theorem 2.2

For the proof of 2.2 it clearly suffices to consider the case where $d(x) = a(x) + b(x)$ for every $x \in V$. Our proof is based on the methodology initiated by Stiebitz [S96]. The proof is by contradiction. We assume that G is a counterexample, i.e., a 4-sparse d-regular graph with no (a, b)-internal partition.

We next define an objective function on vertex partitions of the graph. By optimizing it, we will achieve a contradiction.

For a function $f : V \to \mathbb{N}$ and $S \subseteq V$, we denote $f(S) := \sum_{x \in S} f(x)$. We make substantial use of the following function w that is defined for every partition $\langle A, B\rangle$ of V as follows:

$$w(A, B) = a(B) + b(A) - e(A, B)$$

We calculate the change in w when one vertex changes sides. If $\langle A', B'\rangle = \langle A \cup x, B\backslash x\rangle$, then

$$w(A', B') - w(A, B) = 2\,(b(x) - d_B(x))$$

and if $\langle A'', B''\rangle = \langle A\backslash y, B \cup y\rangle$, then

$$w(A'', B'') - w(A, B) = 2\,(a(y) - d_A(y))$$

We denote by Δw the change in w and call a partition of V *locally maximal* if $\Delta w \leq 0$ whenever a single vertex moves from one part to the other.
For the sake of completeness, we reproduce two easy but crucial propositions, following Stiebitz.

Proposition 2.9. *If G has an (a, b)-internal pair, then it has an (a, b)-internal partition.*

Proof. Consider an (a, b)-internal pair of sets $A, B \subset V$, that maximizes $|A| + |B|$. If $\langle A, B\rangle$ is not an internal partition, then $U := V\backslash(A \cup B)$ is non-empty. By maximality, for every vertex $x \in U$ it holds that $d_A(x) \leq a(x) - 1$, and hence $d_{B\cup U}(x) \geq b(x) + 1$. Thus the partition $\langle A, B \cup U\rangle$ is (a, b)-internal contrary to the assumed maximality. □

Proposition 2.10. *If G has no (a, b)-internal partition, and if $A \subsetneq V$ is not $(a - 1)$-degenerate, then $V\backslash A$ is $(b - 1)$-degenerate.*

Proof. Since A is not $(a - 1)$-degenerate it has a non-empty a-internal subset. If $V\backslash A$ is not $(b - 1)$-degenerate, then it contains a non-empty b-internal subset. This gives an (a, b)-internal pair, contrary to Proposition 2.9. □

Consider the family of non-empty sets $A \subsetneq V$ that are a-degenerate, but not $(a-1)$-degenerate. This family is not empty (for instance, take A to be an inclusion-minimal a-internal subset). Among this family of sets, take a set A such that

- the partition $\langle A, V\backslash A\rangle$ is locally maximal for w, and
- minimizes $|A|$ under these assumptions.

We claim now that the resulting set A *is* a-internal. Otherwise, there is $v \in A$ such that $d_A(v) \leq a(v) - 1$. Then, however, $A\backslash v$ is non-empty a-degenerate and not $(a - 1)$-degenerate. In addition, upon moving v from A to $B = V\backslash A$ it holds that $\Delta w \geq 2$, contradicting the maximality of w.

We next consider the vertices of "low internal-degree" in A and in B. Denote

$$C = \{v \in A \mid d_A(v) = a(v)\} \quad \text{and} \quad D = \{v \in B \mid d_B(v) \leq b(v) - 1\}.$$

Note that $C \neq \emptyset$ since A is a-degenerate. Also $D \neq \emptyset$, since B is $(b-1)$-degenerate by Proposition 2.10. For every $x \in D$ by moving x to A we get $\Delta w \geq 2$. Also, for $y \in C$, by moving y to B we get $\Delta w \geq 0$.

