Why Social Learning May Result in Diversity

Amir Ban

Center for the Study of Rationality, Hebrew University, Jerusalem, Israel

Nati Linial

School of Computer Science and Engineering, Hebrew University, Jerusalem, Israel

---

Abstract

We study social learning by agents located on a social or spatial graph, with emphasis on the question of whether different types, of unequal merit, may persist side by side indefinitely in a constantly learning connected society. In contrast to studies that previously considered this question[3, 4, 1], we bring evidence to answer this in the affirmative, showing that diverse learning societies which are stable exist, making their emergence possible, perhaps even likely.

Keywords: social network, social learning, graph

---

1. Introduction

Agents who face a choice of actions with uncertain payoff rely on their own experience as well as on the opinion of others. We have previously studied the behavior and dynamics of such a system[2] when the latter is available as a *reputation* of the action considered. This is defined as an aggregated record of selectively reported outcomes of the action, which is publicly available to all agents and is often condensed into a single real number enabling a heuristic comparison or ranking of actions based on reputation. We noted that the informal concept of reputation as a society's consensus opinion is formally embodied in reputation systems, also called recommendation systems, of which YouTube, TripAdvisor, Google Scholar and the Google search engine itself are massively-used examples.

In that study all agents were undifferentiated by position or social connections. In the current study, we place the agents on a social or spatial graph. An agent repeatedly chooses among several available experts of uncertain expertise, relying in her[1] choice on the opinion of her neighbors, the neighbors being those directly connected to her vertex with a directed graph edge.

Email addresses: `amirban@netvision.net.il` (Amir Ban), `nati@cs.huji.ac.il` (Nati Linial)

[1] throughout this paper we refer to agents as feminine and to experts as masculine.

The opinion of the neighbors is condensed into an aggregated reputation record of all experts, which, since every agent has a different set of neighbors, is local to that agent. The reputation record counts favorable mentions of each expert (possibly discounted over time and weighted by source) and ranks experts based on that count. The true *expertise*, with full information of the state of the world, is a fixed and agent-independent probability that the expert will provide satisfactory service. An agent wishes to get satisfactory service in the least number of tries, so is motivated to try experts in descending order of their expertise. The expertise, however, is not directly observable, and the reputation order serves as a noisy signal of the expertise order. Since an agent will stop querying experts once she gets satisfactory service, the reputation order is crucial to which experts receive a chance to prove their worth. As the agent will also report successful service to her own neighbors' reputation records, her neighbors' subsequent choices will be affected leading to possible propagation of reputation throughout society's graph.

For simplicity we assume there are only two experts, A and B, whose respective expertise is $\epsilon_A, \epsilon_B \in (0, 1]$. Each agent's reputation record, at any given time, consists of r_A and r_B, real numbers that count favorable mentions (possibly discounted and weighted by source) of A and B, respectively. We furthermore assume that the agent's own experience has also been suitably weighed into the reputation record, so that the agent determines her behavior based on the rank order of her r_A and r_B: We call an agent with a higher or equal r_A an A-type, and an agent with a higher r_B a B-type[2]. An A-type will use expert A primarily, with expert B used as a fallback option. Similarly a B-type will use expert B primarily, using A as a fallback option.

We therefore have a partition of agents into A-types and B-types at any time. Assume all agents seek service from the experts at all integral times $1, 2, \ldots$. A partition is called stable if, once achieved, it does not change subsequently under the dynamics described[3]. A partition is called diverse if not all agents have the same type. In a connected society, is a stable diverse partition possible?

This question has been posed by Bala and Goyal[1] and answered, in the context of their model, in the negative when (rephrased to our terminology) the experts have different expertise. In case experts are equal in expertise, they show a stable diverse partition is possible under some further assumptions, stating "To analyze issues of conformity/diversity in general is quite a difficult problem." (section 5). Ellison and Fudenberg [4] also make the distinction between equal and unequal "technologies" competing for adoption in a learning population while searching for conformity versus diversity outcomes of their model. In the unequal case, they conclude that their model might fail to reach conformity only when it fails to reach any steady state at all, and so apparently answer the diversity question in the negative. Nevertheless, as we discuss below, we believe our result may apply to a network version of their model. De Groot [3] shows

[2] An arbitrary tie-breaking rule is applied
[3] At least with high probability, as the dynamics are probabilistic.

that a committee trying to reach a consensus on a probability distribution by each member observing and weighting other members distributions will indeed converge on a consensus.

Evolutionary dynamics provide a parallel, in form at least, to the questions discussed in social learning. The *fitness* of a type is parallel to our expertise, and it is known that in a population in which two different types compete no diversity is possible unless their fitness is equal (e.g. Nowak[7]). Moreover, the Moran process (ibid.), used to analyze evolutionary dynamics in finite populations is incapable of a diverse steady state by its nature. The key parameter under investigation there is the *fixation probability*, i.e. the probability that one particular type (a mutant) will take over the entire population, the alternative being its extinction. In evolutionary graph theory (Lieberman et. al.[6]) various graphs, where edges denote spatial proximity, are investigated as to whether they increase or diminish the fixation probability of types.

