Rebuilding the Great Pyramids: A method for identifying control relations in complex ownership structures

Gur Aminadav Yoram Bachrach Konstantin Kosenko
Jeffrey S. Rosenschein Yoav Wilf

June 16, 2011

Abstract

Identifying the corporate controller (controlling shareholder, ultimate owner) is an essential prerequisite for any debate on the corporate governance of a specific firm and of entire markets. This paper aims to provide a comprehensive, precise and economically sound method for identifying control relations on the corporate level and especially in complex ownership structures. We apply weighted voting games literature as a theoretical framework for our analysis and use the Shapley-Shubik and the Banzhaf power indices to determine control rights. The core element of the proposed method, distinguishing our study from others, in solving the puzzle of corporate control, is the simultaneous analysis of both the specific ownership map within the corporation and the corporate network in which the firm is embedded. We implemented our algorithm into a Java computer program and tested it on a real-world data set of corporate ownership in the Israeli market. The direct product of the analysis of these data is a comprehensive map of control relations at every time point. We find that the corporate control relations identified by our method are richer and more accurate than those provided by different official sources.

Keywords: corporate governance; control rights; corporate networks; ownership structure; power indices JEL classification: G31, G32, G34

1 Introduction

Identifying the corporate controller (controlling shareholder, ultimate owner) is an essential prerequisite for any debate on the corporate governance of a specific firm and of entire markets. The main reason for such emphasis on this process is that the controller's identity plays a crucial role in mechanisms that policy makers use to overcome various agency problems arising from the separation of ownership and control [10, 16]. An inappropriate identification strategy can result in mechanisms that are, at best, simply inadequate for the challenges facing policy makers or, at worst, counterproductive. Therefore, the purpose of the present study is to provide a comprehensive, precise and economically sound method for identifying control relations that is applicable to a wide range of studies on ownership and corporate governance as well as to practical implementations in any given market.

According to a simple but widely used analytical framework, the systems of governance across the world can be broadly divided into two separate groups [32]. The first, the Anglo-Saxon system (frequently called the "outsider system"), which is prevalent in the US and the UK, is characterized by widely dispersed ownership. The second, the "insider system", is relevant to most of the rest of the world, and relies on the empirical observation of concentrated ownership. While even in the outsider system, which is the simpler of the two systems, identifying the controlling authority is not always a trivial task, the identification process is particularly difficult in the insider system. This complexity stems from the multiplicity of control-enhancing mechanisms, e.g., dual shares, cross- or circular holdings, and especially stock pyramids, which are often used to allow existing blockholders to deviate from the proportionality principle[1] and to enhance control by leveraging their voting power [23, 24, 33, 19]. These complex, multilayered structures, in which an entity, such as an individual, a family or a company, holds significant shares of some firms that in turn hold significant shares in other corporations, make it difficult to identify the corporate controllers and render the market for control less transparent. Control identification instruments able to cope with that complexity are thus required, but as others have previously noted [7], the literature still offers no consensus on which types of instruments are most appropriate.

In the present paper we suggest a new technique for the identification of control relations, on the corporate level and especially in complex ownership structures. In formal terms, the algorithm described in the current study transforms any given input-output matrix of cash-flow (voting) rights[2] into a Boolean matrix of control. The core element of the proposed method, and the element that distinguishes our study from others, is an extensive examination of control relations that incorporates firm-specific characteristics, e.g., the firm's shareholdings distribution, and at the same time considers the firm as embedded in a corporate network, i.e., the algorithm

[1] The "proportionality principle" referring to proportionality between ultimate economic risk and control - means that share capital that has an unlimited right to participate in the profits of the company or in the residue on liquidation, and only such share capital, should normally carry control rights, in proportion to the risk carried.

[2] Unless cash flow rights are not equal voting rights. In such cases explicit voting rights are required.

considers the possibility that a firm's corporate shareholders are controlled by the same ultimate owner and/or that they partake in cross-control relations. In other words, the aspects of control of a certain firm and the control paths that involve that firm are treated simultaneously.

In order to develop control identification techniques, we apply weighted voting games literature as a theoretical framework for our analysis and use the Shapley-Shubik and the Banzhaf power indices to determine control rights. These indices provide measures for the extent to which an agent holding a certain proportion of the voting rights can affect the outcome of a collective decision by casting his vote, given the overall proportion required to win and the distribution of other voters' rights [15]. This approach allows the measurements of control to be extended to cases in which the majority threshold rule[3] is not feasible or valid, and it provides a refined alternative to traditionally used tests, for example, the weakest-link principle (WLP) approach [33, 39, 26, 18]. We develop a power-index-based test for control and show that the holdings required in order to achieve direct control are firm-specific. Afterward, we adapt the control test into a fully integrated algorithm able to analyze control relations in any given complex ownership structure. The main contributions and features of this algorithm are summarized as follows: 1) It provides existence and uniqueness of a solution for any market irrespective of its size or structure; and 2) the method is flexible in parameters and assumptions, facilitating incorporation of additional market-specific information.

Many ownership structures where shareholders make joint decisions use weighted voting (or some weighted voting variant which can be modeled as a simple coalitional game). In such domains, one sometimes wants to obtain many power index results. In large and complex networks computational issues become an obstacle in terms of running time and accuracy [21]. Thus, the practical applications of power indices for measuring shareholder voting behavior in full scale settings have not been widely adopted in the classical corporate finance literature until now. In order to overcome these obstacles in our analysis we incorporated the effective randomized methods for approximating power indices as was recently developed by [43]. This randomized approach outperforms any deterministic algorithm in terms of accuracy, and no other randomized algorithm can achieve significantly better results.

We apply the algorithm to a real database of ownership in the Israeli capital market spanning the period of 1995-2009. For the empirical compatibility test we use an input-output matrix of cash flow rights[4] of approximately 650 listed companies - the whole population of public companies on the Tel Aviv Stock Exchange. The direct product of the analysis of these data is a comprehensive map of control relations at every time point. Moreover, we find that the results obtained through our method are more precise than those achieved with the frequently used WLP method (for a detailed discussion see [38]. The simple WLP criterion, provided by [33], misidentifies shareholders as controllers in many cases[5] , and it also fails to define and empirically

[3]More than half the votes.

[4]Following the "one-share - one-vote" principle legally adopted in Israel since 1994.

[5]For example in a case where three unrelated shareholders A, B, and C hold 20%, 21%, 22% respectively in a certain firm, then the WLP method would misidentify shareholder C as the firm controller.

identify corporate controllers in cases where the firm's position within the corporate network should be considered - i.e., when additional information about the control relations among the firm's direct corporate shareholders is required. Our approach overcomes these obstacles and thus constitutes a more accurate alternative.

The paper is organized as follows. Section 2 presents and discusses a control concept based upon voting (cash flow) rights. Section 3 introduces weighted voting games and power indices in the context of shareholder votes. Section 4 provides a formal definition of control and derives several related concepts. Section 5 states and explains the five central assumptions underlying our model. Section 6 extends our formal definitions by incorporating additional information about the control relations among corporate shareholders. Section 7 puts forward our methodology to identify control relations in corporate networks and intuitively describes the algorithm we developed. Finally, Section 8 demonstrates several results from a real-world database and compares the results of our method with the results derived from the cut-off point method.

2 The concept of "control"

There is a long and rich history attached to such words as influence, authority, power and control. These terms, and especially the latter two, are often used interchangeably in many disciplines as well as in general public parlance. Although for a great many essays distinctions among the concepts are necessary and useful, in this section we seek to explicate the central notion that lies behind these terms and to formulate a uniform conceptual framework. This, in order to supply the key insights underlying our approach, and justify the use of our suggested algorithm as a standalone technique for the identification of control relations.

The question of how "control" is conceptualized is a critical one - apart from the problem of obtaining reliable and valid information and prior to any decision regarding the identification strategy. Generally speaking, control is derived from power - "the capacity to initiate, constrain, circumscribe, or terminate action, either directly or by influence exercised on those with immediate decision-making authority" [2]. Therefore, like the sociological concept of power, the concept of control, by itself, is elusive - it can rarely be sharply segregated or clearly defined [1]. Nevertheless, the attempt should be made.

In our study we refer to the concepts of control and power as they relate to ownership, which is an important basis for obtaining a strategic position, and which constitutes a key factor in the dynamics of decision-making processes within a firm. Power in such settings is defined as the capacity to influence decisions; the degree of influence depends on the complete configuration of holdings allocated to all shareholders. Control refers to a situation in which the ownership stake is so powerful that its influence becomes complete discretion [22]. Thus, control is an absolute property, whereas voting power can be quantified. In the context of corporate control and for practical purposes, as Berle and Means [1] initially put it, control may be said "to lie in the hands of the individual or group who have the actual power to select the board of directors (or its majority), either by mobilizing the legal right to choose them - controlling a majority of the votes (cash-flow rights) directly or through some legal device - or by exerting pressure which influences their choice"[6]. Thus, since direction over the activities of a corporation is exercised through the board of directors, control is restricted to major powers only, i.e. it refers to the power of determining the broad policies for guiding a corporation and to the power of making crucial strategic decisions - for example, choices related to production function and/or markets of distribution, volume and direction of investment, larger commercial and political strategies, and selection of top-management personnel, rather than decisions related to routine, day-to-day activities.

