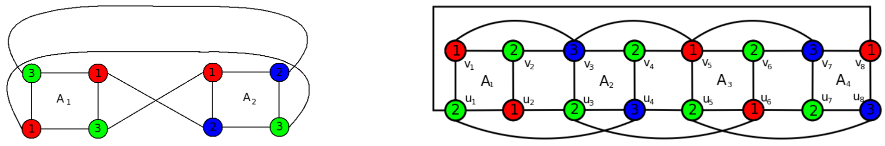
<u>MULTIAGENT GRAPH COLORING</u>
<u>SUPPLEMENT</u>

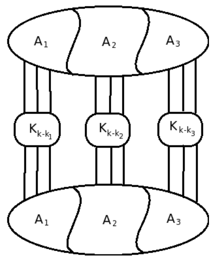
To clarify, the Moebius ladder graph $M_8$ can be divided into two agents and form an irrational partition as described in Figure 1(a). For the general multiagent case, one may construct a global graph from a ladder of squares; on the upper path consecutive odd vertices are connected, in the lower path connect opposite vertices of consecutive agents and finally add an edge between the last upper vertex and the first of the lower path. A four agent example is presented in Figure 1(b).



(a) (b)

Figure 1: Irrational partitions - an extra color propagates throughout the agents.

We add that in order to study partitions of a disconnected global graph, one can add complete graphs $K_{\widehat{\chi}-\chi_i}$ as connectors and the original partition would be individual rational if and only if the chromatic number of the resulting graph remains the same.



Connectors in a three agent partition of a graph with two connected components.

To verify whether a given two-agent coloring of $K_n$ is Pareto optimal, one can proceed in a dynamic programming fashion. To check improvement for agent $A_1$, maintain a table $R[k, i, j]$ to store $A_1$'s optimal value for a $k$-subset of his $i$ favorite colors such that $A_2$'s evaluation of the residue is no more than $j$. Gradually increase the considered set of labels, each time comparing minimal cost and run the procedure once for each of the agents. All can be implemented in running time $O(|\hat{V}|^4)$. This method can be generalized to a constant number of agents, in polynomial running time.
In case of a two agent individual rational partition of a general graph, extra columns are needed in the table, one for the set of joint colors and another for $A_2$'s exclusives. Here, with a more subtle examination of each label in maintaining minmum cost, verification can be processed in $O(|\hat{V}|^6)$ running time.
At the other extreme, if each agent holds a single vertex in $K_n$, it is possible to reduce the problem to finding a maximum matching in a weighted bipartite graph constructed from the sets of agents and labels; The edge between agent $i$ and label $j$ is assigned a weight proportional to $\mathcal{L}_i(j)$ and a given coloring is Pareto optimal if and only if the total weight of the corresponding matching passes a certain threshold. Consequently Pareto efficiency in this case can be verified and an improving allocation, if exists, can be found in $O(n^3)$ running time.

Regarding fair allocations in a public preference model of two agents coloring $K_n$, we distinguish between three possibilities (assuming $A_2$ has less vertices):

- If $A_2$ has an even size, we allocate odds to $A_1$ and evens to $A_2$ up to $l_{\chi_2}$ and vice versa for labels of a cost greater then $l_{\chi_1}$. The rest of the labels are assigned to agent $A_1$.
- If agent $A_2$ has an odd number of vertices while $\chi_1$ is even, we assign the labels surrounding the mean to $A_2$ and the rest for $A_1$.
- If both agents are of an odd size, we combine the previous two methods; allocate odds to $A_1$ and evens to $A_2$ up to $l_{\chi_2}$ and vice versa for labels of a cost greater then $\chi_1 + 1$. Then, we choose an agent at random which incurs a slightly lower cost, and assign to it the mean, keeping the next label for the second agent. The rest of the labels are allocated to $A_1$.

For individual rational partitions of general graphs with $p$ joint colors, the problem reduces to fair coloring $K_{|\hat{V}|-p}$ as described above, while sharing the first $p$ labels between the two agents.