Sum Complexes - a New Family of Hypertrees

by

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Thesis Presented to
The School of Computer Science and Engineering
for the Degree of

Master of Science

The Faculty of Science
The Hebrew University of Jerusalem
Israel, January 2012

Acknowledgments

I would like to thank: Nati Linial for being a wonderful advisor. Especially, for introducing me to so many new fields that I didn't even know exisited; Roy Meshulam, for taking part in this research and for proving most of the proofs.

All my friends in the theory lab, espessialy Ofer Neiman, Golan Pundak, Elazar Goldenberg, and Or Sattath for making our workspace so interesting, fun and homelike.

Chloe Perin and Talia Fernos for helping me understand some Algebraic Topology.

My mother Liba Rosenthal, who always supported me, no matter what I decided to do.

Abstract

A k-dimensional hypertree X is a k-dimensional complex on n vertices with a full $(k-1)$-dimensional skeleton and $\binom{n-1}{k}$ facets such that $H_k(X;\mathbb{Q}) = 0$. Here we introduce the following family of simplicial complexes. Let n, k be integers with $k+1$ and n relatively prime, and let A be a $(k+1)$-element subset of the cyclic group $\mathbb{Z}_n$. The *sum complex* X_A is the pure k-dimensional complex on the vertex set $\mathbb{Z}_n$ whose facets are $\sigma \subset \mathbb{Z}_n$ such that $|\sigma| = k+1$ and $\sum_{x\in\sigma} x \in A$. It is shown that if n is prime then the complex X_A is a k-hypertree for every choice of A. On the other hand, for n prime X_A is k-collapsible iff A is an arithmetic progression in $\mathbb{Z}_n$.

Contents

1 Introduction

This Thesis is based on a published paper: Sum Complexes - A New Family of Hypertrees [1].

1.1 Motivation

It is almost impossible to imagine computer sciences or combinatorics without graphs. Computer scientists first encounter graphs during their first undergraduate year and continue to study and research them throughout their career. But what is the high-dimensional analogue of a graph, or in particular a tree? One possible generalization of graphs are hypergraphs. In a hypergraph an edge can include any number of vertices and not only two as in graphs. In this work we focus on a more natural topological/geometrical generalization of graphs called simplicial complexes. A *simplicial complex* X is a collections of finite sets which we call simplices or faces. This collection satisfies the following conditions: A subset of a simplex from X is also a simplex in X. This applies, in particular, to the intersection of any two simplices $\sigma_1, \sigma_2 \in X$. If $A \in X$ is a simplex, we define its *dimension* as $|A| - 1$ and the dimension of a simplicial complex is defined as the highest dimension of any of its simplices. To construct the *geoemtric realization* of the simplicial complex X we proceed as follows: We associate to every face $A \in X$ a (geometric) simplex of dimension $|A|-1$ in real space of high enough dimension and maintain the intersection pattern among the different faces of X. We note, in particular, that according to this definition a graph is just a 1-dimensional abstract simplicial complex. As was the case with graphs there are several ways to generalize the definition of trees. In this work we follow the lead of Kalai and discuss hypertrees as defined in [3]. We study some of the topological properties of a specific class of hypertrees called sum complexes. This peek in to the world of hypertrees will reveal some of its richness, that goes beyond the known phenomena in the realm of trees.

1.2 Some algebraic topology

In order to continue the discussion, we first need to familiarize ourselves with some algebraic topology. A fundamental problem of topology is that of determining for two spaces whether or not they are homeomorphic. To show that two spaces are homeomorphic, one needs to construct a continuous bijective map, with continuous inverse, mapping one space to the other. To show that two spaces are not homeomorphic one must rule out the existence of such a map. This goal is often harder to accomplish. The usual way to proceed is to find some topological property (i.e. some property that is invariant under homeomorphisms) that distinguishes between the two spaces. For example, the closed unit disc in $\mathbb{R}^2$ is not homeomorphic with the plane $\mathbb{R}^2$, because the closed disc is compact and the plane is not.

The simplest topological property that we will be interested in is collapsibility. Let σ be a face of dimension at most $k-1$ of a simplicial complex X which is contained in a *unique* maximal face τ of X, and let $[\sigma, \tau] = \{\eta \ : \ \sigma \subset \eta \subset \tau\}$. The operation $X \to Y = X - [\sigma, \tau]$ is called an *elementary k-collapse*. X is *k-collapsible* if there exists a sequence of elementary k-collapses

$$X = X_1 \to X_2 \to \cdots \to X_m = \{\emptyset\} \quad .$$

The next few properties are part of the very rich theory of algebraic topology. Algebraic topology originated in the attempt by mathematicians like Poincare and Betti to construct such topological invariants. They found ways to associate certain groups with topological spaces. This includes homotopy groups (in particular the fundamental group) and homology groups which are topological invariants. Thus if the groups associated with the topological spaces X and Y are nonisomorphic, then we can conclude that X and Y are not homeomorphic.

The first homotopy group is *the fundamental group*, which records information about loops in a space. It is relatively easy to describe. Intuitively, homotopy groups record information about the basic shape, or holes, of a topological space. To define the n-th homotopy group, the base point preserving maps from an n-dimensional sphere (with base point) into a given space (with base point) are collected into equivalence classes, called homotopy classes. Two mappings are homotopic if one can be continuously deformed into the other. These homotopy classes form a group, called the n-th homotopy group, $\pi_n(X)$, of the given space X with base point. One can show that a number of familiar spaces, such as the sphere, torus, and Klein bottle can all be told apart topologically, since the fundamental groups are different.

Now we turn to define homology groups. There are several ways to define homology groups. In this work we focus on simplicial homology (and cohomology) which are probably the most tangible variants of homology theory. A simplex can be *oriented* by ordering its vertices. We will abuse notation and denote both the simplex σ and the oriented simplex as $\sigma = \langle v_0, v_1, \ldots, v_k \rangle$. Let X be a simplicial complex. A *simplicial k-chain* is a formal sum of oriented k-simplices $\sum_{i=1}^{N} c_i \sigma^i$ where $c_i \in \mathbb{Z}$, $\sigma^i \in X$ and it holds that $-\sigma = \sigma^{-1}$ the opposite orientation of σ. We denote by C_k the free group of k-chains on X (with respect to the natural addition). Consider a basis element of C_k, a k-simplex, $\sigma = \langle v_0, v_1, \ldots, v_k \rangle$. The *boundary operator* $\partial_k : C_k \to C_{k-1}$ is a homomorphism defined by $\partial_k(\sigma) = \sum_{i=0}^{k} (-1)^i \langle v_0, \ldots, \hat{v}_i, \ldots, v_k \rangle$ where the simplex $\langle v_0, \ldots, \hat{v}_i, \ldots, v_k \rangle$ is the oriented face of σ obtained by deleting its i-th vertex. In C_k, elements of the subgroup $Z_k = \ker \partial_k$ are called *cycles*, and the members of the subgroups $B_k = \operatorname{im} \partial_{k+1}$ are called *boundaries*. Direct computation shows that $\partial_k \circ \partial_{k+1} = 0$, i.e., the boundary of a boundary must be zero. It follows that $B_k \subseteq Z_k$, or, in other words, (C_k, ∂_k) form a *simplicial chain complex*. The k-th homology group H_k of X is defined to be the quotient $H_k(X) = Z_k / B_k$. A homology group H_k is not trivial if the complex at hand contains k-cycles which are not boundaries. This indicates that there are k-dimensional holes in the complex. This definition can be generalized by replacing $\mathbb{Z}$ by an arbitrary abelian group as *coefficient group*. For example we will denote by $H_k(X; \mathbb{Q})$ the k-th rational homology of X.