Proposition 2.11. *For every $x \in D$ there is a subset $A_x \subseteq A$ such that $A_x \cup x$ is $(a+1)$-internal.*

Proof. As mentioned, moving $x \in D$ to A yields $\Delta w \geq 2$. Also, $A \cup x$ is clearly not $(a - 1)$-degenerate. Consequently, by the maximality of $w(A, V\backslash A)$, $A \cup x$ cannot be a-degenerate, and it must contain an $(a + 1)$-internal subset. This $(a + 1)$-internal subset must contain x, as claimed. Note also that A_x is necessarily a-internal. □

Proposition 2.12. *For every $x \in D$ and for every such set A_x it holds that $C \subseteq A_x$.*

Proof. Suppose there is $y \in C\backslash A_x$. Clearly, $A\backslash y$ is a-degenerate. Also, $A\backslash y$ is not $(a-1)$-degenerate since it contains the a-internal subset A_x. In addition $w(A\backslash y, B \cup y) = w(A, B)$, contradicting the minimality of $|A|$. □

Proposition 2.13. *Every vertex in C is adjacent to every vertex in D.*

Proof. Consider some $x \in D$ and $y \in C$. Then $d_A(y) = a(y)$. But y also belongs to the $(a + 1)$-internal set $A_x \cup x$, so that $d_{A\cup x}(y) = a(y) + 1$. The conclusion follows. □

Proposition 2.14. *There are two adjacent vertices in C.*

Proof. We know already that $C \neq \emptyset$. Consider some $y \in C$. Clearly $A \backslash y$ is a-degenerate. Also $w(A\backslash y, B \cup y) = w(A, B)$, therefore, by the minimality of $|A|$, the set $A \backslash y$ must be $(a-1)$-degenerate. In particular there is $z \in A$ such that $d_{A\backslash y}(z) \leq a(x) - 1$, whereas $d_A(z) \geq a(x)$. It follows that $z \in C$ and $yz \in E(G)$. □

The above propositions, and the 4-sparsity of G imply that

- Neither y nor z have an additional neighbor in C.
- The only common neighbor of y and z is $x \in D$.

In other words, $d_{A\backslash\{y,z\}}(v) \geq a(v)$ for every $v \in A\backslash\{y, z\}$. This means that $A' := A\backslash\{y, z\}$ is a-internal. Therefore, A' is a-degenerate, and not $(a-1)$-degenerate. Also, $A' \neq \emptyset$ since $d_A(y) = a(y) \geq 2$. Finally, $w(A', V \backslash A') \geq w(A, V \backslash A)$, contrary to the minimality of $|A|$. □

3 Some Computational and Experimental Results

3.1 Algorithmic Realization

As mentioned, to the best of our knowledge, all previous existence proofs of (a, b)-internal partitions translate into polynomial-time algorithms (under specifc assumptions). This applies as well to Theorem 2.2 and its proof. Thus, let G be a 4-sparse graph, and assume $d(x) = a(x) + b(x)$ for every $x \in V(G)$. We maintain from the proof of the theorem the definitions of the sets C and D. Here is a polynomial-time algorithm that finds an (a, b)-internal partition of G:

Initialization. Find a set A that is a-degenerate but not $(a-1)$-degenerate. Repeatedly remove any vertex $x \in A$ with $d_A(x) < a(x)$ until none remain.

Loop. While $V \backslash A$ is $(b-1)$-degenerate:

1. If there is $x \in D$ such that $A \cup x$ is a-degenerate, update $A := A \cup x$.
2. Else, there is a triangle xyz with $x \in D$ and $y, z \in C$. Update $A := A\backslash\{y, z\}$.

Note that the *initialization* can be done in polynomial time, since (i) for any $\varphi : V \to \mathbb{N}$ it takes linear time to check whether a given $S \subseteq V$ is φ-degenerate, (ii) G is $(a + b)$-internal, (iii) if S is not $(\varphi + 1)$-degenerate, then $S\backslash x$ is not φ-degenerate for every $x \in S$ and (iv) if, moreover, $d(x) \leq \varphi(x)$, then $S\backslash x$ is not $(\varphi + 1)$-degenerate.
The if-else *dichotomy* is proven in Propositions 2.13 and 2.14.
The algorithm *terminates*, since in step (1) $\Delta w > 0$, and step (2) decreases $|A|$ while keeping $\Delta w \geq 0$. Termination is proved by induction on lexicographically-ordered pairs $(w, -|A|)$.
A remains not-$(a-1)$-degenerate throughout, and upon termination $V \backslash A$ is not $(b-1)$-degenerate. Therefore, A and $V \backslash A$ contain a-internal and b-internal subsets, respectively, which is, according to proposition (2.9), a sufficient condition for the existence of an internal partition that can be found in polynomial time (See Proposition 1 in [BTV03, Technical Report]).