Our aim is to show a different result, i.e. that stable diversity is possible even when experts have different expertise. Thus we point out a critical difference between evolutionary dynamics and the dynamics of social learning, which underlies the different result. Evolutionary dynamics progresses by the random selection of one node in the vicinity, and once selected, only that node determines the progeny. In social learning, the entire neighborhood of a node comes into play. When, as is natural and common, social neighborhood is based on spatial proximity, this idea has been epitomized by the saying "It takes a village to raise a child", or by the verse "Man is but the imprint of the landscape of his birthplace" by the Hebrew poet Saul Tchernichovsky.

The reputation model of social learning may be called *learning by the suggestion of alternatives*. Unlike in other models of social learning, alternatives do not differ in their payoff but in their probability for success. All successful outcomes are equally satisfactory, whether provided by the best expert or by an inferior one. This gives a real chance for a "good enough" expert to be adopted and locked in by virtue of being given the first chance to perform. The role of reputation, as a guide to agents in which order they should try alternatives, is therefore critical, and dynamics in which feedback effects perpetuate high reputation and shut out objectively superior experts are common.

Our main result, in graph-theoretical terms, states that every finite social graph has a stable, diverse partition, provided every agent has *self-weight*. Namely, each agent counts itself amongst her neighbors. This is represented by a loop edge from the vertex to itself. As we show, this minimal "inertia" is sufficient to enable a stable and diverse partition regardless of how many real neighbors an agent has.

Furthermore, we conjecture that as the graph becomes infinitely dense, in the sense that the number of neighbors of every agent $\to \infty$, the need for any self-weight loops vanishes, and the infinitely dense graph has a stable diverse partition. We refer to this conjecture as the *Map Conjecture*.

The Map Conjecture has a geometric / geographical interpretation when each agent is seen as a point on a map, and all other agents in its *influence sphere* of radius $R > 0$ are considered neighbors. In other words, the social graph has an (undirected) edge for any pair of agents whose distance is R or less. We say that a point set V is dense if for every point $v \in V$ the set of points from V in its influence sphere has a positive measure. (The reader is encouraged to keep in mind a setting where V resides in the plane and measure means area). Then the map conjecture states that if the point set defined by the agents is dense, it can be partitioned in a stable yet diverse manner.

The geographical interpretation demonstrates the following: Dense population groups of different types may reside in close proximity (at least on their mutual boundary) while continuously interacting through social learning with all their neighbors, with their geographical division remaining stable, even if one of the types is in some sense superior to the other.

The results we develop in this paper apply to our reputation model of social learning, yet they do not depend on all details of this model, and so may well apply to other models. Crucially for our results, the global version of our model allows more than one steady state. That is to say, that, under some restrictions on ϵ_A, ϵ_B, either A or B may emerge as stable reputation leaders under suitable initial conditions. This, as we show, enables a mixed community of A-types and B-types to stably coexist in a connected network. The property of having multiple possible steady states is shared by other models of social learning. In particular, it is a result of Ellison and Fudenberg [4], which leads us to believe that a suitable setting of their model in a social or spatial network will enable diverse steady states. To phrase this in their terms, whether a superior technology is efficiently adopted or an inferior technology is inefficiently adopted may turn out to be a matter of geography.

The rest of this paper is organized as follows:

In section 2 we describe the social learning model through reputation. In section 3 we analyze the behavior of the model and find necessary and sufficient conditions for a diverse steady state. In section 4 we specialize the model to a geographical setting and state the Map Conjecture. In section 5 we discuss and describe the possible solutions, and offer concluding remarks.

2. The Model

A finite social network has N agents, represented by the vertices $V(G)$ of an undirected finite graph $G(V, E)$ with loops. An edge in $E(G)$ represents a mutual influence between a pair of individuals. The incidence matrix $\big(a_{xy}\big)$ is defined as usual. If $xy \in E(G)$ then $a_{xy} = 1$, otherwise $a_{xy} = 0$. The number of edges at each vertex is its degree:

$$\forall y \in V(G), \sum_{x \in V(G)} a_{xy} = d(x) \tag{2.1}$$

Vertices are allowed to have loops, representing the fact that an agent's own experience takes part in influencing her own reputation record.

There are two experts, A and B, each able to provide a service that is repeatedly sought by each of the agents, at integral times $1, 2, \ldots$, called *rounds*.

The agents are determined to get satisfactory service and will try both experts, if necessary, to get it. However, as the interaction with an expert carries a cost, they are motivated to start with the expert with the highest probability of success. Each of the experts has an *expertise*, representing the probability that his service will be satisfactory to the agent for any given agent-expert encounter. This probability is independent of the outcome of any trial by a different agent, or by the same agent in a different round. However, a repeat query of an expert by an agent in the same round yields the same result as the first query, so that it is pointless.