This definition of control encompasses, among others, situations in which control derives from majority ownership. However, we do not restrict our attention to niches in which the binary decision rule of majority control is the leading criterion for identification of the corporate

[6]Nonetheless, "occasionally the major elements of control are made effective through dictation to the management" [1].

controller. The majority control criterion is a conventional, theoretically convenient tool that is consistent with many views regarding the establishment and maintaining of formal/legal control in corporations [7, 17, 24]; however, it seems to be too conservative for many situations. Control in large corporations is more often factual, depending upon a strategic position secured through a measure of ownership. Following the argument of Berle and Means (1932), factual control is less clearly defined than legal forms of control, but it nonetheless "may exist and be maintained over a long period, and as a corporation becomes larger and its ownership more widespread, it tends towards a position of impregnability comparable to that of legal control, a position from which it can be dislodged only by virtual revolution" [1]. Factual, or effective, control, to distinguish it from the majority rule, may involve different degrees of ownership lying below 50% of the cash-flow (voting) rights and may be said to exist when an individual or small group holds a sufficient stock interest to be in a position to dominate a corporation - to attract from scattered owners proxies sufficient to control a majority of the votes during all decision-making processes[7].

Since different corporations are held with different proportions of ownership, it seems to be conceptually important to extend the measurement of control rights, beyond that based on simple majority rule, to more general cases. In practice, however, Berle and Means and their followers have relied on assumptions that eliminate the necessity to address this issue. They have assumed that once a threshold of a minimum specified proportion of the stock[8] is adopted, the corporation can be easily assigned to one of two types: widely held (i.e. under "management control"), meaning that the corporation is not controlled by anyone, or ultimately controlled, i.e., one shareholder owns at least the threshold stock proportion in the corporation. In cross-country studies this approach, undoubtedly, has an enormous advantage, as it allows smoothing of different national disclosure requirements and substantially simplifies the calculation of control rights. On the other hand, this ad hoc rule has no formal justification and does not take into account the variability of ownership concentration - the concept of a universal fixed cutoff point for all firms is not relevant; a specific cutoff point should be applied to each individual firm in each market at each time period, as we will elaborate later. Moreover, this simple empirical rule of thumb constitutes a wide foundation for many serious problems in measurement of control rights and may lead to some paradoxical results (see [38]).

On this background we argue that to completely solve the puzzle of corporate control, it is crucial to analyze the concrete ownership map within the corporation, as well as the corporate network - the web of shareholding relations among corporations - in which the firm is embedded. This must be done before one can begin to understand where control is actually located and how

[7]Even if the controlling shareholder's stake is less than outright majority (50% of the stock) it can still negotiate with other shareholders and try to form a winning coalition, e.g. by 'bribing' them or by offering logrolling. The controlling shareholder has the best chances of forming a winning coalition in the occurrence of sharp divergence of interests vis-a-vis other shareholders [30, 8].

[8]Whether 20 percent - as in the original Berle and Means work - or the 10 percent criteria as adopted in most recent studies.

to identify it. The method of investigation proposed by Berle and Means, their definition and categorization of different kinds of control, provide an essential starting point for the analysis. However, the procedures used in their study do not take into account the complete distribution of ownership within the corporate network, a crucial aspect of control. For this reason, we do not reduce the concept of control to a rigid criterion such as a cutoff point; rather, we propose a more adaptable definition. Inspired by Weber's view of "power" [40], we define "control" as the ability of an object - either an individual or a family - or an identifiable group of objects, given the concrete structure of corporate ownership and inter-corporate relationships, to realize their own will in any action over time, despite the resistance of other participants in that action space. To estimate the probability that a given individual or group controls a corporation, then, we must know who the rivals or potential rivals for control are and what assets they can bring to the struggle [28]. In other words, we must know the exact composition of corporate principal shareholdings.

This definition of control has two obvious immediate implications related to the present study: First, it suggests that a specific arbitrary percentage of ownership in itself can tell us little about the potential for control that it represents. We can discover a threshold of ownership that yields control[9] only by a case study of the exact pattern of ownership within the given corporation. Most importantly, it also implies that focusing our attention on the single corporation may limit our ability to see the pattern of power relationships of which a corporation is merely one element, and it may restrict our understanding of the potential for control represented by a specific bloc of shares in a particular corporation. In many cases, the principal shareholders within the same corporation may themselves be controlled by a common ultimate owner. It is reasonable to assume that in such a situation they will act in unison regarding the decision-making processes in the corporation. Therefore, an individual's or a group's capacity for control must be a function of the distribution of direct ownership and the existence of concerts[10] among the shareholders. Identification of such complex relationships is possible only by incorporating the analysis of the single corporation into the network context and vice versa. In other words the simultaneous analysis of control relations is required.

The methods and procedures in our research, as well as the basic concepts and units of analysis, are not different from those that have been commonly employed in the past [12, 13]. The discussion above established our view of control over a firm and based the manifestation of control upon shareholders' voting power. In order to formalize the concepts of power and control, which will allow us to model decision-making processes in firms and to measure shareholders' voting power, we incorporate weighted voting games into our analytical framework.

[9] If such a threshold exists for a specific shareholder in a given firm.
[10] Groups of shareholders controlled by the same ultimate owner.

3 Weighted voting games and power indices

3.1 Weighted voting games

Weighted voting games are special cases of simple superadditive games[11]. For the purposes of this study, we assume that in each firm, every shareholder is assigned a non-negative number, or weight, represented by its fraction of voting rights in that particular firm. Moreover, for each firm a positive number that is not greater than one is specified as a quota such that a subset of shareholders wins the game if the sum of the weights of its participants exceeds that quota. Denote $[q; w_1, \ldots, w_n]$ where q and $w_1, \ldots, w_n$ are nonnegative real numbers satisfying $0 < q \leq \sum_{i=1}^{n} w_i$. We may think of w_i as the fraction of voting rights, or weight, of shareholder i in the set $N = \{1, \ldots, n\}$ of the direct shareholders in a specific firm, and q as the threshold, or quota, needed for a coalition to win in that firm. Thus $[q; w_1, \ldots, w_n]$ represents the simple game v defined by:

$$v(S) = \begin{cases} 1, & w(S) \geq q \\ 0, & w(S) < q \end{cases}$$

where for $S \subseteq N, w(S)$ means $\sum_{S \subset N} w_i$.

Therefore, each firm is characterized by an idiosyncratic decision rule, which is completely defined by the firm's voting majority quota q and the distribution of its shareholders' voting rights. In this study we concentrate on the component of voting power that derives solely from the decision rule itself - as distinct from real-world interactions. This a priori voting power of a shareholder under a given decision rule is regarded as the capacity of potential influence attributed to the shareholder by virtue of the rule [34]. It is exactly this a priori voting power of shareholders that we want to measure and later to incorporate in an operational test of control in corporate networks.

3.2 Power measurement

A critical shareholder has a strong influence on the result of the voting game, and the prominence of this property is related to various measures of power. The two approaches to measuring the a priori voting power of individual shareholders in weighted voting games, adopted in this

[11] A cooperative transferable utility (TU) game is a pair (N, v), where $N := (1, \ldots, n)$ denotes the set of *shareholders* and v a function which assigns a real number to each nonempty subset or *coalition* of N, and $v(\emptyset) = 0$. A $(0-1)-$*game* is a game in which the function v only takes the values 0 and 1. It is a *simple* game if it is not identically 0, and obeys the condition of monotonicity: $v(T) \geq v(S)$ whenever $T \supseteq S$. In these games a coalition S of shareholders is *winning* if $v(S) = 1$, and is *losing* if $v(S) = 0$. A shareholder i is said to be *critical* in a coalition S if S is winning and $S \backslash \{i\}$ is not. A TU game is superadditive if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$, in simple games superadditivity is equivalent to $v(S) + v(N \backslash S) \leq 1$ *for all* $S \subseteq N$. As a collective decision-making procedure in a firm is specified by the voting body of shareholders and the decision rules, it can be modeled by a $(0-1)-$*game* whose winning coalitions are those that can make a decision without the vote or despite the resistance of the remaining shareholders.

study, are those developed by Shapley and Shubik [36], by Penrose [35] and by Banzhaf [14]. The power indices provided by these studies are functions that associate with each simple superadditive game v a vector or power profile whose ith component is interpreted as a measure of the influence that shareholder i can exert on the outcome [12].