3.2 Improving Previous Experimental Results

In [GK00] some experimental results are presented. They apply a heuristic algorithm in an attempt to find a $\left(\lceil\frac{d(x)}{2}\rceil, \lceil\frac{d(x)}{2}\rceil\right)$-internal partition in random graphs. Their algorithm starts from a random partition and at each iteration minimizes $f(A, B) = \sum_{v\in A}(d_A(v) - d_B(v))^+ + \sum_{v\in B}(d_B(v) - d_A(v))^+$ where the minimum is taken over all partitions which were achieved by switching an unsatisfied vertex. The process can terminate with either an internal partition or a trivial partition. It can also loop indefinitely. In the latter two cases, they restart the process.

We have experimented with a similar algorithm. The main change is that we consider only near-bisections $\langle A, V \backslash A\rangle$, and insist that $\left||A| - \frac{|V|}{2}\right| \leq c(n)$ for $c(n) = \log_d(n)$. When this condition is violated, we move a random vertex from the big part to the small. This algorithm may either output an internal partition or loop forever. However, in extensive simulations with random d-regular graphs ($30 \leq n \leq 10000$ and $4 \leq d \leq \min(50, \frac{n}{2})$) the algorithm has always found an internal partition in fewer than $5n$ iterations. It may be that this phenomenon is barely affected by the choice of c. In 5.4 we present an approach and a conjecture supporting this argument.

4 Discussion and Some Open Problems

4.1 Even vs. Odd Degrees

The problem of internal partition for d-regular graphs seems harder for *odd* degree d; for an even d, a random vertex in a random partition is biased towards being satisfied, while for an odd d, the probability is exactly 0.5. Thus, perhaps, it is no coincidence that we were able to settle the case of $2r$-regular graphs, but are still unable to prove the analogous statement for $2r+1$ regular graphs. To take a case in point, while the cases of 4- and 6-regular graphs were solved, the case of 5-regular graphs is the first unknown case of Conjecture 2.1.

4.2 Further Open Problems

We have mentioned above the analogy between the existence of internal partitions and upper bounds on Cheeger constants. As shown by Alon [A97], for every large d-regular graph the Cheeger constant is at most $\frac{d}{2} - c\sqrt{d}$ for some absolute $c > 0$. Although conjecture 2.1 is still open, this prompts an even more far-reaching possibility.

Problem 4.1. *Is it true that for every integer $\delta \geq 1$ there are integers d and n_0 such that every d-regular graph on $n > n_0$ vertices has a $(\frac{d}{2}+\delta, \frac{d}{2}+\delta)$-internal partition ?*

Also, the upper bound on Cheeger's constant in Alon's paper is actually attained by a bisection (the two parts differ in size by at most one). This suggests:

Problem 4.2. *Does conjecture 2.1 hold also with "near" bisections? E.g., where the cardinalities of the two parts differ by $O_d(1)$.*

How does the computational complexity of the internal partition vary as n grows? So far, existence theorems have gone hand-in-hand with efficient search algorithms. Is this a coincidence or is there a real phenomenon?

Problem 4.3. *Consider the computational complexity of both the decision and the search version of the internal partition problem for d-regular graphs on n vertices. Conjecture 2.1 says that an internal partition exists whenever $n > n_0(d)$. Is there some $n_1(d)$ such that for $n > n_1(d)$ an internal partition can be found efficiently? If so, do n_0 and n_1 coincide?*

5 Other Approaches to the Problem

We collect below some information on other approaches to the problem that we had attempted. Although we were not able to develop any of them to completion, they seem worth recording.