With each simplicial complex X, we have associated a sequence of abelian groups called its homology groups. Now, we associate with X another sequence of abelian groups, called its *cohomology groups*. These groups appeared much later and are geometrically much less natural than homology groups. However, it eventually became clear that these groups are both important in theory and useful in practice. Let X be a finite simplicial complex on the vertex set V. For a set S and a field $\mathbb{K}$, let $\mathcal{L}(S, \mathbb{K})$ denote the $\mathbb{K}$-linear space of all $\mathbb{K}$-valued functions on S. The space $C^m(X; \mathbb{K})$ of $\mathbb{K}$-valued m-cochains of X consists of all functions $\phi \in \mathcal{L}(V^{m+1}, \mathbb{K})$ such that $\phi(v_0, \ldots, v_m) = \operatorname{sgn}(\pi)\phi(v_{\pi(0)}, \ldots, v_{\pi(m)})$ for any permutation π on $\{0, \ldots, m\}$, and such that $\phi(v_0, \ldots, v_m) = 0$ if $\{v_0, \ldots, v_m\}$ is not an m-dimensional simplex of X. (In particular, $\phi(v_0, \ldots, v_m) = 0$ if $v_i = v_j$ for some $i \neq j$.) The coboundary operator $d_m : C^m(X; \mathbb{K}) \to C^{m+1}(X; \mathbb{K})$ is given by

$$d_m \phi(v_0, \ldots, v_{m+1}) = \sum_{i=0}^{m+1} (-1)^i \phi(v_0, \ldots, \hat{v}_i, \ldots, v_{m+1}) \quad .$$

Let $Z^m(X; \mathbb{K}) = \ker d_m$ denote the space of m-cocycles of X over $\mathbb{K}$ and let $B^m(X; \mathbb{K}) = \operatorname{Im}\, d_{m-1}$ denote the space of m-coboundaries of X over $\mathbb{K}$. The

m-dimensional cohomology space of X with coefficients in $\mathbb{K}$ is

$$H^m(X;\mathbb{K}) = \frac{Z^m(X;\mathbb{K})}{B^m(X;\mathbb{K})} \quad .$$

Let $h^m(X,\mathbb{K}) = \dim_{\mathbb{K}} H^m(X;\mathbb{K})$. Then $h^m(X,\mathbb{K}) = h^m(X,\mathbb{F}) = h_m(X;\mathbb{F})$ for any algebraic extension $\mathbb{K}$ of $\mathbb{F}$. In order to establish Theorem 1.2 we may therefore assume that $\mathbb{F}$ already contains a primitive n-th root of unity ω.

From a topological point of view, a spanning tree is a 1-dimensional simplicial complex X such that the following hold:

1. X is collapsible.
2. X contractible (All the Homotopy groups vanish).
3. X is connected and acyclic, namely has trivial H_0 and H_1 homology groups over $\mathbb{Z}$.
4. X is connected and acyclic, namely has trivial H_0 and H_1 homology groups over $\mathbb{Q}$.

The four properties above are equivalent for 1-dimensional simplicial complexes. It can be shown for higher dimensional simplicial comlexes that collapsibility $\Rightarrow$ contractability $\Rightarrow$ vanishing integer homology $\Rightarrow$ vanishing rational homology. However, the reverse implication does not necessarily hold, as we will see when we encounter hypertrees. Indeed a main motivation for the present work was to find an infinite family of examples where these equivalences that hold in one dimension, fail to hold in higher-dimensional simpplicial complexes.

1.3 Hypertrees

We start with some standard notations. All simplicial complexes we consider X have n vertices, and we always identify the vertex set of X with the cyclic group $\mathbb{Z}_n$. The number of i-dimensional faces of X is denoted by $f_i(X)$. We denote by Δ_{n-1} the $(n-1)$-simplex on the vertex set $\mathbb{Z}_n$ and by $\Delta_{n-1}^{(i)}$ the i-dimensional skeleton of Δ_{n-1}. A *k-hypertree* is a simplicial complex $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ such that $f_k(X) = \binom{n-1}{k}$ whose k-th rational homology vanishes $H_k(X;\mathbb{Q}) = 0$. Throughout the paper we assume that $k+1$ is coprime to n. The reason we choose $f_k(X) = \binom{n-1}{k}$ is that we want both $H_k(X;\mathbb{Q}) = 0$ and $H_{k-1}(X;\mathbb{Q}) = 0$.

Claim 1.1. *Let X be a simplicial complex s.t. $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ then $h_k(X;\mathbb{Q}) = h_{k-1}(X;\mathbb{Q})$ iff $f_k(X) = \binom{n-1}{k}$.*

Proof. The *Euler characteristic* of a simplicial complex X is defined to be $\chi(X) = \sum_{i\geq 0}(-1)^i f_i(X)$. The well-known Euler-Poincare relation [2] can be viewed as a generalization of the familiar Euler formula for planar graphs

$$v - e + f = 2.$$

The Euler-Poincare relation asserts that

$$\sum_{i\geq 0}(-1)^i f_i(X) = \sum_{i\geq 0}(-1)^i h_i(X;\mathbb{Q}). \tag{1}$$

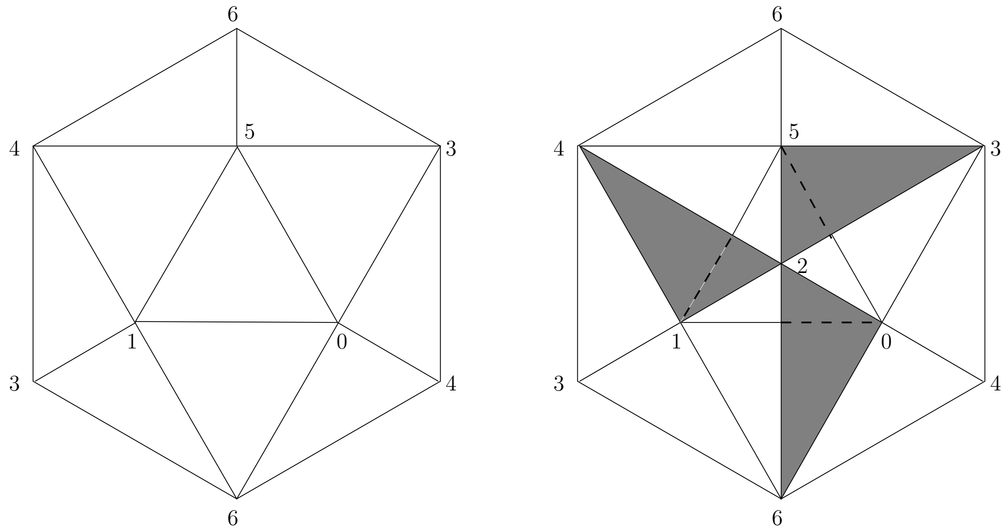
Since $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$, all h_j vanish for $j < k-1$, so that the right side of Equation (1) equals $(-1)^{k-1}(h_{k-1} - h_k)$. On the ohter hand, the left side equals $\sum_0^k (-1)^i \binom{n}{i} + (-1)^{k+1}\binom{n-1}{k}$, which can be readily verified to be zero. The claim follows.