3.2.1 The Shapley-Shubik power index

The Shapley-Shubik power index [36] may be interpreted as a prior estimate of a voter's expected relative share in a fixed prize available to the winning coalition as a measure of voting power. For calculation of this index, it is assumed that shareholders join a coalition in a particular order. A *pivotal* shareholder for a given ordering is the member whose joining turns a developing coalition from a losing coalition into a winning coalition. The Shapley-Shubik power index is simply the Shapley value when applied in a setting of the simple weighted voting game v. Denote by π a permutation (reordering) of the shareholders, so π is a reversible function $\pi : \{1,\ldots,n\} \rightarrow \{1,\ldots,n\}$, and by Π the set of all possible such permutations. Denote by $S_\pi(i)$ the predecessors of i in π , so $S_\pi(i) := \{j|\pi(j) < \pi(i)\}$. The Shapley-Shubik index is given by $Sh(v) = (Sh_1(v),\ldots,Sh_n(v))$, where $Sh_i(v) := \frac{1}{n!}\sum_{\pi\in\Pi}[v(S_\pi(i)\cup\{i\}) - v(S_\pi(i))]$. In other words, if we denote by piv_i the number of orderings where shareholder i is pivotal, the Shapley-Shubik power index of shareholder i is: $Sh_i(v) := \frac{piv_i}{n!}$.

3.2.2 The Banzhaf power index

The Banzhaf power index [14, 31] may be interpreted as the probability that a given shareholder will change an outcome of a vote by swinging, where a swing for shareholder i corresponds to a winning coalition S containing i, from which i's defection would change the coalition from winning to losing. For a game v, the Banzhaf index[13] of shareholder i is given by: $\beta(v) = (\beta_1(v),\ldots,\beta_n(v))$, where $\beta_i(v) := \frac{1}{2^{n-1}}\sum_{S\subset N, i\in S}[v(S) - v(S\backslash\{i\})]$. In other words, using the notation of $swing_i$ for the number of swings for shareholder i, the Banzhaf power index of shareholder i is: $\beta_i(v) := \frac{swings_i}{2^{n-1}}$.

In the next section we propose a basic definition of control over a firm by using voting games and power indices.

3.3 Power Indices and Computational Complexity

The applicability of the power indices to measuring power in various domains has raised the question of finding tractable ways to compute them. Unfortunately, this problem has a high computational complexity, e.g. calculating the Shapley-Shubik power index in a weighted voting

[12] It is possible to generalize the power indices that we apply by using any member of the class of indices that fulfill certain relevant axioms (see e.g. [6]). They may serve for the same purpose of power measurement.

[13] Banzhaf actually considered the percentage of coalitions in which the shareholder is a swinger out of all winning coalitions. This is called the normalized Banzhaf index.

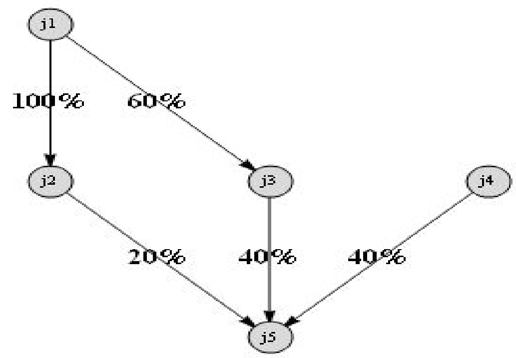
game. A naive algorithm for computing the Shapley-Shubik power index simply examines all the permutations of the n players, and averages the marginal contribution[14] of a player in each of them. Although calculating the marginal contribution of the player in a single permutation is easy, there are $n!$ permutations to consider. This is intractable even for $n = 50$ players, as even the strongest supercomputers available today cannot enumerate over such a high number of permutations. Computer scientists generally consider algorithms whose running time is polynomial in the size of the input as reasonably efficient algorithms, and those whose running time grows super-polynomially (for example exponentially) as inefficient. In the case the Shapley-Shubik power index computation, the input to the problem is the list of the weights of the players, or n integers. The naive method examines $f(n) = n!$ permutations, and since $f(n)$ grows faster than any polynomial this method is considered inefficient. Consequently, a prominent question is whether there exists a more efficient algorithm for computing the Shapley-Shubik power index in a weighted voting game. Although such improved algorithms do exist, a desirable goal is to find an algorithm whose running time is polynomial in the input size, or in other words an algorithm that computer scientists would consider efficient. Unfortunately, it seems such an algorithm is very unlikely to exist[15]. A key way of overcoming these negative results is using an approximation algorithm which works for any weighted voting game, but only computes the power index approximately, providing a certain guarantee regarding the error in the computation. In this paper we have used the power index approximation method proposed and analyzed by [43]. This algorithm estimates the power indices and returns, with high probability, a result that has a very low approximation error[16]. This allows us to compute power indices for highly complex ownership structures with many firms, while guaranteeing almost no error in the computed power indices.

[14] If a player is pivotal for a permutation of the players, we say it has a marginal contribution of one, and otherwise we say she has a marginal contribution of zero. One definition of the Shapley-Shubik power index is simply the average marginal contribution a player has across all permutation.

[15] [42] show that calculating the the Banzhaf index and Shapley value in weighted voting games are NP-complete problems, and [41] showed an even stronger negative result regarding the Shapley value, proving it to be a $\#P$-complete.

[16] More formally, given a game in which a player's true power index is ψ, and given a target accuracy level ϵ and confidence level δ, the algorithm returns an approximation $\hat{\psi}$ such that with probability at least $1 - \delta$ we have $|\psi - \hat{\psi}| \leq \epsilon$ (i.e. the result is approximately correct, and is within a distance ϵ of the correct value). This algorithm works by drawing a sample of k permutations (or coalitions), and testing whether the target player is pivotal in them. The total running time is logarithmic in the confidence and quadratic in the accuracy, so this approach provides an extremely accurate approximation and operates quickly even for games with many thousands of agents.

Figure 1: Simple corporate network



4 Defining control through voting games and power indices

4.1 Corporate Networks

Let the network of shareholding relationship among a set V of n firms and shareholders[17] be described by the $n \times n$ adjacency matrix $H \in [0,1]^{(n \times n)}$.

H is defined such that $h_{i,j} > 0$ represents the fraction of the stock of firm j owned (directly) by shareholder i. When $h_{i,j} > 0$ we say that i is a *direct shareholder* of j. For each shareholder/firm $j \in V$ we denote by $\Gamma^{-}(j)$ the set of direct shareholders of j i.e. $\Gamma^{-}(j) = \{i \in V : h_{i,j} > 0\}$. When $\Gamma^{-}(j) = \emptyset$ it means that j is an individual shareholder or a group of shareholders attached by family ties. For the sake of simplicity, in both cases we will call j a *family* and we denote by F the set of families in the network. Thus $\sum_{i=1}^{n} h_{i,j} = 1$ for every $j \in V \backslash F$, i.e. if j is not a family, then it is a firm and thus its shareholders jointly hold the entire stock of j. To each firm $j \in V \backslash F$ corresponds a *weighted voting game* g_j, which is defined by the shareholders $\Gamma^{-}(j)$ with their weights $\{h_{i,j} : h_{i,j} > 0\}$[18]. Thus, the weighted voting game g_j provides a model for the decision-making process of firm j.

In Figure 1 we present a simple market with participants $V = \{j1, j2, j3, j4, j5\}$ and an ownership structure represented by the corresponding input-output matrix of ownership:

[17] Firms themselves can be and usually are shareholders of other firms (corporate shareholders) (La Porta et al., 1998).

[18] In other words, g_j is completely represented by j's column of the input-output matrix of ownership together with firm j's majority quota q_j.

$$H := \begin{pmatrix} 0 & 100 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The set of families in this example is $F = \{j1, j4\}$, the set of firms is $V \backslash F = \{j2, j3, j5\}$, and the set of direct shareholders of $j5$, for example, is $\Gamma^{-}(j5) = \{j2, j3, j4\}$.

4.2 Direct control

We consider the class of weighted voting games g_j, where each shareholder is endowed with a certain number of weights, represented by the shareholder's holding fraction in a given firm $j \in V \backslash F$. For each firm with its specific voting rule, we use the power index as a measure of the a priori voting power of each of its direct shareholders. A controller will be a) the most powerful player; and b) powerful enough to dominate votes.

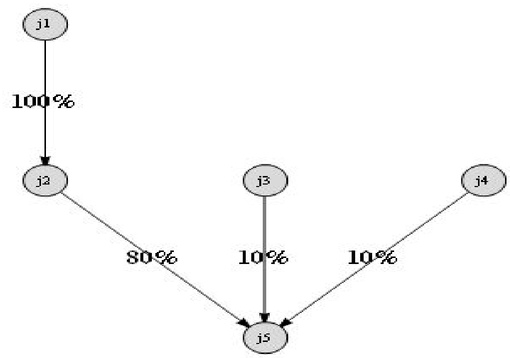
Definition 4.1 *Let $j \in V \backslash F$ be a firm. Denote the majority quota by q_j, with $0.5 \leq q_j \leq 1$; and denote the control threshold by θ, with $0.5 < \theta \leq 1$. A party $i \in V$ with stock holdings in the firm j is said to* directly control j*, if and only if the power index of i, given majority quota q_j, is at least as large as the control threshold θ. The power index is calculated for the shareholders of j, $\Gamma^{-}(j)$, as a player-set in a weighted majority game g_j with weights equal to their fraction of voting rights in j.*

If for a given firm there is no shareholder with direct holdings that fulfills the conditions above, then we say that this firm is not directly controlled, i.e., the firm is *widely held*.