A's expertise is $\epsilon_A \in (0, 1]$ while B's expertise is $\epsilon_B \in (0, 1]$. Agents do not know the true expertise of the experts, and use the signals provided by their reputation to make inferences on it.[4]

Each agent x, at time t, sees a *reputation record* consisting of real numbers $r_A(x, t), r_B(x, t)$: reputations of A and B respectively. The reputation record is a running aggregate of feedback reported by the agent and her neighbors on the experts' performance, and its value at the start of round t represents the entire information an agent has to guide her behavior in that round: The agent has no recall of past reputation records, no information on unsuccessful encounters with experts and no access to any reputation record but her own.[5].

Under these circumstances, agents' belief on who the better expert is depends on which expert has the higher reputation: If $r_A(x, t) \geq r_B(x, t)$, that expert is A and x is called *A-type* (at time t). Otherwise that expert is B and x is called *B-type*.

At each round, a Z-type ($Z \in \{A, B\}$) agent behaves as follows: She queries expert Z. If not satisfied, she will query the other expert. Finally, if any of the experts was satisfactory, she will provide feedback to her own reputation record and to all her neighbor agents to which she has an edge using the following procedure.

At the end of round t, and for each agent x:

- The reputation record for the next round $t + 1$ is initialized from the existing reputation record, applying a discounting factor $\alpha \in [0, 1]$:

$$r_A(x, t + 1) \leftarrow \alpha r_A(x, t) \tag{2.2}$$

$$r_B(x, t + 1) \leftarrow \alpha r_B(x, t) \tag{2.3}$$

- Subsequently, if expert $Y \in \{A, B\}$'s service was satisfactory in the round, x updates her neighbor's reputation records. For each $y \in V(G)$:

$$r_Y(y, t + 1) \leftarrow r_Y(y, t + 1) + a_{xy} \tag{2.4}$$

[4] That reputation is indeed a valid signal for expertise is proven, in a slightly altered context, in the paper "Market Share Indicates Quality" which is part of this thesis

[5] As in the global case, this mechanism may be explained by the presence of a reputation system that provides personalized, localized services for the agent

When all reputation records for round $t+1$ have been updated by round t's results, round $t+1$ will start, and so on.

The reputation records at round 1 are *initial conditions*. Their origin is extraneous to the model.

3. Steady-States of the Model

We investigate the behavior of the model, and specifically ask under what conditions it will reach a steady state. We define a steady state as follows:

Define $V_A(t) \subset V(G)$ to be the set of agents who are A-types at time t. $V_B(t) = V(G) - V_A(t)$ is the set of agents who are B-types at time t.

A (strict) steady state of the social graph is said to be reached at round T if for all rounds $t > T$, $V_A(t) = V_A(T)$. Since our dynamics are probabilistic, strict steady states are hard to find. We instead define *quasi-stability* or stability *by expectation*:

An agent x is said to be *quasi-stable* at time T if either:

- It is A-type and for all $t > T$ $\mathbb{E}[r_A(x,t)] \geq \mathbb{E}[r_B(x,t)]$, or:
- It is B-type and for all $t > T$ $\mathbb{E}[r_A(x,t)] < \mathbb{E}[r_B(x,t)]$.

If all agents are quasi-stable at time T we say that the social graph is quasi-stable at time T.

At each round, each A-type agent has probability ϵ_A of positive feedback for A, and probability $(1-\epsilon_A)\epsilon_B$ of positive feedback for B. Each B-type agent has probability ϵ_B of positive feedback for B, and probability $(1-\epsilon_B)\epsilon_A$ of positive feedback for A.

Let $u_{xZ}(t)$ be the expectation of total feedback on agent x's reputation of expert $Z \in \{A, B\}$ at time t, then:

$$u_{xA}(t) = \epsilon_A \sum_{y \in V_A(t)} a_{yx} + (1-\epsilon_B)\epsilon_A \sum_{y \in V_B(t)} a_{yx} \tag{3.1}$$

$$u_{xB}(t) = (1-\epsilon_A)\epsilon_B \sum_{y \in V_A(t)} a_{yx} + \epsilon_B \sum_{y \in V_B(t)} a_{yx} \tag{3.2}$$

Subtracting and dividing by $\epsilon_A \epsilon_B$:

$$\frac{u_{xA}(t) - u_{xB}(t)}{\epsilon_A \epsilon_B} = \left[1 + \frac{1}{\epsilon_B} - \frac{1}{\epsilon_A}\right] \sum_{y \in V_A(t)} a_{yx} - \left[1 + \frac{1}{\epsilon_A} - \frac{1}{\epsilon_B}\right] \sum_{y \in V_B(t)} a_{yx} \tag{3.3}$$

Denote $q_A := \frac{1}{2}(1 + \frac{1}{\epsilon_B} - \frac{1}{\epsilon_A})$ and $q_B := \frac{1}{2}(1 + \frac{1}{\epsilon_A} - \frac{1}{\epsilon_B})$.