Let $G = (V, E)$ be a graph. In what follows, we view a vertex partition of G as a mapping $x : V \to \{-1, 1\}$, and for $i \in V$ we denote $x(i)$ by x_i. We denote by M the adjacency matrix of G.

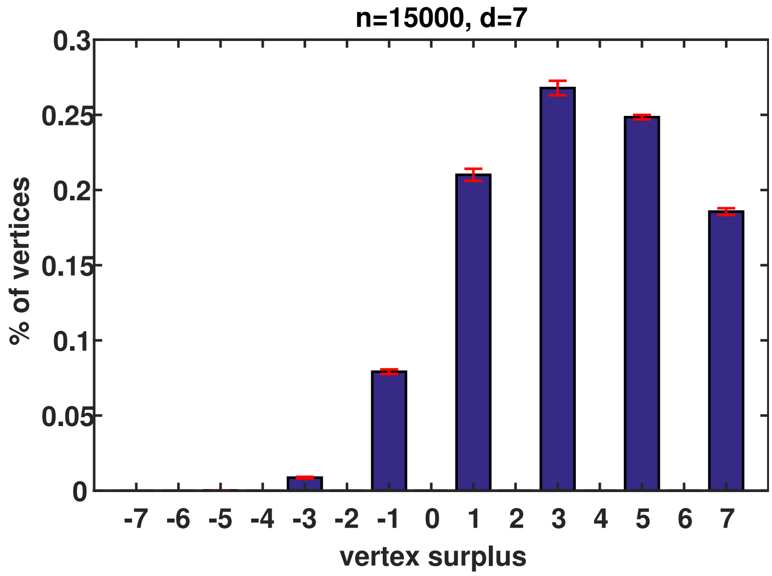
5.1 Formulation as a fixed point problem

For simplicity, let us momentarily assume that all vertex degrees in G are odd. Then x is an internal partition of G iff $\text{sign}(Mx) = x$. If we allow G to have vertices of even degree as well, that condition is relaxed and is considered satisfied even if it holds only at the nonzero coordinates of Mx. Viewed from this perspective, the internal partition problem is the search for fixed points of the nonlinear map $x \to \text{sign}(Mx)$. It is conceivable that ideas from fixed-point theory can be used to this end.

5.2 Second eigenfunction partitions

Let $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of M with $u_1, \ldots, u_n$ as corresponding l_2-normalized eigenfuctions. By standard linear algebra (Perron-Frobenius Theorem and the variational definition of eigenvalues), we know the following facts: The function u_1 is positive, the function u_2 is orthogonal to u_1, and under orthogonality and normality conditions, u_2 minimizes the quadratic form uMu^T. This intuitively suggests that partitioning V according to the sign of u_2, namely, *the second eigenfunction partition*, gives a fairly balanced partition with a small edge cut, seemingly,

Figure 1: A histogram of vertex *surpluses* for random 7-regular graphs with $n = 15,000$ vertices. Note that only fewer than 9% of the vertices have a negative surplus. The red error bars show one standard deviation of the sample.



a good indication of being close to internal.

To measure "how internal" a partition x is, we define the *surplus* of every vertex $i \in V$ as $s(i) = \sum_{j \quad i} x_i \cdot x_j$. Clearly, a vertex is satisfied if it's surplus is non-negative. The *surplus* of a partition is the (normalized) distribution of the vertex surpluses.

We have experimented with random d-regular graphs, and have found that the vast majority of vertices are satisfied in the second eigenfunction partition. For odd d, more than 91% of the vertices are satisfied, while for even d the number is around 97%. These numbers hardly change over the range $3 \leq d \leq 13$ and n going up to 30000. See e.g., figure 1.

In fact, an even more intriguing phenomenon emerges. Not only does the fraction of unsatisfied vertices seem to converge, but also, for every $-\sigma \leq \sigma \leq d$, the percentage of vertices with surplus σ seems to converge. In other words, for fixed d, as n grows, the *surplus distribution* is converging.

As can be seen in Appendix A, this phenomenon persists for different values of d, and the convergence as n grows is rapid.