□

1.4 Sum complexes

In this work we construct a family of complexes called sum complexes, which we will define via a simple arithmetic rule. We investigate some topological properties of sum-complexes. In particular we show that they are hypertrees, namely that their k and $k-1$ rational homologies vanish. Then we will fully characterize when sum-complexes collapse. This indicates that in most cases the topological structure of sum-complexes is richer than the simple construction of $\tilde{X}$ in the proof of Claim 1.1. Kalai's k-dimensional Cayley's formula [3] suggests that most k-hypertrees are not $\mathbb{Z}$-acyclic. We conjecture that when sum-complexes do not collapse they also have a torsion, namely the integer homology does not vanish. Another interesting question is to characterize the conditions under which hypertrees with vanishing integer homology collapse. In order to define sum-complexes we first define for $a \in \mathbb{Z}_n$, the set X_a to be the following collection of subsets of $\mathbb{Z}_n$:

$$X_a = \{\sigma \subset \mathbb{Z}_n : |\sigma| = k+1 \ , \ \sum_{x\in\sigma} x = a\} \ .$$



(a) The 6-point triangulation of $\mathbb{RP}^2$ (b) X_A for $A = \{0, 1, 3\} \subset \mathbb{Z}_7$

Figure 1

For a subset $A \subset \mathbb{Z}_n$ of cardinality $k+1$, define the *Sum Complex* X_A by

$$X_A = \Delta_{n-1}^{(k-1)} \cup (\cup_{a \in A} X_a) \ .$$

Example: Let $n = 7$, $k = 2$ and $A = \{0, 1, 3\} \subset \mathbb{Z}_7$. The 2-dimensional complex X_A (figure 1b) can be obtained from the standard 6-point triangulation of the real projective plane $\mathbb{RP}^2$ on the vertices $\{0, 1, 3, 4, 5, 6\}$ (figure 1a) by replacing the face $\{0, 1, 5\}$ with the three faces $\{0, 1, 2\}$, $\{0, 2, 5\}$, $\{1, 2, 5\}$, and adding the faces $\{2, 3, 5\}, \{0, 2, 6\}$ and $\{1, 2, 4\}$. Since the last 3 can be collapsed, X_A is clearly homotopy equivalent to $\mathbb{RP}^2$. We already know that $H_1(\mathbb{RP}^2; \mathbb{Z}) = \mathbb{Z}_2$ and $H_2(\mathbb{RP}^2; \mathbb{Z}) = 0$. It follows that both X_A and the standard 6-point triangulation of the real projective plane $\mathbb{RP}^2$ are hypertrees, and that both have non-trivial torsion.

In this paper we are concerned with topological and combinatorial properties of X_A. Let $\mathbb{F}$ be a field and let $h_i(X_A; \mathbb{F}) = \dim_{\mathbb{F}} H_i(X_A; \mathbb{F})$. Since $X_A \supset \Delta_{n-1}^{(k-1)}$ it follows that $h_0(X_A; \mathbb{F}) = 1$ and $h_i(X_A; \mathbb{F}) = 0$ for $1 \leq i \leq k-2$. Since $k+1$ is coprime to n, it follows that for any $y \in \mathbb{Z}_n$, the number of $\sigma \subset \mathbb{Z}_n$ of cardinality $k+1$ that satisfy $\sum_{x \in \sigma} x = y$ is $\frac{1}{n}\binom{n}{k+1}$. Therefore $f_k(X_A) = \frac{k+1}{n}\binom{n}{k+1} = \binom{n-1}{k}$. The Euler-Poincaré relation $\sum_{i \geq 0}(-1)^i f_i(X_A) = \sum_{i \geq 0}(-1)^i h_i(X_A; \mathbb{F})$ then implies that $h_{k-1}(X_A; \mathbb{F}) =$

$h_k(X_A;\mathbb{F})$. In the sequel we assume that the characteristic of $\mathbb{F}$ does not divide n.

Let ω be a fixed primitive n-th root of unity in the algebraic closure $\overline{\mathbb{F}}$. For $x \in \mathbb{Z}_n$ let $e(x) = \omega^x$. The $n \times n$ Fourier matrix M over $\overline{\mathbb{F}}$ is given by $M(u,v) = e(-uv)$ for $u, v \in \mathbb{Z}_n$. For a subset $B \subset \mathbb{Z}_n$ of cardinality $k+1$ let $M_{A,B}$ denote the $(k+1)\times(k+1)$ submatrix of M determined by A and B. Let $\mathcal{B}_{n,k}$ denote the family of all $(k+1)$-element subsets of $\mathbb{Z}_n$ that contain 0.

Theorem 1.2.

$$h_{k-1}(X_A;\mathbb{F}) = h_k(X_A;\mathbb{F}) = \frac{1}{k+1} \sum_{B \in \mathcal{B}_{n,k}} \dim \ker M_{A,B} \ . \qquad (2)$$

The Fourier transform matrix $M = (M_{uv})$ of $\mathbb{Z}_n$ over $\overline{\mathbb{Q}} = \mathbb{C}$ is given by $M_{uv} = \exp(-2\pi i uv/n)$. A classical result of Chebotarëv (see e.g. [5]) asserts that if n is prime then any square submatrix of M is nonsingular. Theorem 1.2 therefore implies

Corollary 1.3. *If n is prime then X_A is a k-hypertree.*

If A is an arithmetic progression in $\mathbb{Z}_n$ then $M_{A,B}$ is a Vandermonde matrix for all $B \in \mathcal{B}_{n,k}$. Hence, by Theorem 1.2, X_A is $\mathbb{F}$-acyclic for any $\mathbb{F}$ whose characteristic is coprime to n. More is in fact true. Note that if $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ is k-collapsible and $f_k(X) = \binom{n-1}{k}$, then X is $\mathbb{Z}$-acyclic.

Theorem 1.4. *Let n be a prime and let A be a subset of $\mathbb{Z}_n$ of cardinality $k+1$. Then X_A is k-collapsible iff A is an arithmetic progression in $\mathbb{Z}_n$.*

Theorems 1.2 and 1.4 are proved in Sections 2 and 3. In Section 4 we compute the homology of X_A for $A = \{0,1,3\}$. We conclude in Section 5 with some remarks concerning possible extensions and open problems.

2 Homology of X_A

The Fourier transform of a function $\phi \in \mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$ is the function $\mathcal{F}(\phi) = \widehat{\phi} \in \mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$ given by

$$\widehat{\phi}(u_1,\ldots,u_k) = \sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k} \phi(x_1,\ldots,x_k) e(-\sum_{j=1}^{k} u_j x_j) \ .$$

The Fourier transform is an automorphism of $\mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$.

The proof of Theorem 1.2 involves computing the image of $H^{k-1}(X;\mathbb{F})$ under the Fourier transform. We first consider the Fourier image of the $(k-1)$-coboundaries.