By construction, the Shapley-Shubik and the normalized Banzhaf power indices of all shareholders in a specific game g_j sum to 1. Therefore, the restriction $\theta > 0.5$ in the definition implies the uniqueness of a controller of a firm, if a controller exists. For each firm j and for each of j's shareholders $i \in \Gamma^{-}(j)$, given the majority quota q_j, the power index threshold for control θ, and the complete configuration of holdings allocated to all other shareholders in this firm $\Gamma^{-}(j) \backslash \{i\}$, we can determine the minimum level of holdings required for shareholder i to achieve direct control in j (if such a minimum exists)[19], is at least as large as the control threshold θ. Thus, we say that the **voting rights (holdings)** required in order to achieve direct control are **firm-specific** in this model, and cannot be fixed for all firms as suggested, for example, in the WLP model.

[19] By solving the inverse problem: Given $\Gamma^{-}(j) \backslash \{i\}$ and their corresponding voting rights, and given the majority quota q_j, what are the minimal holdings a 'newly added' shareholder i must possess such that the power index of i calculated for the weighted majority game, played by i and the other shareholders $\Gamma^{-}(j) \backslash \{i\}$

Figure 2: Indirect control and an ultimate owner



4.3 Indirect control

Controlled firms may and in many real-world cases indeed control other firms as dominant corporate shareholders. We provide an extension to Definition 4.1 of direct control, by using transitivity, in which a shareholder *indirectly controls* a firm through a "chain of direct control relations".

Definition 4.2 *For a shareholder/firm $j_1 \in V$ and a set of distinct firms $\{j_2, \ldots, j_m\} \subseteq V$, if for all $k := 1, \ldots, m-1$ j_k directly controls j_{k+1}, then we say that j_1* indirectly controls *j_k for all $k := 2, \ldots, m$.*

We shall call the set $P(j_1, j_m) := (j_1 j_2, \ldots, j_{m-1} j_m) \subseteq V \times V$ the *control path* from j_1 to j_m. A shareholder that indirectly controls every firm along a control path but itself is not controlled is defined as the *ultimate owner* of each of its indirectly controlled firms.

Definition 4.3 *Given a firm $j \in V$ and a shareholder $u \in V$, we say that u is the* ultimate owner *of j if the following two conditions hold:*

1. *u controls j (directly or indirectly); and*
2. *u is not controlled*

Since a controller of a firm is uniquely determined, a firm can have one and only one ultimate owner.

The ownership structure from Figure 2 is represented by the following adjacency matrix:

$$H := \begin{pmatrix} 0 & 100 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 80 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In Figure 2 we recognize the set of firms to be $V \backslash F = \{j2, j3, j5\}$ and assume that the majority quota $q_j = 50\%$ for every firm in $V \backslash F$, which means that a simple majority is required to win a vote. Furthermore, we will use the Shapley-Shubik power index (SS) and fix the power threshold for control $\theta = 0.9$. Family $j1$ controls firm $j2$, since this family holds 100% of the voting rights of firm $j2$. Next we will observe firm $j5$; it has three direct shareholders $\Gamma^-(j5) = \{j2, j3, j4\}$, whose voting rights are taken from $h_{\bullet,5}$[20] and are represented by the voting rights vector $(80, 10, 10)$, together with $q_{j5} = 50\%$ the game g_{j5} is completely determined. Calculating the Shapley-Shubik power index of the shareholders in game g_{j5} yields the corresponding power index vector $(1, 0, 0)$, and since $Sh(j2) = 1 > \theta = 0.9$ we identify the corporate shareholder $j2$ as the direct controller of firm $j5$. In sum, since $j1$ directly controls $j2$ and since $j2$ directly controls $j5$, then according to Definition 4.2 of indirect control we will say that $j1$ indirectly controls $j5$; the control path from $j1$ to $j5$ is given by $P(j1, j5) = (j1j2, j2j5)$. Moreover, since $j1$ is not controlled by anyone (it is a family), according to Definition 4.3 $j1$ is the ultimate owner of $j2$ and of $j5$.

4.4 concert of shareholders

There are cases where several shareholders of a certain firm are directly or indirectly controlled by the same ultimate owner. We name each such (maximal) subset *a concert of shareholders*[21]. In the next section we will impose an assumption about the voting patterns of concerts of shareholders. The following example illustrates this case.

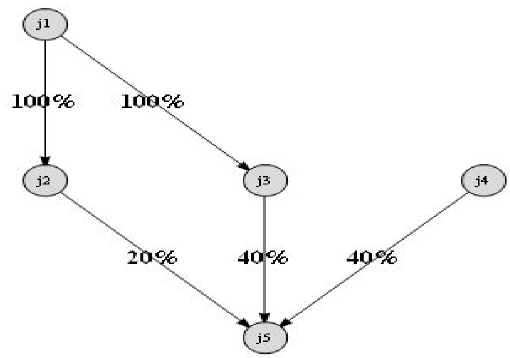
In Figure 3 assume as in Figure 2 a majority quota $q_j = 50\%$ for every firm $j \in V \backslash F$, and a Shapley-Shubik power index with a fixed power threshold for control $\theta = 0.9$. Using similar argumentation as in the analysis of Figure 2, we can say that $j1$ is the ultimate owner of firm $j2$ and of firm $j3$. When we observe firm $j5$'s set of shareholders $\Gamma^-(j5) = \{j2, j3, j4\}$, we recognize that $\{j2, j3\}$ is a concert of shareholders, since the two firms have the same ultimate owner ($j1$) and since it is a maximal set, as no other shareholder of $j5$ has $j1$ as its ultimate owner.

Definition 4.4 *Let C be a subset of shareholders in some firm j. We say that C is a* concert of shareholders *if the following three conditions hold:*

[20] The column of H representing the voting rights of the shareholders of firm $j5$. An entry of 0 in column $j5$ means that the shareholder in that particular row has no direct holdings in $j5$, e.g. $j1$.

[21] Inspired by the definition in the City Code on Takeovers and Mergers (the Code), persons acting in concert are persons who, pursuant to an agreement or understanding (whether formal or informal), co-operate to obtain or consolidate control of a company or to frustrate the successful outcome of an offer for a company.

Figure 3: Concert of Shareholders



1. *There exists one and only one vertex u, where u is the ultimate owner of each member c_i of C.*

2. *For each $c_i \in C$, there exists a control path $P(u, c_i)$ from u to c_i such that $P(u, c_i)$ does not contain the firm j.*

3. *C is a maximal set for which conditions* (1.) *and* (2.) *hold.*

The set of shareholders of a certain firm may contain several concerts of shareholders. However, given the uniqueness of control relations and of the ultimate owner, these concerts must be disjoint sets.

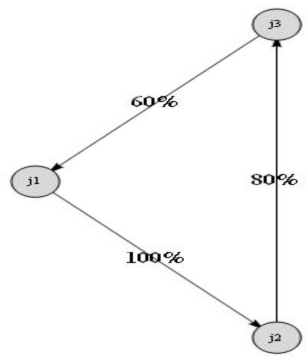
Remark: We can weaken condition (1.) in Definition 4.4 by requiring that u only indirectly control each member c_i of C, i.e. u is not required to be an ultimate owner, and thus can be controlled. In this case we say that C is a *weak concert of shareholders*.

4.5 Cycles

Real corporate networks may contain cross shareholdings (two companies that posses a stake of each other's shares), or in the general case be cyclic with respect to shareholdings. In some cases the shareholdings among the firms in a cycle are significant enough to be considered control relations by Definition 4.1. This setting might give rise to a *simple closed control cycle* of firms having direct control relations, where none of the firms contained in the closed cycle is controlled directly or indirectly by a shareholder outside the cycle. Formally we define:

Definition 4.5 *A* simple closed control cycle *is a set of firms $\{j_1, \ldots, j_n\} \subseteq V \backslash F$ for which the following conditions hold:*

Figure 4: A closed control cycle



1. *For all* $k, l := 1, \ldots, m$*,* j_k *indirectly controls* j_l

2. *For all* $k := 1, \ldots, m$*,* j_k *has* no *ultimate owner.*

It is clear from Figure 4 that firm $j1$ controls firm $j2$, which controls firm $j3$, and that $j3$ in turn controls $j1$. The presence of closed control cycles creates an ambiguity about the actual power of a corporate shareholder $j1$ to dominate votes in a firm $j2$ that it controls, if in addition $j2$ controls $j1$ directly or indirectly. We address this issue in the next section by imposing an assumption about the validity of control in such cases.