In our experiments we sampled only a small number of graphs for each pair (d, n), where $3 \leq d \leq 15$ and $n \in \{1500, 10000, 15000, 30000\}$. The tiny standard deviation that we observe, strongly suggests that this is a real and robust phenomenon. Clearly, further investigation and experiments are needed to understand and prove this phenomenon.

5.3 A Quadratic Programming approach

The internal partition problem can be formulated as a Quadratic Program (QP). We maintain the notation that a vertex partition is a map $x : V \to \{-1, 1\}$, i.e., $x \in \left(S^0\right)^n$. The problem takes the form:

$$\begin{aligned} \text{Find} \quad & x \in \left(S^0\right)^n \\ \text{s.t.} \quad & \forall i \in V : \sum_{j \sim i} x_i x_j \geq 0 \\ & n > \sum_{i \in V} x_i > -n \end{aligned}$$

The last inequality captures the requirement for a non-trivial partition.

In a natural relaxation of the above problem, we replace the condition $x_i \in \{-1, 1\} = S^0$ by the condition $x_i \in S^{n-1}$. In other words, we seek n unit vectors, one for each vertex. The conditions $\sum_{j\sim i} x_i x_j \geq 0$ are now translated into the inner product conditions $\sum_{j\sim i} \langle x_i, x_j\rangle \geq 0$.
It is harder to capture the non-triviality condition in the relaxed problem. This can be achieved, e.g., by requiring the existence of two vertices $i, j \in V$ such that $\langle v_i, v_j\rangle < 0$.

Hence, we can define the SDP relaxation for the internal partition problem:

$$\begin{aligned} \text{Find} \quad & Y = X^T X \in SDP_n \\ \text{s.t.} \quad & \forall i \in [n] : \sum_{j\sim i} y_{ij} \geq 0 \\ & \exists i, j \in [n] : y_{ij} < 0 \end{aligned}$$

Note that for an odd d, $\sum_{j\sim i} y_{ij}$ never equals zero, and therefore we can impose a stronger condition $\sum_{j\sim i} y_{ij} \geq 1$. This is true for both the QP problem and the SDP relaxation.

Various constraints that hold for every solution in $\left(S^0\right)^n$ can be added, and translated to the language of SDP problems.

Practically, the feasibility problem of those SDPs was found to be hard.

Clearly, if there is a solution for the integral problem, there is a solution for the SDP problem. The other way is not necessarily true. Nevertheless, in simulations we found that projecting the SDP solution onto one dimension and rounding to ± 1 results in a good approximation of an internal partition in the sense that most of the vertices were satisfied.

5.4 Other approaches to asymptotic statements

Recall Conjecture 2.1 according to which, for every d there are only finitely many d-regular graphs with no internal partition. A weaker version of the same idea speaks about the asymptotic almost sure existence of internal partitions, as in our main Theorem 2.2. Below is an approach that we had tried without success, but still believe to be worth mentioning. This method is intended to work in two steps. In the first, we prove that if a partition $\left\langle A, \bar{A}\right\rangle$ is both locally maximal and nearly balanced, it is internal. In the second step, we had tried to prove that asymptotically almost every d-regular graph has a partition that is both locally maximal and nearly balanced. We will start by defining *nearly balanced* partition and objective function.

Definition 5.1. *A nearly balanced partition $\left\langle A, \bar{A}\right\rangle$ is a partition such that*

$$\frac{(d-\pi)(n-1)+2d}{2d} < |A| < \frac{(d+\pi)(n-1)}{2d}$$

where $\pi = \pi(d)$ is 1 or 2 if d is odd or even, respectively.

Definition 5.2. *Let $\left\langle A, \bar{A}\right\rangle$ be a partition of a d-regular graph. The excess of the partition is defined as*

$$f(A, \bar{A}) = \frac{d}{n-1} \cdot |A| \cdot \left|\bar{A}\right| - e\left(A, \bar{A}\right) \quad .$$

Excess is essentially the same as discrepancy, a notion of great importance in many parts of combinatorics. We say that $\left\langle A, \bar{A}\right\rangle$ is *locally maximal* if its excess does not increase as we move a single vertex between A and $\bar{A}$.