Claim 2.1.

$$\mathcal{F}(B^{k-1}(X_A;\mathbb{F})) = \{g \in C^{k-1}(X_A;\mathbb{F}) : \textit{support}(g) \subset \mathbb{Z}_n^k - (\mathbb{Z}_n - \{0\})^k\} \quad .$$

Proof: Let $\psi \in C^{k-2}(X_A;\mathbb{F})$. Then

$$\widehat{d_{k-2}\psi}(u_1,\ldots,u_k) = \sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k} d_{k-2}\psi(x_1,\ldots,x_k)e(-\sum_{j=1}^k u_jx_j) =$$

$$\sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k} \Big(\sum_{i=1}^k(-1)^{i+1}\psi(x_1,\ldots,\hat{x}_i,\ldots,x_k)\Big)e(-\sum_{j=1}^k u_jx_j) =$$

$$\sum_{i=1}^k(-1)^{i+1}\sum_{x_i}e(-u_ix_i)\sum_{x_1,\ldots,\hat{x}_i,\ldots,x_k}\psi(x_1,\ldots,\hat{x}_i,\ldots,x_k)e(-\sum_{j\neq i}u_jx_j) =$$

$$n\sum_{i=1}^k(-1)^{i+1}\delta(0,u_i)\sum_{x_1,\ldots,\hat{x}_i,\ldots,x_k}\psi(x_1,\ldots,\hat{x}_i,\ldots,x_k)e(-\sum_{j\neq i}u_jx_j)$$

where $\delta(0,u_i)=1$ if $u_i=0$ and is zero otherwise. Therefore

$$\mathcal{F}(B^{k-1}(X_A;\mathbb{F})) \subset \{g \in C^{k-1}(X_A;\mathbb{F}) : \text{support}(g) \subset \mathbb{Z}_n^k - (\mathbb{Z}_n - \{0\})^k\} \quad .$$

Equality follows since both spaces have dimension $\binom{n-1}{k-1}$ over $\mathbb{F}$.

□

We next study the Fourier image of the $(k-1)$-cocycles of X_A. Fix a $\phi \in C^{k-1}(X_A;\mathbb{F})$. For $a \in \mathbb{Z}_n$ define a function $f_a \in \mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$ by

$$f_a(x_1,\ldots,x_k) = d_{k-1}\phi\big(a - \sum_{i=1}^k x_i, x_1,\ldots,x_k\big) =$$

$$\phi(x_1,\ldots,x_k) + \sum_{i=1}^k(-1)^i\phi\big(a-\sum_{j=1}^k x_j, x_1,\ldots,\hat{x}_i,\ldots,x_k\big) \quad .$$

Let T be the automorphism of $\mathbb{Z}_n^k$ given by

$$T(u_1,\ldots,u_k)=(u_2-u_1,\ldots,u_k-u_1,-u_1)\ .$$

Then $T^{k+1}=I$ and for $1\leq i\leq k$

$$T^i(u_1,\ldots,u_k)=(u_{i+1}-u_i,\ldots,u_k-u_i,-u_i,u_1-u_i,\ldots,u_{i-1}-u_i).$$

Claim 2.2. *Let $u=(u_1,\ldots,u_k)\in\mathbb{Z}_n^k$. Then*

$$\widehat{f_a}(u)=\widehat{\phi}(u)+\sum_{i=1}^k(-1)^{ki}e(-u_ia)\widehat{\phi}(T^iu)\quad . \tag{3}$$

Proof: For $1\leq i\leq k$ let $\psi_i\in\mathcal{L}(\mathbb{Z}_n^k,\mathbb{F})$ be given by

$$\psi_i(x_1,\ldots,x_k)=\phi\big(a-\sum_{j=1}^k x_j,x_1,\ldots,\hat{x}_i,\ldots,x_k\big)\quad .$$

Then

$$\widehat{\psi_i}(u)=\sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k}\phi\big(a-\sum_{j=1}^k x_j,x_1,\ldots,\hat{x}_i,\ldots,x_k\big)e(-\sum_{j=1}^k u_jx_j)\quad .$$

Substituting

$$y_j=\begin{cases} a-\sum_{\ell=1}^k x_\ell & j=1\\ x_{j-1} & 2\leq j\leq i\\ x_j & i+1\leq j\leq k\end{cases}$$

it follows that

$$\sum_{j=1}^k u_jx_j=(a-y_1)u_i+\sum_{j=2}^i(u_{j-1}-u_i)y_j+\sum_{j=i+1}^k(u_j-u_i)y_j\ .$$

Therefore

$$\widehat{\psi_i}(u)=e(-u_ia)\sum_{y=(y_1,\ldots,y_k)\in\mathbb{Z}_n^k}\phi(y)e(u_iy_1-\sum_{j=2}^i(u_{j-1}-u_i)y_j-\sum_{j=i+1}^k(u_j-u_i)y_j)=$$

$$e(-u_ia)\widehat{\phi}(-u_i,u_1-u_i,\ldots,u_{i-1}-u_i,u_{i+1}-u_i,\ldots,u_k-u_i)=$$
$$e(-u_ia)(-1)^{i(k-i)}\widehat{\phi}(T^iu)\ . \tag{4}$$

Now (3) follows from (4) since $f_a=\phi+\sum_{i=1}^k(-1)^i\psi_i$.

□

For $u \in \mathbb{Z}_n^k$ let $E_u = \{T^i u : 0 \le i \le k\}$ and let

$$L_u = \bigcap_{a \in A} \{g \in \mathcal{L}(E_u, \mathbb{F}) : g(u) + \sum_{i=1}^{k} (-1)^{ki} e(-u_i a) g(T^i u) = 0\}. \qquad (5)$$

Let $\phi \in Z^{k-1}(X_A; \mathbb{F})$. Then for all $a \in A$ and $(x_1, \ldots, x_k) \in \mathbb{Z}_n^k$

$$f_a(x_1, \ldots, x_k) = d_{k-1}\phi\big(a - \sum_{i=1}^{k} x_i, x_1, \ldots, x_k\big) = 0 \ .$$

Eqn. (3) then implies that for all $a \in A$ and $u \in \mathbb{Z}_n^k$

$$\widehat{\phi}(u) + \sum_{i=1}^{k} (-1)^{ki} e(-u_i a) \widehat{\phi}(T^i u) = 0 \ .$$

Writing $\widehat{\phi}_{|E_u}$ for the restriction of $\widehat{\phi}$ to E_u we obtain

Corollary 2.3. *Let $\phi \in C^{k-1}(X_A; \mathbb{F})$. Then $\phi \in Z^{k-1}(X_A; \mathbb{F})$ iff $\widehat{\phi}_{|E_u} \in L_u$ for all $u \in \mathbb{Z}_n^k$.*

□

Let the symmetric group S_k act on $\mathbb{Z}_n^k$ by

$$\sigma\big((u_1, \ldots, u_k)\big) = (u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(k)})$$

and let $G_{n,k}$ denote the subgroup of $\mathrm{Aut}(\mathbb{Z}_n^k)$ generated by T and S_k. The subset

$$D_{n,k} = \{(u_1, \ldots, u_k) \in (\mathbb{Z}_n - \{0\})^k : u_i \ne u_j \text{ for } i \ne j\}$$

is clearly invariant under $G_{n,k}$.

Claim 2.4.
(i) Let $\sigma \in S_k$ and $1 \le i \le k$. Then $\eta = T^i \sigma T^{-\sigma^{-1}(i)} \in S_k$ and $\mathrm{sgn}(\eta) = (-1)^{k(i+\sigma^{-1}(i))}\mathrm{sgn}(\sigma)$.
(ii) Any element of $G_{n,k}$ can be written uniquely as σT^i where $\sigma \in S_k$ and $0 \le i \le k$. $G_{n,k}$ acts freely on $D_{n,k}$.
(iii) $L_u = L_{T^j u}$ for all $u \in D_{n,k}$ and $0 \le j \le k$.