5 Assumptions

So far we have introduced a single-firm-level mechanism for identifying control relations and provided auxiliary definitions for indirect control, control path, concert of shareholders and closed control cycle. In this section we shift our attention to the corporate-network level of analysis and extend our mechanism to complex ownership structures. Our approach relies on the following five central assumptions:

Assumption 5.1 *Control relations are effectively transitive.*

We assume that indirect control relations effectively function as direct control relations. That is, if a shareholder is powerful enough to control a certain firm, and that firm in turn has direct control over another firm, we assume that the controller of former has the equivalent ability to realize its own will in any decision-making process in either firm. This concept of transitivity of control is assumed to extend further along possible control paths, such that the ultimate owner has the dominant say in major issues under vote in each firm that it indirectly controls.

Assumption 5.2 *The unobserved minority shareholders do not participate in votes.*

Specifically, we assume that the total unobserved voting rights are dispersed over an infinitely large number of 'minor' owners. Then, following the argument of Grossman and Hart [37], we postulate that shareholders with a very small stake would in most cases abstain from participating in voting and company meetings, since the social benefit to monitoring by participation is far larger than the private benefit to any individual. In other words, because of the 'free rider' problem not all shareholders will participate in the process of decision-making - it is thus mainly a contest between the large shareholders. Therefore, we normalize to 1 the fraction of total observed holdings in every firm $j \in V \backslash F$.Consequently, if firm j is not a family, then its *observed* shareholders are assumed to hold together the entire stock of j.[22]

Assumption 5.3 *Shareholders within the same concert vote in the same direction.*

The meaning of this assumption is that we impose a departure from the equal probability assumption for coalitions in which some shareholders in a firm are controlled by the same ultimate owner. In other words, a pre-coalition structure is assumed to exist in this case with a priori unions of shareholders, where each such union votes as one bloc. We assume that the obligation to vote in a certain direction is dictated by the ultimate owner. Thus, if a certain corporate shareholder is a member of a concert, a deviation from the concert into another coalition is likely to be correlated with severe negative payoffs. The concept of a priori unions or pre-coalitions implies that certain coalitions will not form at all, i.e. will have a probability

[22]It is possible to incorporate models for the voting patterns of the unobserved dispersed shareholders instead of normalizing (see [44, 11]).

of zero of forming (because shareholders that are controlled by the same ultimate owner cannot appear in separate coalitions). However, an underlying assumption for the indices we use to measure the power of an agent presupposes equal probability for each coalition. In order to avoid deviating from this independence assumption regarding power indices we will refer to such blocs of shareholders (concerts) as one voter, i.e., a bloc whose weight is equal to the sum of the weights of its members. Thus, for each bloc we will calculate the power index of the entire bloc rather than the individual index of each member [5, 4, 20].

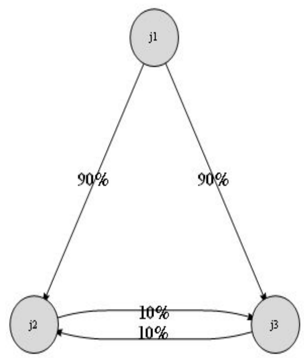
Assumption 5.4 *Shareholders outside concerts and different concerts vote independently.*

The only systematic pre-determined coalition structure in our model is the one derived from control relations, i.e., the concert of corporate shareholders (in which shareholders controlled by the same ultimate owner vote in the same direction). In our technical analysis we will not consider non-systematic ad-hoc unions or alliances based on incidental or temporary convergence of interests. Such non-systematic occurrences are random, and consequently any further correlation in the pattern of voting by shareholders requires imposing extra assumptions and restrictions on the distribution of the coalition's formation. In cases where voting behavior is not dictated by an external authority, and since interests are usually hidden, not measurable, and diverge and converge randomly, extra assumptions about the temporary voting tendencies of shareholders are extremely difficult to justify without further data about the preferences of these shareholders. Therefore, we shall adopt the independence assumption regarding the voting patterns outside concerts of shareholders.

Assumption 5.5 *Control relations in a "closed control cycle" cancel each other.*

Closed control cycles raise a difficult issue in analyzing control relations, since they create ambiguity about the corporate shareholders' power to dominate votes in firms belonging to the same cycle. We assume that the closed control cycle structure neutralizes the effects of the direct control relations composing the cycle. Otherwise, if we take j_1 and j_2 to be corporate shareholders such that j_1 indirectly (directly) controls j_2 and vice versa, then regarding the relations between these two firms as control relations according to our conceptual framework would be counterintuitive and ambiguous, especially when the interests of j_1 and j_2 diverge. Therefore, we will not consider the control relations of closed control cycles of firms in our results. The exception is the case where at least two firms within a control cycle are directly controlled by the same ultimate owner. Since this ultimate owner is not subordinated to any of the cycle firms, it can thus impose decisions through direct and indirect control relations. This case can reflect the interpretation that companies in dominated cross-ownership structures are linked by horizontal cross-holdings of shares that reinforce and entrench the power of a 'central' controller [24], which in our case is the ultimate owner.

Figure 5: A control cycle with an ultimate owner



In Figure 5 $j1$ is the ultimate owner of firms $\{j2, j3\}$, $\{j1, j2\}$ is the controlling concert[23] of shareholders in firm $j3$, and $\{j1, j3\}$ is the controlling concert of shareholders in firm $j2$.

[23] For a detailed discussion of control by a concert, see Section 6.1.

6 Extending the definitions by using concerts of shareholders

In this section we refine our formal model to accommodate control by a concert of shareholders. We apply the assumptions stated in the previous section to control by a concert of shareholders in order to provide extended definitions for indirect control, control path and closed control cycle. These definitions will become useful in Section 7 when we present our method.

6.1 Control by a concert of shareholders

We stated in Assumption 5.3 that concerts of shareholders vote together and that we regard them as one bloc when measuring the power of those shareholders. A firm can be controlled by a concert of shareholders, if the power index of the concert (bloc) is at least as large as the control threshold θ. Since the concert must be controlled by an ultimate owner, by the transitivity of control relations stated in Assumption 5.1, it is effectively this ultimate owner that dominates votes in the controlled firm through the concert. In this case we say that the firm is *controlled by a concert*; each member of the concert will be called a *controlling concert member*. The definition of indirect control extends accordingly, i.e., if a firm $j1$ controls each member of a concert $C \subseteq V \backslash F$, and another firm $j2$ is controlled by that concert C, then we say that $j1$ indirectly controls $j2$ (through a concert). Consequently, *the ultimate owner of the controlling concert is the ultimate owner of the firm controlled by that concert.* For illustration, in Figure 3 concert $j2, j3$ controls the firm $j5$, and $j1$ is the ultimate owner of $j5$. This observation allows us to extend the definitions of the simple control path and of the simple control cycle to accommodate control by a concert.

Definition 6.1 *Let $\{j_1, \ldots, j_m\} \subseteq V$ be a set of m shareholders/firms. A* compounded control path *from j_1 to j_m, is an acyclic sub-graph, denoted by $CP(j_1, j_m) \subseteq V \times V$ such that $CP(j_1, j_m) := \{j_i j_k | j_i\}$ is a controlling concert member of $\{j_k, i\}_{k \leq m}$;*

Moreover, j_1 indirectly controls (through a concert) j_i for all $i = 2, \ldots, m$ and j_m is indirectly controlled (through a concert) by all $\{j_k\}_{k=1,\ldots,m-1}$.

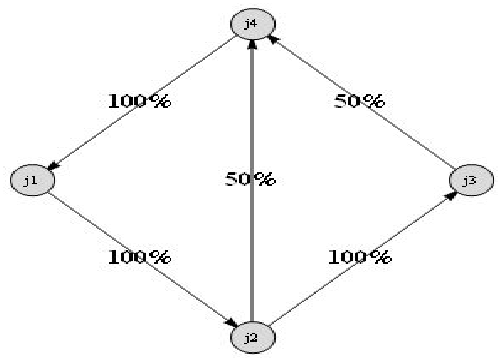
The definition above covers the simple case of a control path (see Definition 4.2) as well, since a concert of shareholders may contain a single member. The same argumentation is applied in order to extend simple closed control cycles. We use the variation of a weak concert of shareholders presented at the end of Section 4.4.

Definition 6.2 *A* compounded closed control cycle *is a set of firms $\{j_1, \ldots, j_m\} \subseteq V \backslash F$ for which the following conditions hold:*

1. *For all $i, k := 1, \ldots, m$, j_i indirectly controls j_k (through a* weak *concert of shareholders).*
2. *For all $i := 1, \ldots, m$, j_i has* no *ultimate owner.*

Hereafter, when we mention control paths and closed cycles we refer to the extended definitions.