Lemma 5.3. *A partition $\left\langle A, \bar{A}\right\rangle$ is locally maximal iff*

$$\forall x \in A \quad d_A(x) \geq \frac{|A|-1}{n-1} \cdot d \quad \textit{and} \quad \forall y \in \bar{A} \quad d_{\bar{A}}(y) \geq \frac{n-|A|-1}{n-1} \cdot d \quad .$$

Proof. Let us spell out the condition that $f(A) \geq f\left(A \backslash \{x\}\right)$:

$$\frac{d}{n-1} \cdot |A| \cdot \left|\bar{A}\right| - e\left(A, \bar{A}\right) \geq \frac{d}{n-1} \cdot (|A|-1) \cdot \left(\left|\bar{A}\right|+1\right) - e\left(A \backslash \{x\}, \bar{A} \bigcup \{x\}\right) \quad .$$

The claim follows by straightforward calculation. The argument for $y \in \bar{A}$ is identical. $\square$

For $A \subseteq V$ of cardinality $|A| = \frac{(d-\pi)(n-1)+2d}{2d}$ we find that $\langle A, \bar{A} \rangle$ is locally maximal iff

$$\forall x \in A \;\; d_A(x) \geq \frac{d-\pi}{2} \;\; \text{and} \;\; \forall y \notin A \;\; d_{\bar{A}}(y) \geq \frac{d+\pi}{2} \;\; .$$

The second calculation yields $d_{\bar{A}}(y) \geq \frac{d+\pi}{2} - \frac{d}{n-1}$ which we can round up, since $\frac{d+\pi}{2}$ and $d_{\bar{A}}(y)$ are integers and $\frac{d}{n-1} < 1$.

Lemma 5.4. *Every locally maximal nearly balanced partition $\langle A, \bar{A} \rangle$ of a graph $G = (V, E)$ is internal.*

Proof. By Lemma 5.3 $d_A(x) \geq d \cdot \frac{|A|-1}{n-1}$ for every $x \in A$. Also $|A| > \frac{(d-\pi)(n-1)+2d}{2d}$, since $\langle A, \bar{A} \rangle$ is nearly balanced. Together, this implies that $d_A(x) > \frac{d-\pi}{2}$. But both sides are integers, so $d_A(x) \geq \frac{d-\pi}{2} + 1 \geq \frac{d}{2}$.

Similarly, by the local maximality (Lemma 5.3) $d_A(y) \geq d \cdot \frac{n-|A|-1}{n-1}$ for every $y \in \bar{A}$. The partition is nearly balanced, so that $|A| < \frac{(d+\pi)(n-1)}{2d}$. Combining these two inequalities, we find that $d_A(y) > \frac{d-\pi}{2}$. But both sides are integers, so $d_A(y) \geq \frac{d-\pi}{2} + 1 \geq \frac{d}{2}$. □

The following conjecture could replace the second step in our approach and yield a proof of Theorem 2.2.

Conjecture 5.5. *For every $d \geq 2$, asymptotically almost every d-regular graph has the following property: every locally maximal partition is nearly balanced.*

Even less would suffice:

Conjecture 5.6. *For every $d \geq 2$, asymptotically every d-regular graph has a locally maximal nearly balanced partition.*

Despite considerable efforts, these conjectures remain open, while substantial numerical experimentation suggests their correctness.

Regardless of whether or not these two conjectures are valid, Lemma 5.4 suggests an algorithmic approach. Namely, to greedily search for a locally maximal nearly balanced partition. As the lemma shows, such a partition is internal. As described in Section 3.2, we have experimented with a similar greedy algorithm which efficiently succeeded in finding internal partitions in all the sparse graphs in our sample. We have toyed with other objective functions, but have not experimented with them extensively enough to draw any solid conclusions. It is also interesting to see what happens if we modify the definition of a *local maximum* by allowing several vertices at once to switch sides. Getting better insights on the power of such methods is of great interest.