Proof: (i) For $1 \le \ell \le k$ let $\tau_\ell \in S_k$ be given by

$$\tau_\ell(i) = \begin{cases} k-\ell+1+i & 1 \le i \le \ell-1 \\ k-\ell+1 & i=\ell \\ i-\ell & \ell+1 \le i \le k \ . \end{cases}$$

It can be checked that

$$\eta = T^i \sigma T^{-\sigma^{-1}(i)} = \tau_{k-i+1}^{-1} \sigma \tau_{k-\sigma^{-1}(i)+1} \ .$$

Noting that $\mathrm{sgn}(\tau_\ell) = (-1)^{k\ell+1}$ it thus follows that

$$\mathrm{sgn}(\eta) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau_{k-i+1})\mathrm{sgn}(\tau_{k-\sigma^{-1}(i)+1}) = (-1)^{k(i+\sigma^{-1}(i))}\mathrm{sgn}(\sigma) \quad .$$

(ii) It follows from (i) that

$$G_{n,k} = \{\sigma T^i : \sigma \in S_k \ , \ 0 \le i \le k\} \ .$$

Let $u = (u_1, \dots, u_k) \in D_{n,k}$ and let $v = (v_1, \dots, v_k) = \sigma T^i u$. If $i \neq 0$ then

$$\sum_{j=1}^{k} v_j = \sum_{j=1}^{k} u_j - (k+1)u_i \neq \sum_{j=1}^{k} u_j$$

and therefore $\sigma T^i u \neq u$. It follows that $G_{n,k}$ acts freely on $D_{n,k}$ and that the representation of an element of $G_{n,k}$ as σT^i is unique.
(iii) Let $g \in L_u$ and $a \in A$. Then

$$g(T^j u) + \sum_{i=1}^{k} (-1)^{ik} e(-(T^j u)_i a) g(T^{i+j} u) =$$

$$g(T^j u) + \sum_{i=1}^{k-j} (-1)^{ik} e(-(u_{i+j} - u_j)a) g(T^{i+j} u) +$$

$$(-1)^{(k-j+1)k} e(u_j a) g(u) + \sum_{i=k-j+2}^{k} (-1)^{ik} e(-(u_{i-k+j-1} - u_j)a) g(T^{i+j} u) =$$

$$(-1)^{jk} e(u_j a)\Big(g(u) + \sum_{i=1}^{k} (-1)^{ik} e(-u_i a) g(T^i u)\Big) = 0. \tag{6}$$

Hence $g \in L_{T^j u}$.

□

Proof of Theorem 1.2: Let $R \subset D_{n,k}$ be a fixed set of representatives of the orbits of $G_{n,k}$ on $D_{n,k}$. Then $|R| = \frac{|D_{n,k}|}{|G_{n,k}|} = \frac{1}{k+1}\binom{n-1}{k}$. Consider the mapping

$$\Theta : Z^{k-1}(X_A; \mathbb{F}) \to \bigoplus_{u \in R} L_u$$

given by

$$\Theta(\phi) = \left(\widehat{\phi}_{|E_u} \ : \ u \in R\right) \quad .$$

Claim 2.5.

$$\ker \Theta = B^{k-1}(X_A; \mathbb{F}) \quad .$$

Proof:

$$\ker \Theta = \{\phi \in Z^{k-1}(X_A; \mathbb{F}) : \widehat{\phi}_{|E_u} = 0 \text{ for all } u \in R\} =$$

$$\{\phi \in Z^{k-1}(X_A; \mathbb{F}) : \widehat{\phi}(u) = 0 \text{ for all } u \in D_{n,k}\} = B^{k-1}(X_A; \mathbb{F})$$

by Claim 2.1.

□

Claim 2.6. Θ *is surjective.*

Proof: Let $(g_u : u \in R) \in \bigoplus_{u \in R} L_u$. Define $g \in C^{k-1}(X_A; \mathbb{F})$ by

$$g(v) = \begin{cases} 0 & v \notin D_{n,k} \\ \mathrm{sgn}(\sigma) g_u(T^j u) & v = \sigma T^j u \text{ where } u \in R. \end{cases}$$

Clearly $\Theta(\mathcal{F}^{-1}(g)) = (g_u : u \in R)$. To show that $\mathcal{F}^{-1}(g) \in Z^{k-1}(X_A; \mathbb{F})$ it suffices by Corollary 2.3 to check that $g \in L_v$ for all $v \in \mathbb{Z}_n^k$. If $v \notin D_{n,k}$ then $g_{|E_v} = 0$. Suppose then that $v = \sigma T^j u \in D_{n,k}$ where $u \in R$ and $0 \leq j \leq k$. Combining Claim 2.4(i) and Eq. (6) it follows that

$$g(v) + \sum_{i=1}^{k} (-1)^{ik} e(-v_i a) g(T^i v) =$$

$$g(\sigma T^j u) + \sum_{i=1}^{k} (-1)^{ik} e(-(\sigma T^j u)_i a) g(T^i \sigma T^j u) =$$

$$\text{sgn}(\sigma)g_u(T^j u)+\sum_{i=1}^{k}(-1)^{ik}e(-(T^j u)_{\sigma^{-1}(i)}a)(-1)^{k(i+\sigma^{-1}(i))}\text{sgn}(\sigma)g_u(T^{\sigma^{-1}(i)+j}u) =$$

$$\text{sgn}(\sigma)\big(g_u(T^j u) + \sum_{i=1}^{k}(-1)^{ik}e(-(T^j u)_i a)g_u(T^{i+j}u)\big) =$$

$$= (-1)^{jk}\text{sgn}(\sigma)e(u_j a)\big(g_u(u) + \sum_{i=1}^{k}(-1)^{ik}e(-u_i a)g_u(T^i u)\big)= 0.$$

□

Claims 2.5 and 2.6 imply that

$$H^{k-1}(X_A, \mathbb{F}) \cong \bigoplus_{u\in R} L_u \ . \qquad (7)$$

For $u = (u_1, \ldots, u_k) \in D_{n,k}$ let $B_u = \{0, u_1, \ldots, u_k\}$. Then $\dim L_u = \dim\ker M_{A,B_u}$. Combining (7) with Claim 2.4(iii) it thus follows that

$$h^{k-1}(X_A; \mathbb{F}) = \sum_{u\in R} \dim L_u =$$

$$\frac{1}{k+1}\sum_{u\in R}\sum_{j=0}^{k} \dim L_{T^j u} = \frac{1}{k+1}\sum_{B\in\mathcal{B}_{n,k}} \dim\ker M_{A,B} \ .$$

□

3 When is X_A collapsible?

In this section we prove Theorem 1.4, so that in this section n is prime. We find it convenient to maintain the vertices in a face sorted according to the order induced from $\mathbb{N}$, and also refer to subsets of $\mathbb{F}_n$ as sorted vectors and not only as sets.

3.1 Equivalence

Let $\phi : \mathbb{F}_n \to \mathbb{F}_n$ be the linear map $\phi(x) = \alpha x + \beta$. It is clear that the image of X_a under ϕ is X_t where $t = \alpha a + (k+1)\beta$. We say that the complexes $X_{a_0,\dots,a_k}$ and $X_{b_0,\dots,b_k}$ are *equivalent* iff there exist a permutation π on $\{b_0, \dots, b_k\}$ and α, β s.t. $\pi(b_i) = \alpha a_i + (k+1)\beta$ for every $0 \le i \le k$. Equivalent complexes are clearly isomorphic.