Figure 6: A compounded closed control cycle

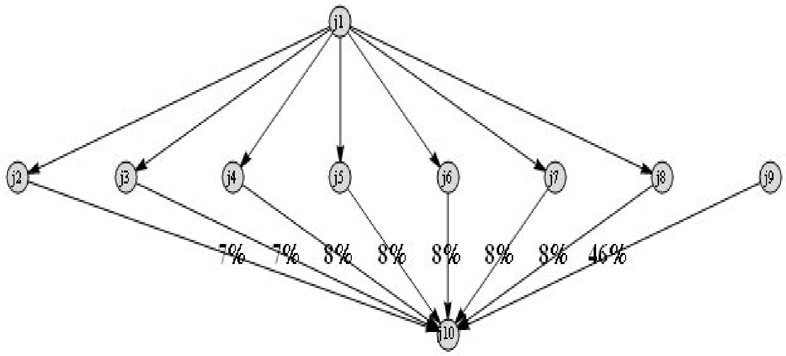


6.2 Changes in the control threshold

Incorporating the analysis of the single corporation into the network context depends on the complete distribution of ownership within the corporate network. There are cases in which a concert controls a firm for some control threshold θ_{High}, but does not control the same firm for a lower threshold θ_{Low}. This is possible since a low enough control threshold like θ_{Low} may induce the creation of other concerts of shareholders whose voting rights are sufficiently large to rival the original concert (that controlled the firm for the higher threshold condition θ_{High}) and thus to eliminate its control or even to become the controller instead.

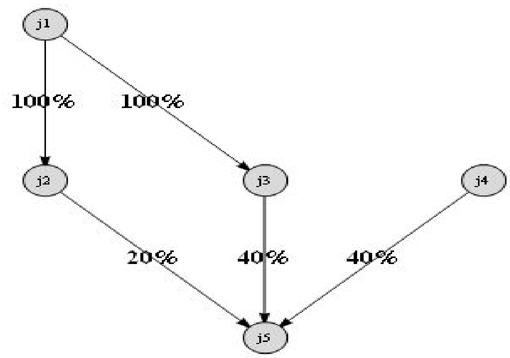
We focus on the extraction of a corporate network presented in Figure 7, where $\{j2, j3, j4, j5, j6, j7, j8, j10\}$ are firms and $\{j1, j9\}$ are families. Assume that for some control threshold $\theta_{Low} = 0.6$ we identify family $j1$ as the controller of all the firms $\{j2, j3, j4, j5, j6, j7, j8\}$ which thus constitute a concert of shareholders C in firm $j10$. Since the concert C holds together 54% of the voting rights, it has a power index of 1 which is greater than θ_{Low} in firm $j10$. Therefore, this concert controls the firm $j10$ and it follows that in this case $j1$ is the ultimate owner of firm $j10$. Let us now choose a higher control threshold $\theta_{High} = 0.7$ (such that $\theta_{High} = 0.7 > \theta_{Low} = 0.6$) and such that the power index of family $j1$ would be too small to control any of the firms $\{j2, j3, j4, j5, j6, j7, j8\}$. Those firms will no longer constitute the concert C in $j10$ as before and thus the voting distribution within the firm $j10$ is different from the case of θ_{Low} by being more scattered (since the concert C is eliminated) and thus presenting a weaker opposition to family $j9$. Given this more scattered distribution, family $j9$ has a power index of 0.75 in firm $j10$ which is greater than $\theta_{High} = 0.7$, thus it would be identified as the controller (and ultimate owner) of firm $j10$ as opposed to the case of θ_{Low}. The effect of a control threshold change on the control status of a firm, when taking the corporate network into consideration, is more complex to analyze than the case of a single firm and its shareholders when neglecting the network

Figure 7: Extraction of a corporate network to illustrate the network effect of a change in the control threshold



structure. This is because changing the control threshold involves not only the direct effect - the control threshold of the voting game within a specific firm, but also the indirect network effect - through voting games in other firms and their influence on the creation of concerts of shareholders in that firm. In other words, the original voting game in a firm j, $g_j\,(W_1, q, \theta_1)$, changes into the voting game $g_j\,(W_2, q, \theta_2)$, where $W_i, i = 1, 2$, is the set of weights of the direct concerts of shareholders, and q is the majority quota in this firm. The direct effect is due to the change of θ_1 into θ_2, whereas the indirect effect is represented by the change of W_1 into W_2. Following the example of Figure 7 we have $g_j 10\,(W_1, 50\%, 0.6)$ where $W_1 = \{54\%, 46\%\}$, and $g_j 10\,(W_2, 50\%, 0.7)$ where $W_2 = \{7\%, 7\%, 8\%, 8\%, 8\%, 8\%, 8\%, 46\%\}$.

Figure 8: Example of a simple market



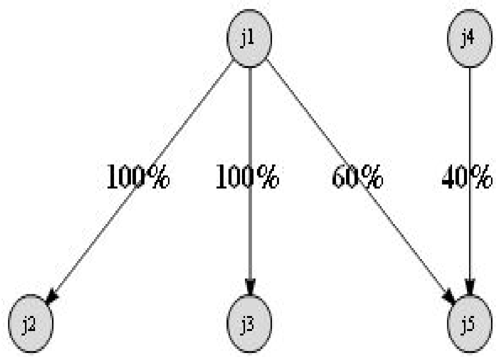
7 The algorithm

We present here a fully integrated algorithm that is able to analyze control relations in any complex ownership structure, and to transform a given input-output matrix of ownership (cash-flow rights) into a corresponding Boolean matrix of control. For this purpose we will use the specific market model from Figure 8 as a visual simplification to accompany an intuitive presentation of our method. For a formal description see Appendix A. The input-output matrix of ownership describing the market in Figure 8 is: $H := \begin{pmatrix} 0 & 100 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Step 1 - Power index test for control: Assume a majority quota $q_j = 50\%$ for every firm $j \in V \backslash F$ and a Shapley-Shubik power index test with $\theta = 0.9$. By considering that shareholders outside concerts and different concerts vote independently, as stated in Assumption 5.4), we calculate the power index[24] of the shareholders in each of the firms $\{j2, j3, j5\}$, which yields the matrix of power P, defined such that $p_{i,j}$ represents the power index of shareholder i in firm j.

[24] Although power analysis has been very restricted due to computational considerations, recent work [43] has provided means to overcome this difficulty using tractable approximating power indices with a very high accuracy. We have used the algorithms suggested in that work to perform our analysis, which is heavily based on power index computation.

Figure 9: Updated network after clustering and replacing concerts of shareholders



$$P := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From column 2 we learn that family $j1$ controls firm $j2$, since $p_{1,2} = 1 > \theta = 0.9$. By the same argumentation, family $j1$ controls firm $j3$. However, since none of the shareholders of firm $j5$ has a power index that is at least as large as $\theta = 0.9$, this company is not controlled.

Thus, the temporary Boolean matrix of control C after step 1 is:

$$C := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

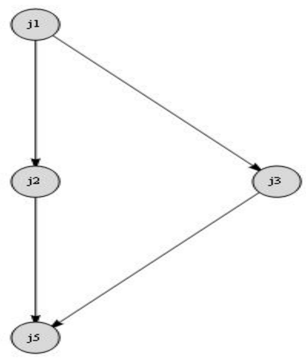
C is defined such that $c_{i,j} = 1$ if and only if shareholder i is a member of a concert that controls the firm j.

Step 2 - Ultimate owners: For each controlled firm we find its ultimate owner by climbing up the (compounded) control paths that lead to the controlled firm. Once the ultimate owner is identified, we update the pair $(Firm, UltimateOwner)$ to an ultimate-owners information set U, defined by $U := \{(j,u) \subseteq V \times V : u \textit{ is the ultimate owner of firm } j\}$. Thus, in our example, $U = \{(j2, j1), (j3, j1)\}$. If a firm is involved in a closed control cycle then it is not added to the information set, since it has no ultimate owner.

Step 3 - Concerts of shareholders:

For the set of shareholders of each firm j, i.e. $\Gamma^{-}(j)$, we use the ultimate-owners information set U to find concerts of shareholders. We then calculate the voting power of each

Figure 10: Identified Control Relations



concert by summing the shareholders' weights, assuming that they vote in the same direction (Assumption 5.3). Furthermore, we replace the concert with its ultimate owner, which effectively controls the concert's decisions, as suggested in Assumption 5.1. In our example, when observing firm $j5$'s set of shareholders $\Gamma^{-}(j5) = \{j2, j3, j4\}$ as well as the set $U = \{(j2, j1), (j3, j1)\}$ we can qualify $\{j2, j3\}$ as a concert of shareholders. Summing up the voting rights of $j2$ and $j3$ and replacing them with $j1$ yields the updated network represented in Figure 9 and in the corresponding updated matrix $H^{updated}$.

$$H^{updated} := \begin{pmatrix} 0 & 100 & 100 & 0 & 60 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We then carry out a second iteration, repeating steps 1 and 2, which results in:

$$P := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$U = \{(j2, j1), (j3, j1), (j5, j1)\}$

Repeating step 3 in order to update the ownership matrix for concerts will not change

$H^{updated}$, since the concerts' structure is the same as before. Therefore, P, C and U will also not change, since they are derived from $H^{updated}$. This outcome will thus remain fixed for all subsequent iterations, and therefore the algorithm has converged into a solution. The control relations in the market in our example are represented by Figure 10 and by the corresponding Boolean matrix of control B[25],[26].