We call q_A, q_B the *qualities* of expert A, B respectively. Note that $q_A + q_B = 1$. Then we have:

$$\frac{u_{xA}(t) - u_{xB}(t)}{2\epsilon_A \epsilon_B} = q_A \sum_{y \in V_A(t)} a_{yx} - q_B \sum_{y \in V_B(t)} a_{yx} \tag{3.4}$$

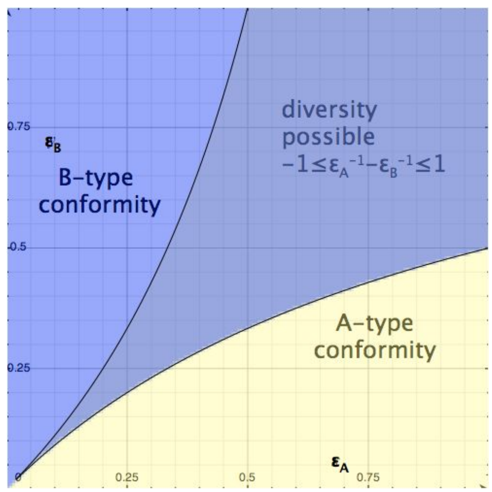

Figure 1: Conformity/diversity depending on expertise levels

Observe in (3.4) that if $V_A(t)$ is constant for all $t \geq T$ then so is $u_{xA}(t) - u_{xB}(t)$. In a steady state, it is therefore impossible for x to be an A-type if $u_{xA}(t) - u_{xB}(t) < 0$, because each round $r_A(x,t) - r_B(x,t)$, initially positive at T, would decrease on average by at least $u_{xB}(t) - u_{xA}(t) > 0$ (discounting decreases it even more). Therefore with probability 1 at some future t_0 $r_A(x,t_0) < r_B(x,t_0)$ and x will no longer be an A-type, contradicting the assumption of a steady state.

Similarly in a steady state it is impossible for x to be a B-type if $u_{xA}(t) - u_{xB}(t) > 0$.

Note again in (3.4) that if q_A and q_B are of opposite sign, then $u_{xA}(t) - u_{xB}(t)$ has the same sign for all agents $x \in V$. By the above argument, in this case if a steady state occurs it cannot be diverse, because such a steady state cannot hold both A-types and B-types.

As our interest is in diverse steady states, we turn our attention to the case where q_A and q_B have the same sign. As both cannot be negative we must have $q_A \geq 0$ and $q_B \geq 0$, or (see figure 1):

$$-1 \leq \frac{1}{\epsilon_A} - \frac{1}{\epsilon_B} \leq 1 \tag{3.5}$$

Referring once more to (3.4) we see that the social graph is quasi-stable at time t if, and only if:

- For each A-type agent x:

$$q_A \sum_{y \in V_A(t)} a_{yx} \geq q_B \sum_{y \in V_B(t)} a_{yx} \tag{3.6}$$

- For each B-type agent x:

$$q_A \sum_{y \in V_A(t)} a_{yx} \leq q_B \sum_{y \in V_B(t)} a_{yx} \tag{3.7}$$

Conclusion 1. *A social graph is quasi-stable iff its partition into A-types and B-types has every agent with at least as many neighbors (weighed by type) on its own side of the partition as on the other side.*

More specifically: Quasi-stability requires that every A-type agent has at least $q_B = 1 - q_A$ of its neighbors from A-types, and that every B-type agent has at least $q_A = 1 - q_B$ of its neighbors from B-types.

When expertise levels are equal, qualities are equal, $q_A = q_B = \frac{1}{2}$ and each agent requires at least half of its neighbors from its own kind.

When expertise levels are unequal, each A-type agent must have at least a $q_B = 1 - q_A$ fraction of her neighbors to be A-type. Likewise for type B. Note that this means that "high-quality" types can be stable in an environment with relatively few similar neighbors, while "low-quality" types need a large majority of similar neighbors to "survive".

While the criterion for quasi-stability is simply stated, it has no easy solutions, and indeed the existence of a solution appears unlikely, especially when no loops are present, and expertise levels are different.

Here is an example for which no stable diverse partition exists:

Example 1. *Let G be the complete graph with N agents, and without loops, i.e. $\forall x \in V(G), a_{xx} = 0$, and $\forall x, y \in V(G), x \neq y, a_{xy} = 1$.*

For any $q_A \in [0, 1], q_B = 1 - q_A$, no stable diverse partition exists for this social network. For let the set of A-types be called $V_A \subset V(G)$. By (3.6) each member of V_A is quasi-stable iff

$$q_A(|V_A| - 1) \geq q_B(N - |V_A|) \tag{3.8}$$

While by (3.7) each member not in V_A is quasi-stable iff:

$$q_A|V_A| \leq q_B(N - |V_A| - 1) \tag{3.9}$$

But (3.8) and (3.9) contradict each other.

If a loop is added to every vertex of G, a stable diverse partition is possible: (assuming $q_B N$ to be an integer) any partition in which $|V_A| = q_B N$ is stable.