It is an easy observation that $a_0, \dots, a_k$ is an arithmetic progression iff $X_{a_0,\dots,a_k}$ is equivalent to the complex $X_{0,\dots,k}$. We show that $X = X_{0,\dots,k}$ is collapsible whence $X_{a_0,\dots,a_k}$ is collapsible for $a_0, \dots, a_k$ an arithmetic progression.

3.2 Proof of sufficiency

To show that X is collapsible we introduce an order $\prec_R$ by which we remove the k-faces from X. We need first some preliminary definitions. With every k-face $u \in X$ we associate a vector $h(u)$ of dimension $\lceil \frac{k}{2} \rceil$. The i-th coordinate in h counts how many integers in the interval $[u_i, u_{k-i}]$ do not belong to $\{u_i, \dots, u_{k-i}\}$. Namely, the i-th coordinate of $h(u)$ is:

$$h_i(u) := u_{k-i} - u_i - (k - 2i)$$

Clearly $h_i(u)$ is non-increasing in i. For every two k-faces $u, v \in X$ we say that $u \prec_L v$ if $h(u)$ is lexicographically smaller than $h(v)$. When $h(u) = h(v)$ we say that $u \equiv_L v$. It should be clear that h is invariant under set reversal i.e. $x \to n - x$. It is also invariant under shifts that "do not overflow" in the obvious sense, but we will not be using this fact. If $u \not\equiv_L v$ for some $u, v \in X$, we denote by $\delta_L(u, v)$ the first index for which $h(u)$ and $h(v)$ differ. Thus if $u \prec_L v$ and $\delta_L(u, v) = i$ then $h_j(u) = h_j(v)$ for all $j < i$ and $h_i(u) < h_i(v)$.
For $i, j \in \mathbb{F}_n$ it is convenient to define $\rho(i, j)$ as $i - j$ if $i > j$ and as $j - i$ otherwise. This is extended as usual to: $\rho(i, A) = \min\{\rho(i, a) \mid a \in A\}$ and $\rho(A, B) = \min\{\rho(a, b) \mid a \in A, b \in B\}$.
If $u \in X_i$ and $v \in X_j$ we say that $u \prec_I v$ if i is closer than j to $\{0, k\}$, i.e., if $\rho(i, \{0, k\}) < \rho(j, \{0, k\})$. We say that $u \equiv_I v$ when $\rho(i, \{0, k\}) = \rho(j, \{0, k\})$, namely, $i = j$ or $i = k - j$. Letting $i' = \rho(i, \{0, k\})$, it is clear that $u \prec_I v$ iff $i' < j'$. If $u \not\equiv_I v$, we denote by $\delta_I(u, v) = \min\{i', j'\} = \rho(\{i, j\}, \{0, k\})$.

We are now ready to define the relation $\prec_R$. This is done in terms of the relations $\prec_L$ and $\prec_I$. To begin, $u \equiv_R v$ iff $u \equiv_L v$ and $u \equiv_I v$. If $u \preceq_L v$ and $u \preceq_I v$ and at least one inequality is proper, then $u \prec_R v$. Finally,

when $u \prec_L v$ and $u \succ_I v$, the order $\prec_R$ is determined according to the smaller of $\delta_I(u,v), \delta_L(u,v)$. Namely, if $\delta_I(u,v) < \delta_L(u,v)$ then $u \succ_R v$ and if $\delta_I(u,v) \geq \delta_L(u,v)$ then $u \prec_R v$.
To sum up, for $u, v \in X$:

1. If $u \equiv_I v$ and $u \equiv_L v$ then $u \equiv_R v$.
2. If $u \equiv_I v$ and $u \prec_L v$ then $u \prec_R v$.
3. If $u \prec_I v$, then $u \prec_R v$ unless
   (a) $u \succ_L v$ and
   (b) $\delta_L(u,v) \leq \delta_I(u,v)$

   In which case $u \succ_R v$.

To clarify this definitions a little bit more, we present an example from the complex $X_{0,1,2,3}$ over $\mathbb{F}_7$. Let $u = \{0,1,2,5\}$, $v = \{1,2,5,6\}$. The set u has two missing integers between 0 and 5 and no missing integers between 1 and 2, hence $h(u) = h(\{0,1,2,5\}) = (2,0)$. Similarly $h(v) = h(\{1,2,5,6\}) = (2,2)$. Also, $u \prec_L v$ because $(2,0)$ is lexicographically smaller than $(2,2)$. Furthermore, $\delta_L(u,v) = 1$ because the first coordinate the vectors differ is the second coordinate (and we start indexing coordinates from zero). Now $u \in X_1$ since $0+1+2+5 \equiv 1 \mod 7$. Similarly $v \in X_0$. We next calculate that $1' = 1 = \rho(1, \{0,7\})$ and $0' = 0$. Hence $v \prec_I u$ because $0' < 1'$, and $\delta_I(u,v) = \min\{0,1\} = 0$. To recap, $u \prec_L v$ and $v \prec_I u$, so we turn to compare $0 = \delta_I(u,v) < \delta_L(u,v) = 1$, it follows that in this case the order R is determined by I, hence $\{0,1,2,5\} = u \succ_R v = \{1,2,5,6\}$. A full description of the order R on $X_{0,1,2,3}$ over $\mathbb{F}_7$ is shown in Figure 2 and Figure 3:

A few words are in order about Figure 2. The rows are sorted by the lexicographic order of $h(\cdot)$. The columns on the right include all facets of X sorted by value of i'. Note that for each value of h and each i' there are two facets that attain this pair of values. The leftmost column gives the value of $\delta_L(x,y)$ for every two consecutive lines in the table.

We now turn to show that X can indeed be collapsed in the order $\prec_R$. That is, for every $x \in X$ it is possible to apply an elementary collapse step to x if all the $\prec_R$-predecessors of x have already been collapsed. In order to show this, we need to point out an free$(k-1)$-face that is contained in x. What we will show is that for $x \in X_a$, the face $\hat{x} := x \setminus \{x_a\}$ is free. (Note

Figure 2: $X_{0,1,2,3}$ parameters over $\mathbb{F}_7$

$\delta_L(x,y)$	$h(x)$	$i'=0$	$i'=1$
	$(0,0)$	$\{2,3,4,5\},\{1,2,3,4\}$	
0			
	$(1,0)$	$\{0,1,2,4\},\{2,4,5,6\}$	$\{2,3,4,6\},\{0,2,3,4\}$
1			
	$(1,1)$		$\{0,1,3,4\},\{2,3,5,6\}$
0			
	$(2,0)$	$\{1,3,4,6\},\{0,2,3,5\}$	$\{0,1,2,5\},\{1,4,5,6\}$
1			
	$(2,1)$		$\{1,3,5,6\},\{0,1,3,5\}$
1			
	$(2,2)$	$\{1,2,5,6\},\{0,1,4,5\}$	
0			
	$(3,0)$		$\{0,4,5,6\},\{0,1,2,6\}$
1			
	$(3,1)$	$\{0,3,5,6\},\{0,1,3,6\}$	