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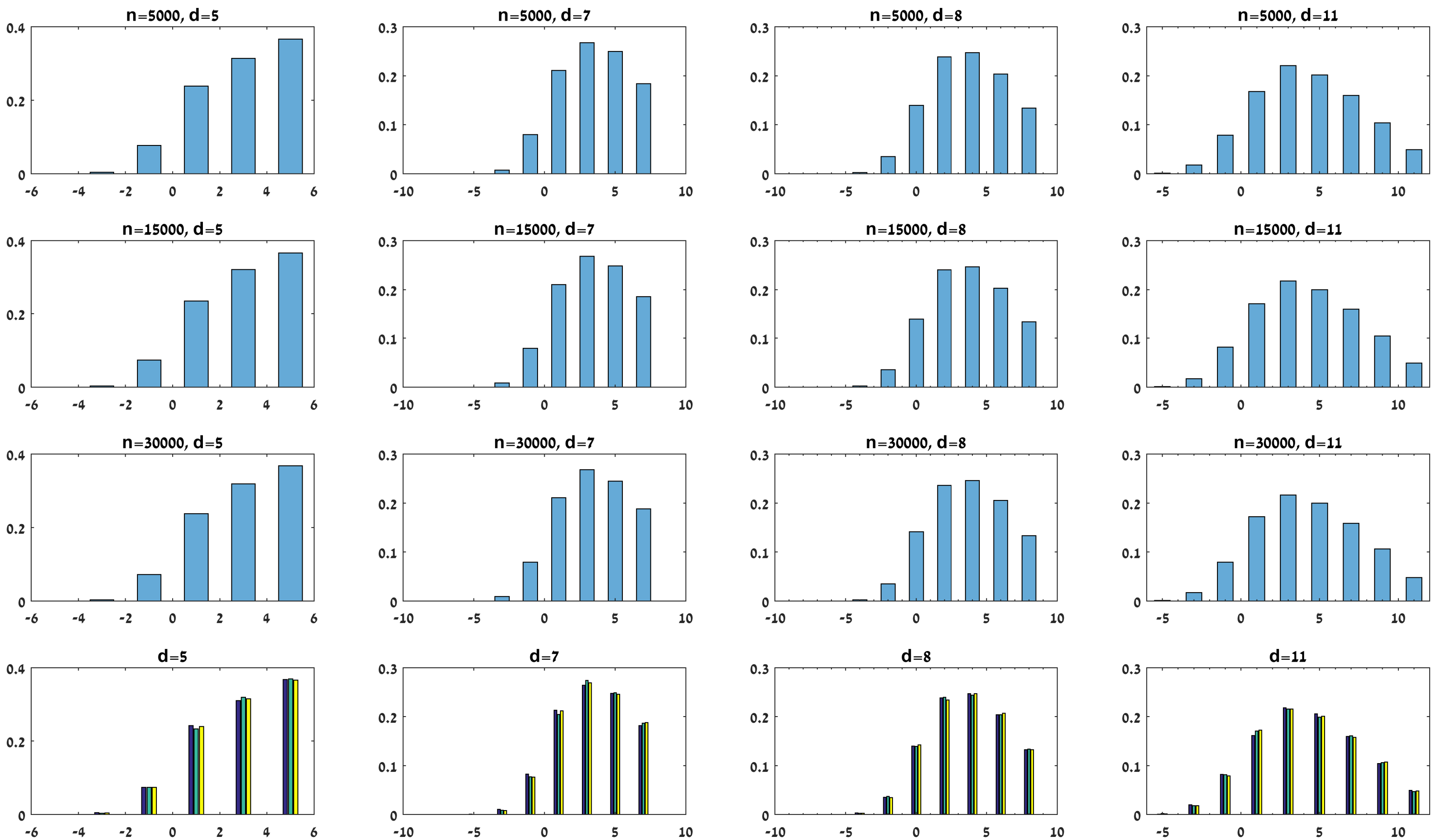
Appendix A : The surplus histogram for different values of (n,d)



Different vertex *surplus* histograms for random d-regular graphs of n vertices. Each figure represents an average of multiple simulations. The standard deviation was too small to be represented (similar to figure 1). The bottom row of figures shows the histograms for different n visually organized next to each other, displaying the fact that the surplus distributions converge relatively fast.