This effect of loops is not restricted to the above example. In fact, *any* social graph in which a loop is attached to every vertex has a stable and diverse partition:

Theorem 1. *Every social graph $G(V, E)$ in which $a_{xx} = 1$ for every $x \in V(G)$ has a stable and diverse partition for every pair of qualities q_A, q_B satisfying $q_A + q_B = 1$.*

Proof. This is a consequence of result by Stiebitz[8] showing that for every graph $G(V, E)$ and functions $a, b : V \mapsto \mathbb{N}$ satisfiying $d_G(x) \geq a(x) + b(x) + 1$ for every vertex $x \in V$, there is a diverse partition (A, B) of V such that

1. $d_A(x) \geq a(x)$ for every vertex $X \in A$, and
2. $d_B(x) \geq b(x)$ for every vertex $X \in B$

Setting $a(x) = \lfloor q_A d(x) \rfloor$, $b(x) = \lceil q_B d(x) \rceil - 1$, the result applies to $G(V, E)$, so there exists a diverse partition (A, B) of V. $\square$

4. The Social Graph on a Map

We now take the social graph described in Section 2 and put it on a map, defining what we will call a *social map*. To this end we specialize the model in the following ways:

- Every agent is placed at some point on $\mathcal{R}^2$, the two-dimensional plane, endowed with a metric μ. A constant *influence radius* $R > 0$ is associated with the social map to define neighborhood: Two agents are connected by an edge if their mutual distance (according to μ) is R or less.
- We let the agent population be large, so that it can be idealized as a planar domain V, where for every point x the set $B(x, R) \cap V$ has a positive two-dimensional measure.
- No loops: an agent's self-weight is zero.

The most obvious choice for μ is the Euclidean planar metric. But it may also be defined by any metric, to take account of geographical obstacles and available modes of transportation: E.g. the travel time, or economic cost of traveling from one point to another is a possible choice of the μ metric.

Let us spell out the model, adapted from Section 2, for social maps:

A social map $M(V, \mu, R)$ is a compact[6] λ-measurable[7] set V of points in the plane $\mathcal{R}^2$, equipped with a metric μ and an influence radius $R > 0$ to define neighborhood: Points c, d are neighbors if $\mu(c, d) \leq R$.

As in Section 2 there are two experts A and B, with expertise levels ϵ_A, ϵ_B respectively, and qualities $q_A := \frac{1}{2}(1 + \frac{1}{\epsilon_B} - \frac{1}{\epsilon_A})$ and $q_B := \frac{1}{2}(1 + \frac{1}{\epsilon_A} - \frac{1}{\epsilon_B})$. We assume $q_A > 0, q_B > 0$ since, as we have seen, these are necessary conditions for a stable, diverse partition.

With each $v \in V$ and at any round t there are associated reputation values $r_A(v, t), r_B(v, t)$ of experts A, B respectively.

At any time t, V is partitioned into $V_A(t)$, the subset of V whose agents are A-type: $V_A := \{v \in V | r_A(v, t) \geq r_B(v, t)\}$, and $V_B(t) = V \setminus V_A(t)$, the subset of V whose agents are B-type.

[6] In the topology induced by the metric μ.

[7] λ is the planar Lebesgue measure.

The *neighborhood* of $v \in V$ is denoted $N(v)$, and is the subset of V in v's influence radius, i.e. agents whose distance from v is R or less: $N(v) := \{y \in V | \mu(y, v) \leq R\}$.

A social map is called *dense* if the neighborhood of each agent has positive Lebesgue measure, i.e. for all $v \in V$:

$$\lambda(N(v)) > 0 \tag{4.1}$$

We can then rephrase the criterion for quasi-stability of social graphs for social maps: A dense social map $M(V, \mu, R)$ is quasi-stable at time t, if, and only if:

- For each $v \in V_A(t)$

$$q_A \lambda(N(v) \cap V_A(t)) \geq q_B \lambda(N(v) \cap V_B(t)) \tag{4.2}$$

- For each $v \in V_B(t)$

$$q_A \lambda(N(v) \cap V_A(t)) \leq q_B \lambda(N(v) \cap V_B(t)) \tag{4.3}$$

We now make the Map Conjecture:

Conjecture 1. *Let $M(V, \mu, R)$ be a dense social map. Let q_A, q_B be the qualities of type A, B, respectively, with $q_A + q_B = 1$. Then M has a stable and diverse partition for q_A, q_B. I.e. V can be partitioned into $V_A(t), V_B(t)$ satisfying* (4.2) *and* (4.3).

We can at present offer no complete proof of the conjecture. Instead we sketch an outline of a proof which suggests that it is true:

1. We can construct increasingly fine grids covering V:
   - The social graph $G_n(V_n, E_n)$ is constructed by covering the map by a grid of squares with side $\epsilon_n = o_n(1)$. The vertex set V_n consists of the grid squares which are wholly in V (see Figure 2).
   - The edge set E_n connect those grid squares of V_n whose square centers are in each other's neighborhood, as defined by μ, R.
   - G_n is then a simple, undirected graph.
2. Since every vertex has a loop (every grid square centre is in its own neighborhood), by Theorem 1 there exists a stable, diverse partition (A_n, B_n).
3. If the series of partitions $(A_n, B_n), n = 1, 2, \ldots$ converges to a definite partition (A, B) of V, then that partition satisfies the conjecture.
   - By convergence we mean convergence according to the symmetric-difference ($\triangle$) pseudo-metric: The $\triangle$ distance between two sets U, V is the Lebesgue measure of their symmetric difference: $\triangle(U, V) = \lambda((U \setminus V) \cup (V \setminus U))$. A series $\triangle$-converges to a limit if its $\triangle$ distance from the limit converges to 0.