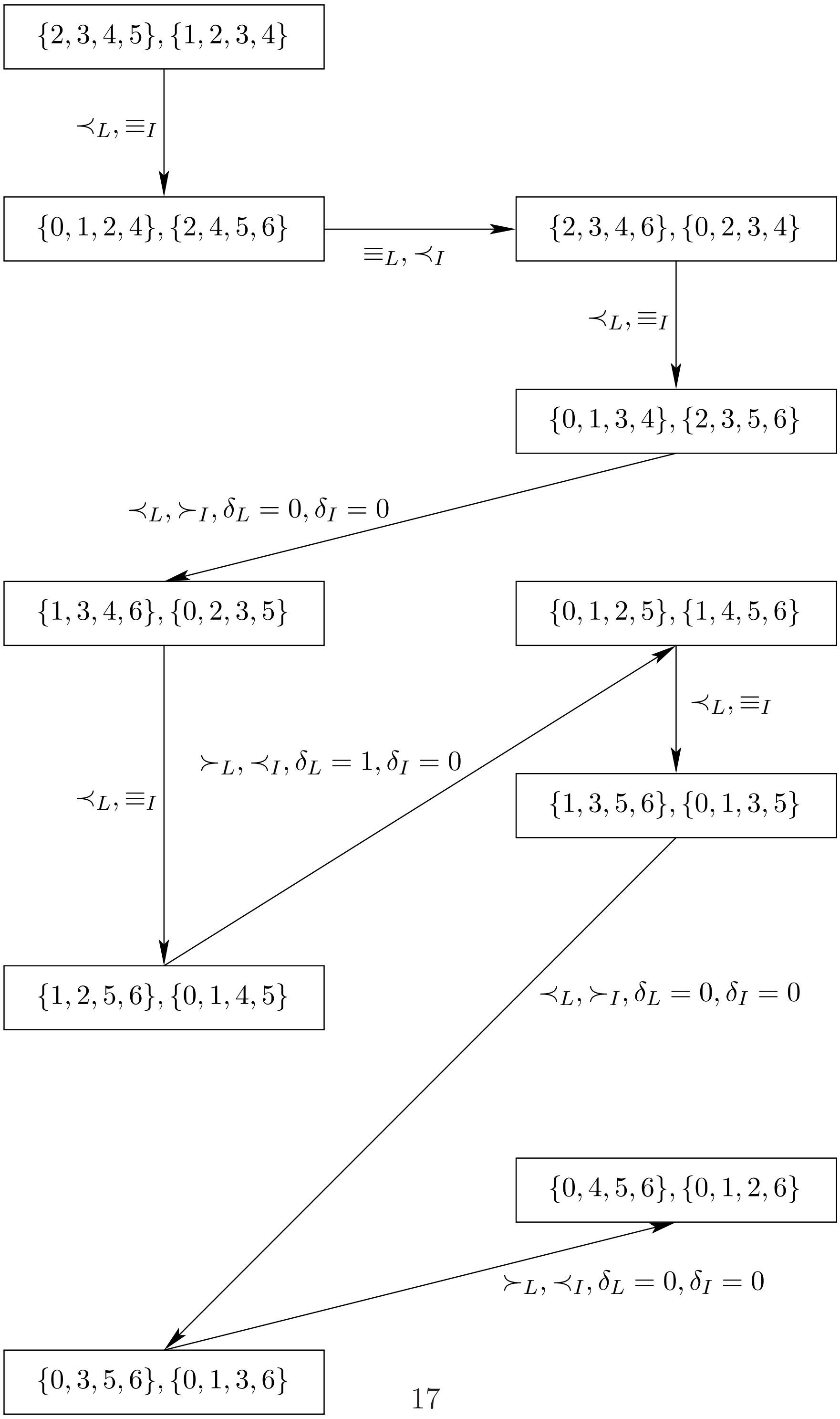
that since $x \in X = X_{0,\ldots,k}$, there indeed must exist some $a \in \{0,\ldots,k\}$ s.t. $x \in X_a$). It may be helpful to mention that a plays a double role here. It is an index in the vector x as well as the sum of the elements of x. Being free means that all the k-faces containing $\hat{x}$, precede x in the order $\prec_R$. A k-face that contains $\hat{x}$ has the form $y^{(b)} := \hat{x} \cup \{x_a + (b-a)\}$ with $0 \le b \le k$ and $b \ne a$. Clearly, $y^{(b)}$ is a k-face in X iff $x_a + (b-a) \notin \hat{x}$. Also, in this case $y^{(b)} \in X_b$, as we assume below.

The proof that $y^{(b)} \prec_R x$ has two cases:

1. We first consider the case where $y^{(b)} \succeq_I x$. Since $y^{(b)} \in X_b$ and $x \in X_a$, the meaning of $y^{(b)} \succeq_I x$ is that $b' \ge a'$. Therefore $\delta_I(y^{(b)}, x)$ which is the smaller of a' and b' equals a'. This means that b lies between a and $k-a$ (whether a or $k-a$ is bigger is immaterial here).

   - Consequently, $x_a + (b-a)$ is in the interval $[x_{a'}, x_{k-a'}]$. It follows that the first and last $a'-1$ elements of x and $y^{(b)}$ are identical. In particular, $h_i(y^{(b)}) = h_i(x)$ for $i < a'$.
   - We recall that $y^{(b)}$ is created by removing x_a from x and replacing it by the term $x_a + (b-a)$. Thus the interval $[y^{(b)}_{a'}, y^{(b)}_{k-a'}]$ is shorter than $[x_{a'}, x_{k-a'}]$. It follows that the first coordinate where $h(y^{(b)})$

Figure 3: The order of collapse determined by $\prec_R$



and $h(x)$ differ is the a'-th coordinate, where $h_{a'}(y^{(b)}) < h_{a'}(x)$. Consequently, $y^{(b)} \prec_L x$ and $a' = \delta_L(y^{(b)}, x)$.

- If $y^{(b)} \equiv_I x$ then we are done, because we already know that $y^{(b)} \prec_L x$. By definition of $\prec_R$ this yields the desired conclusion $y^{(b)} \prec_R x$.
- If $y^{(b)} \succ_I x$ then from the previous points we conclude that $a' = \delta_L(y^{(b)}, x) = \delta_I(y^{(b)}, x)$. To sum up, $y^{(b)} \prec_L x$ and $\delta_L(y^{(b)}, x) = \delta_I(y^{(b)}, x)$, which yields by definition, $y^{(b)} \prec_R x$, as claimed.

2. Now consider the case $y^{(b)} \prec_I x$. This means that $b' < a'$. Therefore $b' = \delta_I(y^{(b)}, x)$. Consequently b does not lie between a and $k - a$.

   - It follows that $x_a + (b - a) \in [x_{b'}, x_{k-b'}]$. Consequently, the first and last $b' + 1$ elements of x and $y^{(b)}$ are identical. In particular, $h_i(y^{(b)}) = h_i(x)$ for $i \leq b'$. Thus $\delta_L(y^{(b)}, x) > b'$.
   - If $y^{(b)} \preceq_L x$ then $y^{(b)} \prec_R x$ and we are done.
   - If $y^{(b)} \succ_L x$ then from the previous points we conclude that $b' = \delta_I(y^{(b)}, x) < \delta_L(y^{(b)}, x)$. Hence $y^{(b)} \succ_I x$ and $\delta_I(y^{(b)}, x) < \delta_L(y^{(b)}, x)$. Again, by definition, $y^{(b)} \prec_R x$, as claimed.

This completes the proof that $X_{0,\ldots,k}$ is collapsible and hence that $X = X_{a_0,\ldots,a_k}$ is collapsible whenever $a_0, \ldots, a_k$ is an arithmetic progression.