$$B := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The algorithm is not additive in the sense that during the iterative process the number of "adjusted" corporate shareholders in the updated matrix of ownership does not necessarily diminish as a result of clustering firms into concerts of shareholders and replacing them with their corresponding ultimate owners. An intuitive reason is that that it might take several iterations until a concert is identified, and a concert that, in early iterations, appears to control a firm might "lose" its control in a later iteration. This might occur because of the formation of another rival concert of shareholders. Furthermore, losing control over a certain firm might cause other concerts to break, form, or lose or gain control in other firms that are connected to the original firm throughout the network. However, the total number of identified control relations in each iteration cannot be smaller than the number of that iteration. Since the number of shareholders within the network is finite (which is the case in real-world situations) and thus the number of possible control relations is bounded, our algorithm always converges to a solution that can be obtained after a finite number of iterations. Under the assumptions stated in section 5, we formulate the theoretical principle of our method as follows: Let the network of shareholding relationships among a set V of n firms and shareholders be described by the $n \times n$ adjacency matrix $H \in [0,1]^{n \times n}$ as defined in Section 4.1. Given a majority quota $q \in (0,1]$ and power index threshold $\theta \in (0.5,1]$, denote by Ψ a mapping $\Psi : [0,1]^{n\times n} \times (0,1] \times (0.5,1] \to \{0,1\}^{n\times n}$: $(H,q,\theta) \mapsto \Psi(H,q,\theta) := F$ from the initial input-output matrix of holdings H to a sub-network of control relations F described by the $n \times n$ adjacency matrix $F \in \{0,1\}^{n\times n}$. The set $V_C \subseteq V$ is the set of firms and shareholders participating in control relations. If a firm is not controlling or controlled, then it is represented as a disconnected vertex, i.e. for all $s \in V \backslash V_C$ we have $F_{sj} = 0$ and $F_{js} = 0$, $j = 1,\ldots,n$. F satisfies the following: A firm j is controlled by shareholder i if and only if i is a member of a concert of shareholders controlling firm j. Formally, $F_{ij} = 1$ if and only if there exists $B := \Gamma^-_{V_C}(j) \subseteq \Gamma^-_V(j)$ - a concert of shareholders (possibly with only one shareholder) in firm j such that $PowerIndex_j(B) > \theta$ and $i \in B$. In other words, the power index of the concert B to which shareholder i is associated, when calculated as one bloc in the game g_j with majority quota q, is greater than θ. The control matrix F, which represents all

[25]A firm/shareholder that does not participate in control relations (family $j4$ in this example) will have all entries in its corresponding row and column equal zero, and will not be included in the solution.

[26]Notice that $h_{2,5} = 1$ and also $h_{3,5} = 1$, which means that firm $j5$ is controlled by a concert.

control relations, is a directed network where each edge links each shareholder of a controlling concert to the controlled firm in a direct control relation. This network is composed of disjoint maximal connected components. Each maximal connected component is arranged as a directed tree with a specific shareholder as the root (the ultimate owner).

Proposition 7.1
Given the ownership matrix H,a majority quota q and a power index threshold θ, as defined above, then the algorithm proposed in this section and formally in Appendix A:

1. *Converges to a solution (control network) after a finite number of iterations.*
2. *This solution is exclusive (or correct) in the sense that every identified control relation between a shareholder (concert) and a firm is valid, and no other shareholder (concert) can be the controller of that firm (it also excludes the possibility that the firm is widely held).*
3. *The solution is exhaustive, in the sense that it identifies all control relations.*

In other words our algorithm accepts H, q and θ as inputs and its output is the control matrix F_{Algo} with the above properties. Thus, the mapping Ψ exists and is well defined, and $\Psi(H, q, \theta) = F$ can be identified with our algorithm's output of control matrix, F_{Algo}. That is, $F = F_{Algo}$.

Proof: In Appendix B.

8 Application and Results

We implemented our algorithm into a Java computer program and tested it on a real-world data set of holdings in the Israeli market between the years 1995 and 2009. In this section we provide several results based on our algorithm.

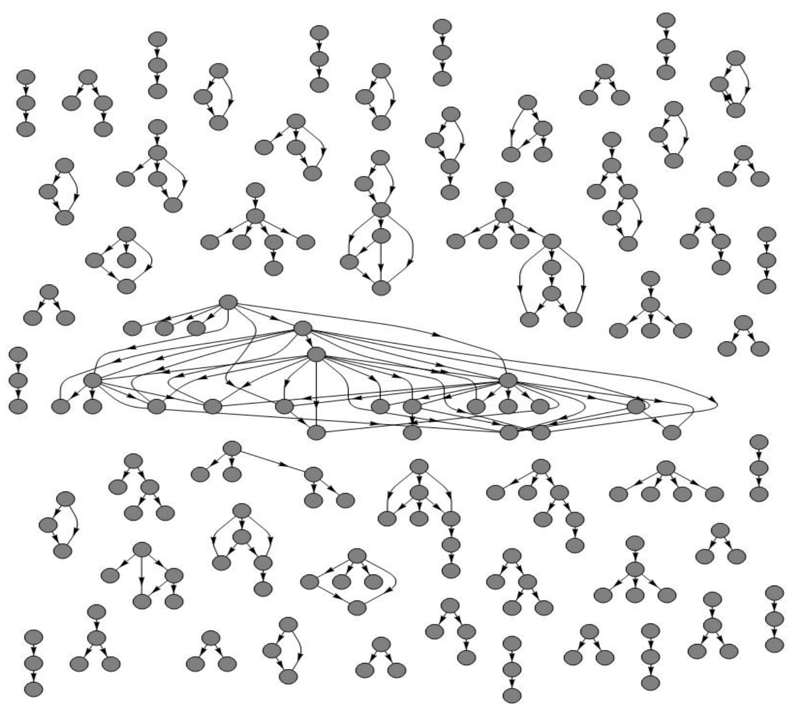
Our source is the ownership data for 650 listed Israeli companies - the whole population of publicly traded firms on the Tel Aviv Stock Exchange. Our sample comes from the Tel Aviv Stock Exchange electronic reports on ownership distribution in each listed company, and specifically the data regarding the percentage holdings of the principal shareholders (parties of interest with a holding over 5 percent). At the first stage we identified on average 2300 owners (i.e., potential controllers) per month. These owners are heterogeneous, and include local individuals (55%), private companies (21%), foreign owners (10%), publicly traded companies (8%) and trusts and partnerships (6%). Since the available official information does not include holdings in private companies we added data on ultimate owners of these companies[27], which we collected manually from the Registrar of Companies' database in the Ministry of Justice. In cases where identification was successful, we re-attributed the shareholdings held by private firms to their ultimate owners. If we could not identify the ultimate owner, then the private companies' shareholdings remained unchanged. Next, we examined the extended set of all individual shareholders and clustered them, when possible, into groups defined by family ties. We assumed that every family votes and owns its shares collectively SAF2. The adjusted set of corporate owners comprised 2700 parties of interest on average for each month. According to assumption 1 the unobserved minority shareholders do not participate in votes; therefore, we normalized the total shareholdings of parties of interest in each firm to 100 percent[28].

8.1 Controlled public firms for different control thresholds

We examined the percentage of controlled public firms out of the total number of public firms for each month between April 1995 and April 2009. We ran the algorithm four times, with control thresholds of $\theta = 1$ and $\theta = 0.5 + \epsilon$, the highest and lowest boundaries for the control threshold respectively: two runs with the Shapley-Shubik power index (denoted SS in the figure); as well as two runs with the Banzhaf power index (denoted B in the figure). Figure 11 summarizes the results of those runs across consecutive time points.

Figure 11: Percentage of controlled public firms for different control thresholds

Figure 12: Ultimate owners and controlled public firms in maximal control components containing at least two public firms in the Israeli market for April 2009



8.2 Maximal control components containing at least two public firms in the Israeli market for April 2009.

In Figure 12 the vertices represent firms and shareholders, and the edges represent control relations[29]. Shareholders within the same tree that are located above a firm and are connected to it by edges are the controlling concert of that firm, and the root of the tree is the ultimate owner. Since we identified many firms participating in control relations, for the clarity of illustration, we chose to visualize only ultimate owners and public firms in control components that contained at least two controlled public firms (and thus removed the private firms from the figure).

8.3 Comparison with the cut-off point method

In this section we examine the results of our method vs. the results derived from cut-off point method on a real-world data set of holdings in the Israeli market. For the application of our method, we chose the Shapley-Shubik power index with control threshold parameter $\theta = 0.75$; and for the cut-off method we followed the widely-used 20% rule - i.e. the corporate controller (ultimate owner) was identified if this shareholder's direct and indirect voting rights in the firm exceeded 20 percent [33]. The following table summarizes the comparison of the main results and refers to the control status of listed Israeli companies as of April 2009:

Table 1: Comparison of the empirical results provided by the power indices based method and the 20% cut-off point rule, April 2009.