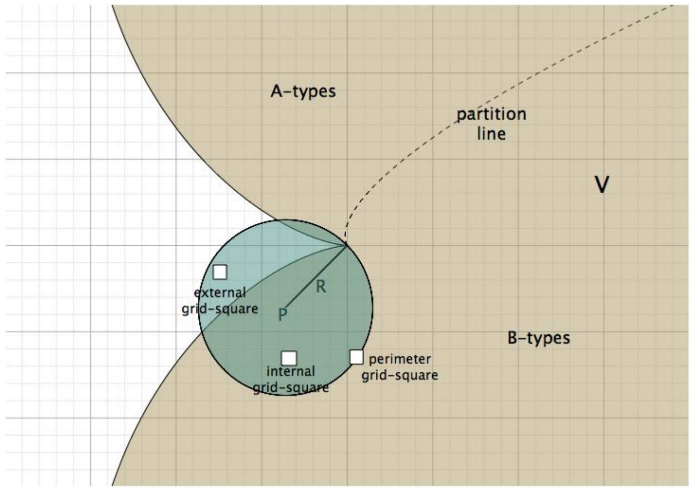

Figure 2: Grid coverage of map

4. Therefore the conjecture is true whenever the series A_n (or B_n) $\triangle$-converges.

That every social map has such a convergent grid partition remains to be proven. We note that not every series of partitions necessarily converges: For example, an increasingly fine checkerboard partition of a square does not $\triangle$-converge, as the $\triangle$ distance between any two elements of the series is constant at half the square's measure. Stiebitz' existence proof is non-constructive and of little use in ruling out non-convergence.

5. Discussion

5.1. Some Solutions

The Map Conjecture merely states the existence of diverse stable partitions, but gives no clue as to how such a partition might look. Deriving an actual solution, whether in analytic or numeric form, for any specific case is difficult. Therefore we turn now to some special cases for which we are able to explicitly describe a solution. In the following, we restrict the metric used to the Euclidean distance.

The case of equal qualities $q_A = q_B = \frac{1}{2}$ is of special interest. Here we note that any set that has an axis of symmetry may be partitioned along the axis of symmetry: E.g. an isosceles triangle may be partitioned along one of its altitudes, a circular disc may be partitioned by any of its diameters, etc.

The case of unequal qualities $q_A \neq q_B$ lends itself to a simple solution if the set to be partitioned contains a large enough circular disc:

Theorem 2. *Let a social map $M(V, \mu, R)$ be given, and assume unequal qualities $q_A \neq q_B$. Assume w.l.o.g. that $q_A > q_B$. Then, if a disc of sufficiently* *large radius lies entirely within V, a stable diverse partition may be constructed as follows:*

- *When $\frac{1}{2} < q_A < \frac{3}{4}$: Let $\theta \in (0, \pi)$ be implicitly given by:*

$$q_A = \frac{1}{2} + \frac{1}{2\pi}\left[\frac{\theta}{2} - \frac{\theta}{2\tan^2\frac{\theta}{2}} + \frac{1}{\tan\frac{\theta}{2}}\right] \tag{5.1}$$

  Set $r = \frac{R}{2\sin\frac{\theta}{2}}$. If a disc $B(O, r+R)$ is wholly contained in V, partition V into the inside and the outside of a circle of radius r and centre O, with the inside disc $B(O,r)$ populated by A-types and the outside populated by B-types.

- *When $\frac{3}{4} \leq q_A < 1$: If a disc $B(O, \frac{3R}{2})$ exists which is wholly contained in V, partition V into a point set V_A populated by A-types, of measure $\pi R^2 q_B$, which is wholly contained in $B(O, \frac{R}{2})$, and a point set $V \setminus V_A$, populated by B-types.*

Proof. For $\frac{1}{2} < q_A < \frac{3}{4}$, such a partitioned map is shown in figure 3. Clearly the partition is stable if the criteria for quasi-stability (4.2), (4.3) hold with equality on the partition boundary. Due to radial symmetry, if this is true for one point on the boundary, say X, it holds for all.

At X, this means that:

$$\lambda(B(X,R) \cup B(O,r)) = q_B\lambda(B(X,R)) = \pi R^2 q_B \tag{5.2}$$

Observe in figure 3 that $sin\frac{\theta}{2} = \frac{R}{2r}$. And:

$$\lambda(B(X,R) \cup B(O,r)) = \frac{\pi - \theta}{2}R^2 + \theta r^2 - R\frac{R}{2\tan\frac{\theta}{2}} \tag{5.3}$$

Therefore:

$$q_B = \frac{1}{2} - \frac{1}{2\pi}\left[\theta - \frac{\theta}{2\sin^2\frac{\theta}{2}} + \frac{1}{\tan\frac{\theta}{2}}\right] = \tag{5.4}$$

$$= \frac{1}{2} - \frac{1}{2\pi}\left[\frac{\theta}{2} - \frac{\theta}{2\tan^2\frac{\theta}{2}} + \frac{1}{\tan\frac{\theta}{2}}\right] \tag{5.5}$$

$$q_A = 1 - q_B = \tag{5.6}$$

$$= \frac{1}{2} + \frac{1}{2\pi}\left[\frac{\theta}{2} - \frac{\theta}{2\tan^2\frac{\theta}{2}} + \frac{1}{\tan\frac{\theta}{2}}\right] \tag{5.7}$$

where $r = \frac{R}{2sin\frac{\theta}{2}}$ as claimed. This solution is valid so long as $B(X,R)$ and $B(O,r)$ intersect, i.e. for $r > \frac{R}{2} \Rightarrow \theta < \pi \Rightarrow q_A < \frac{3}{4}$.