3.3 Proof of necessity

We now turn to show that if $a_0, \ldots, a_k$ is not arithmetic, then $X_{a_0,\ldots,a_k}$ is not collapsible. In fact we show that in this case exactly $k + 1$ elementary collapse steps can be carried out.
For $X \subseteq \mathbb{F}_n$ we denote as usual by $X + a$ the a-shift of X, namely, the set $\{x + a | x \in X\}$. We start with the following simple observation.

Observation 3.1. *Let n be a prime. A subset $X \subsetneq \mathbb{F}_n$ is an arithmetic progression iff there is an element l for which $|(X + l) \setminus X| = 1$.*

When is the $(k - 1)$-face $x_1, \ldots, x_k$ free a free face? This is the case iff, for each $k \geq i \geq 1$ the element $x_i + \sum_{j=1}^{k} x_j$ belongs to the set $\{a_0, \ldots, a_k\}$. If $x_i + \sum x_j = a_l$ it means that $x_1, \ldots, x_k$ cannot be extended to a k-face in X_{a_l}. This translates into a linear system of equations in $x_1, \ldots, x_k$ whose matrix has 2's along the main diagonal and 1's elsewhere. Such a matrix is nonsingular, so the solution is unique. Also, all the k terms $x_i + \sum x_j$ are distinct, so the only choice we have in constructing this linear system is which of the $k+1$ elements in $\{a_0, \ldots, a_k\}$ to omit. There are $k+1$ such choices which yields $k+1$ distinct collapse steps that can be carried out.

We now explicitly describe the $k+1$ collapse steps that can be carried out. Each of these collapsible faces has the form $x^{(t)} := \{a_0+l_t, \ldots, a_k+l_t\} \in X_{a_t}$ for some $l_0, \ldots l_k$

The condition $x^{(t)} \in X_{a_t}$ determines l_t via $l_t = \frac{a_t - \sum_{i=0}^{k} a_i}{k+1}$. We claim that the face $y := x^{(t)} \setminus \{a_t + l_t\}$ is free. The sum of y's elements is $-l_t$, so that for every $i \neq t$ we need to add the term $\{a_i + l_t\}$ to y in order to attain the sum a_i. This is, however, impossible since $\{a_i + l_t\}$ is a member of y.

We turn to show that after these first $k+1$ collapse steps are carried out, there remain no free $(k-1)$-faces in X. In order for a $(k-1)$-face y to be free following the above collapses, y has to be contained in exactly one of these $k+1$ collapsed faces. Since $\{a_0, \ldots, a_k\}$ is not an arithmetic progression, by Observation 3.1, the intersection of any two of the $x^{(t)}$ contains at most $k-1$ elements. In particular there is no $(k-1)$-face that they both contain. Thus we have to consider only $(k-1)$-faces y which are contained in one of the $x^{(t)}$ and exactly one more k-face.

It follows that y must be of the form $x^{(t)} \setminus \{a_j + l_t\}$ for some j and t. The sum of y's elements is $a_t - a_j - l_t$. If y is contained as well in a k-face $z \in X_{a_i}$, then necessarily $z_i = z = y \cup \{a_i - a_t + a_j + l_t\}$. We are assuming that y becomes free with the collapse of $x^{(t)}$, so there must be exactly one index i for which z_i is a legal k-face different from $x^{(t)}$. It follows that $x^{(t)}$ and $x^{(t)} + (a_j - a_t)$ must have k elements in common. Again by Observation 3.1 this means that the elements in $x^{(t)}$ form an arithmetic progression, a contradiction. The proof of Theorem 1.4 is now complete.

4 Example: Homology of $X_{\{0,1,3\}}$

For a prime p and an integer n indivisible by p, let $U_{p,n}$ be the group of n-th roots of unity in $\overline{\mathbb{F}_p}$.

Proposition 4.1. *Let $k = 2$, $A = \{0, 1, 3\}$. Let p be a prime and suppose n*

is coprime to $3p$*. Then*

$$h_1(X_A;\mathbb{F}_p) = \frac{1}{3}\left|\left\{\{u,v\} \subset U_{p,n} - \{1\} : u \neq v \text{ and } 1+u+v=0\right\}\right| .$$

Proof: Let $B = \{0,k,\ell\}$ with $0 < k < \ell < n$ and let $u = \omega^{-k}, v = \omega^{-\ell}$. Then

$$\det M_{A,B} = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & u & v \\ 1 & u^3 & v^3 \end{bmatrix} =$$

$$uv^3 - vu^3 + u^3 - u - v^3 + v = (u-1)(v-1)(v-u)(u+v+1) \quad .$$

It follows that

$$\text{rk } M_{A,B} = \begin{cases} 2 & 1+u+v=0 \\ 3 & \text{otherwise.} \end{cases}$$

Thus the Proposition follows directly from Theorem 2.

□

Corollary 4.2. *Let* $k = 2$*,* $A = \{0,1,3\}$*. Let* p *be a prime and suppose* $n = p^m - 1$ *is coprime to 3. Then*

$$h_1(X_A;\mathbb{F}_p) = \begin{cases} \frac{n-1}{6} & p=2 \\ \frac{n-2}{6} & p=3 \\ \frac{n-4}{6} & p>3. \end{cases}$$

Proof: Clearly $\mathbb{F}_{p^m}^* = U_{p,n}$. Therefore, by Proposition 4.1

$$h_1(X_A;\mathbb{F}_p) = \frac{1}{6}\left|\left\{u \in \mathbb{F}_{p^m}^* - \{1\} : -(1+u) \notin \{0,1,u\}\right\}\right|.$$

The Corollary now follows since

$$\left\{u \in \mathbb{F}_{p^m}^* - \{1\} : -(1+u) \notin \{0,1,u\}\right\} = \begin{cases} \mathbb{F}_{2^m}^* - \{1\} & p=2 \\ \mathbb{F}_{3^m}^* - \{\pm 1\} & p=3 \\ \mathbb{F}_{p^m}^* - \{\pm 1, -2, -\frac{1}{2}\} & p>3. \end{cases}$$

□

5 Concluding Remarks

Theorem 1.2 provides an explicit description of the homology of the sum complex X_A over fields of characteristic coprime to n. In particular, it follows via Chebotarëv's Theorem that if n is prime then X_A is $\mathbb{Q}$-acyclic, i.e. X_A is a k-hypertree. When A is an arithmetic progression, X_A was shown to be k-collapsible, and in particular $\mathbb{Z}$-acyclic. One natural question is whether there exist other A's for which X_A is $\mathbb{Z}$-acyclic. Kalai's k-dimensional Cayley's formula [3] suggests that most k-hypertrees are not $\mathbb{Z}$-acyclic. Likewise we conjecture that X_A is not $\mathbb{Z}$-acyclic for most $(k+1)$-subsets $A \subset \mathbb{Z}_n$. One possible approach to the question of $\mathbb{F}_p$-acyclicity of X_A for primes $p \nmid n$ is via the following reduction. Let $S_{\mathbb{F}}(A)$ be the $\mathbb{F}$-linear space of polynomials in $\mathbb{F}[x]$ spanned by the monomials $\{x^a : a \in A\}$. Theorem 1.2 then implies that X_A is $\mathbb{F}_p$-acyclic iff $\deg \gcd(f(x), x^n - 1) \leq k$ for all $0 \neq f(x) \in S_{\overline{\mathbb{F}_p}}(A)$.

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