For $q_A \geq \frac{3}{4}$, see figure 4 for illustration. Observe that if V_A, the community of A-types (exemplified by the peanut-shaped region in figure 4), is entirely within $B(O, \frac{R}{2})$, then every agent in V_A has all of V_A within her influence

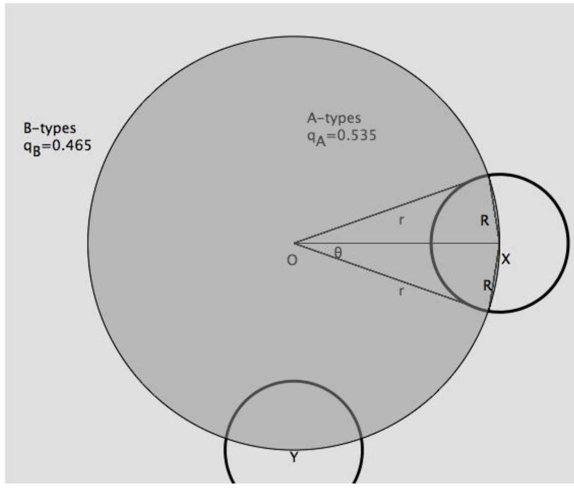

Figure 3: Partition into inside and outside of a circle

radius. Furthermore, since $B(O, \frac{3R}{2})$ is entirely within the map, every agent in $B(O, \frac{R}{2})$ has her entire influence radius within the map and so the measure of her influence region is πR^2. As the measure of V_A is $\pi R^2 q_B$, (4.2) holds for all A-type, while (4.3) holds for all B-types, so such a partition is stable. □

The social interpretation for the partition with $q_A \geq \frac{3}{4}$ is that a strong minority can survive in an arbitrary geographical setting, provided they reside close enough to influence each other. When this occurs, their proportion of the population in their area of prevalence is not too low for survival, nor too high, as they would take over the entire population. Their weaker neighbors are also stable by virtue of their larger numbers.

5.2. Dynamics

The existence of diverse steady states raises the question of whether such steady states are likely to arise from a random partition, and whether one can point to real-world situations which approximate the model and the dynamics that arise out of it.

We developed a program that simulates the dynamics of the model on a map of the continental USA (see figures 5-9). The map is divided into RED-types and $BLUE$-types, starting at some arbitrary initial division, from which the partition evolves, ending in either conformity, where one of the types populates the entire map, or in diversity, where a stable partition of the two types is reached. Two instructive examples demonstrate each of the alternatives: In figure 5, RED is the superior type with quality 0.55. After many rounds, the $BLUE$-types retreat, but manage to hold their ground in a stable frontier shown in figure 6. Caution should be used when interpreting these results: The agents on the map are simulated by finite-sized pixels, and pixelation effects are not

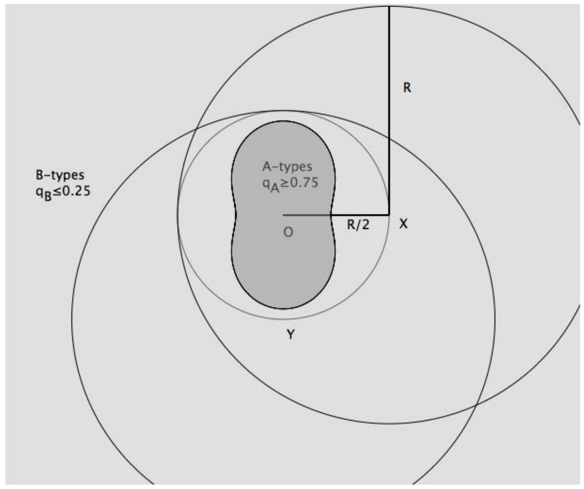

Figure 4: Partitions for strong minorities

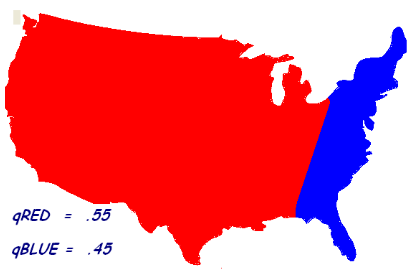

Figure 5: initial $q_{BLUE} = 0.45$

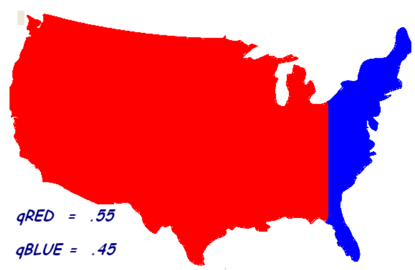

Figure 6: final $q_{BLUE} = 0.45$

negligible (e.g. they allow the red-blue border to be mostly straight, which would be impossible in the continuous case). An inspection of the final steady state also makes clear that the particular position of the final border hinged on a fortunate angle it made with map boundaries at its "Great Lakes" and "Caribbean" ends.

In figure 7 RED's quality advantage is larger, and the $BLUE$-types eventually (see figure 9) become extinct. An intermediate partition shown in figure 8 shows that in advanced stages of retreat the $BLUE$-types are concentrated in the peninsulas of New England, Florida and Delaware, where they are relatively insulated from contact with the dominant RED-types.

The pattern of $BLUE$ retreat is similar to the 2000+ year retreat of the Celtic languages in Europe, as they are being displaced by Romance and Germanic languages. The diachronic distribution seen in figure 10 [9, 5], shows that the Celtic languages that once dominated central and western Europe, have gradually retreated towards western peninsulas and islands, and at present take refuge (perhaps only temporarily) in several relatively insulated regions includ-

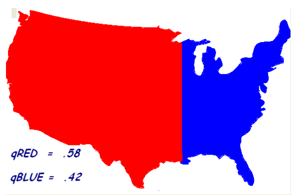

Figure 7: initial $q_{BLUE} = 0.42$

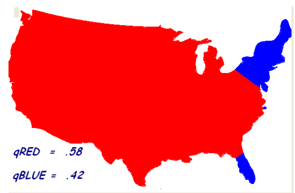

Figure 8: intermediate $q_{BLUE} = 0.42$

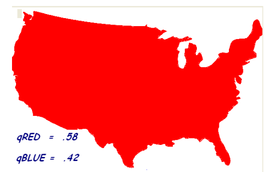

Figure 9: final $q_{BLUE} = 0.42$

ing the peninsulas of Bretagne, Cornwall and Wales.

The retreat of the Celtic languages is, in general, not due to the retreat of the Celtic people themselves, at least not in recent history. Indeed the Scots and the Irish, for example, retain their national and cultural identity, but have abandoned their Celtic tongues in favor of English. The retreat of the Celtic languages is due, not to the extinction or migration of their speakers (which would better be described by demographics or population dynamics), but to the adoption of a language in place of another, a form of social learning.

[1] Bala, V. and S. Goyal: Learning from Neighbors. *Review of Economic Studies*, 65, 595-621, 1998.

[2] Ban, A. and N. Linial: The Dynamics of Reputation Systems. *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge*, pp. 91-100, 2011.

[3] de Groot, M. H. Reaching a consensus, *J. Amer. Statist. Assoc.* 69, 118-121 (1974).

[4] Ellison, G., and D. Fudenberg Word of mouth communication and social learning, *Quarterly Journal of Economics* 110, 93-126 (1995).

[5] Haywood, J.: Atlas of the Celtic World, London Thames & Hudson Ltd., 2001, pp.30-37

[6] Lieberman E, Hauert C, Nowak M.A.: Evolutionary dynamics on graphs. *Nature*. 2005;433:312-316.

[7] Nowak, M.A.: Evolutionary Dynamics, Exploring the Equations of Life. Harvard University Press, 2006.

[8] Stiebitz, M.: Decomposing graphs under degree constraints. *Journal of Graph Theory* 23, no. 3 (1996): 321-324.

[9] *http://commons.wikimedia.org/wiki/File:Celts_in_Europe.png*

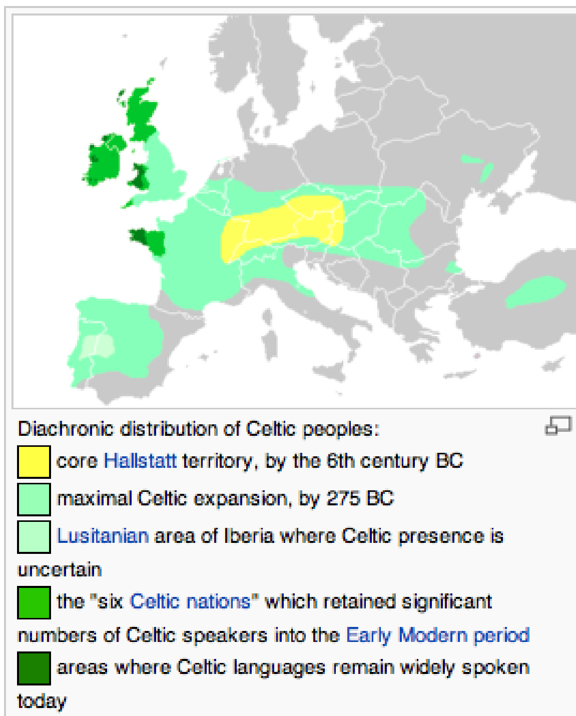

Figure 10: Diachronic distribution of Celtic languages (source: Wikipedia)