<table>
<tr><th>Category</th><th></th><th>Algorithm SS 0.75</th><th>20% cut-off</th></tr>
<tr><td>1</td><td>Total number of identified controlled firms</td><td>425</td><td>534</td></tr>
<tr><td>2</td><td>Identified as controlled in ONLY one method</td><td>13</td><td>122</td></tr>
<tr><td>3</td><td>Identified as controlled in both methods</td><td colspan="2">412</td></tr>
<tr><td>4</td><td>- out of which (3) firms with different ultimate owner</td><td colspan="2">31</td></tr>
<tr><td>5</td><td>- Identified with the same controller and ultimate owner</td><td colspan="2">381</td></tr>
</table>

The analysis of the results shows that both methods identify the same controllers and ultimate owners for 381 (category 5) firms. However, for a total of 166 firms (categories 2 and 4) the results are significantly different. Such difference, at the sampled time point (April 2009), accounted for 25% of the total market value[30]. A more detailed analysis of the discrepancy between two methods is provided below. As can be seen from Table 1, the "cut-off method" identifies

[27] This procedure is not essential for using our method. Running the algorithm without incorporating the "adjusted" shareholding data would still yield correct results with respect to the input information.

[28] Precise knowledge or different assumptions regarding the voting patterns of the unobserved minority can be easily incorporated instead of normalizing.

[29] After running the algorithm for $\theta = 0.8$

[30] Not including "Teva".

Figure 13: First example of a weakness of the 20% cut-off-rule in accurately identifying control

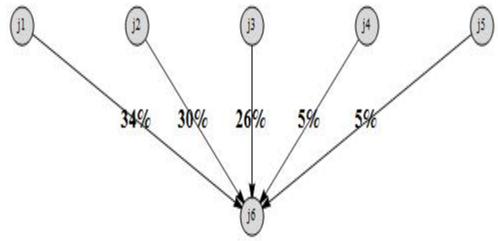
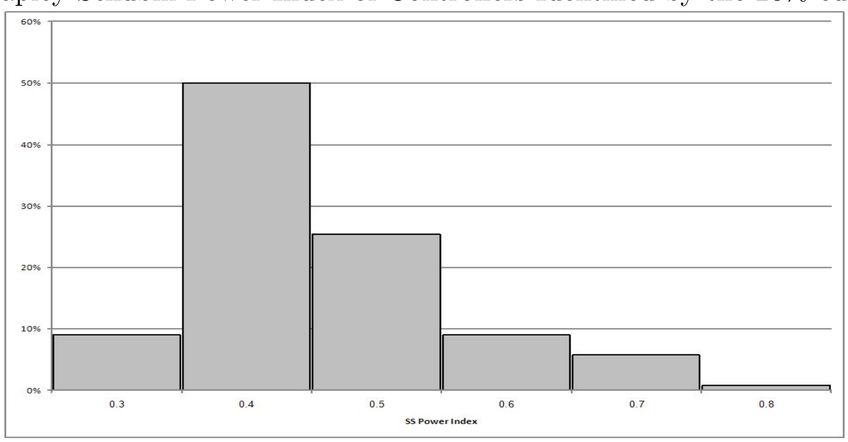


Figure 14: Shapley-Schubik Power Index of Controllers Identified by the 20% cut-off rule only

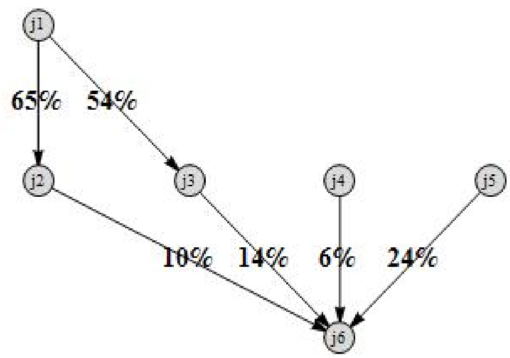


122 controlled firms, or 8.6% in terms of market value, that are considered non-controlled by the method presented in this paper. A careful examination of the ownership structure of these firms reveals that in most cases despite the fact that controller holds a significant stake (more than 20% by construction) there are other shareholders that hold large blocks as well and thus are presenting a potential strong opposition to the controller. In these situations the distribution of power among the rest of the shareholders questions the possibility of the identified controller (by the cut-off method) to effectively exercise control.

Figure 13 shows a true representative case in which the 20% cut-off method identified a controller and our method did not.

The histogram in Figure 14 shows the power index of the controllers in the 122 firms that the cut-off method identifies as controlled, which are considered non-controlled by our method. The most important implication of Figure 14 is that in about 85% of those firms the controller, who was identified (only) by the 20%-rule, has a power index lower than 0.5. In other words, due to the distribution of power in those firms, the assumed controller does not achieve a pivotal

Figure 15: Second example of a weakness of the 20% cut-off-rule in accurately identifying control. The cut-off method identified j5 as the controller and ultimate owner of j6, however since j1 controls the concert j2,j3 they are likely to vote as one block with 24% which matches exactly the stake of j5. Therefore, our method identified firm j6 as not controlled.



position in more than 50% of the cases. Accepting the fact that such a shareholder controls a firm by virtue of voting rights (holdings) is not in accord with our definition of effective control. Moreover, the cut-off method does not examine the entire sub-networks in which firms are embedded, and thus misses the fact that some non-controlling shareholders may form concerts and thus vote in the same direction. This leads again to a different distribution of power and to a greater potential opposition to the shareholder who was identified as a controller.

Figure 15 shows a true representative case in which the 20% cut-off method identified a controller and our method did not.

Regarding the 13 companies which are identified as controlled by using our method and as widely-held by the cut-off rule, we find that the total market value of these firms exceeds 6.3% - this in comparison with 8.6% of 122 companies identified as controlled by cut-off method only - and that their market activity is concentrated primarily in banking, computers and electronic sectors. A fascinating observation, however, is that the list of these companies includes one of the two biggest Israeli banks - Bank "Hapoalim". In this case as well as in the rest of the cases in which our method uniquely identifies corporate controllers and the cut-off method does not, the controller holds less than 20% of the corporate voting rights but, in line with the proposed methodology, is powerful enough to obtain effective control of the firm.

The last bone of contention between two methods is the discrepancy about the identity of ultimate owners. We find that among 31 firms (10.25% of total market value) four relatively big companies (7.23% of total market value) are affiliated with 3 different business groups, while in 27 other cases the ultimate owners are not the same. Besides the numerical differences provided in this section, the implications of choosing the identification strategy is of concern.

The discrepancy about the controller's identity between the two methods [31] in our particular example raises concerns about the robustness of empirical analysis related to corporate governance in other cases. Thus, for example, identifying the largest blockholder as a controller in the presence of other large shareholders is questionable. The phenomena of multiple dominant shareholders which is well documented in different studies [33, 25] and appears also in 122 companies in our example above has different implications than the existence of a single ultimate owner. These implications are linked strongly to shareholders power, corporate performance, valuation and governance issues in companies with concentrated ownership [27, 9, 29, 3] In such cases, the misspecification of the corporate control structure, while using the cut-off method, can lead, at best, to biased or insignificant empirical results. In other scenarios, however, an inappropriate identification strategy can result in mechanisms that are simply inadequate for the challenges facing policy makers while coping with the consequences of principal-agent problems, or even counterproductive. Despite the fact that by using the cut-off method one can simplify the calculation process of identifying control relations and especially those in complex ownership structures, adopting more accurate and methodologically based approach - as provided in this paper - seems to significantly reduce many potential sources of bias and to avoid serious measurement errors.

[31] In the Israeli case we find disagreement between the two methods for 166, or 25% from the total population of listed companies and 25% of total market capitalization.

9 Summary and conclusions

In the present study we provide a new method for identifying control relations on the corporate level and especially in complex ownership structures. We argue that a complete identification of such relations is possible only by incorporating the analysis of the single corporation into the network context and vice versa. In other words, in order to obtain the exact and comprehensive map of corporate control in any given market, the aspects of control of a particular firm and the control paths that involve that firm must be treated simultaneously. For this purpose we apply weighted voting games as a theoretical framework for our analysis and use the Shapley-Shubik and the Banzhaf power indices to determine the voting power of shareholders. Given the exact distribution of voting rights in a specific firm we define the direct controller as the shareholder who is pivotal in most decision making processes. Subsequently, from our definition of direct control we derive the following essential concepts: indirect control, ultimate owner, concert of shareholders and control cycle. These concepts allow us to shift our focus to the corporate-network level of analysis and extend our mechanism to complex ownership structures. Our approach for the identification strategy relies on five central assumptions: 1) control relations are effectively transitive; 2) the unobserved minority shareholders do not participate in votes; 3) shareholders within the same concert vote in the same direction; 4) shareholders outside concerts and different concerts vote independently; and 5) control relations in a "closed control cycle" cancel each other out. We integrate our analytical framework and assumptions to present an algorithm that is able to analyze control relations in any complex ownership structure, and prove that if the number of firms within the network is finite (which is the case in real-world situations) then there always exists a solution that can be obtained after a finite number of iterations, and it is unique. Thus, our algorithm is well defined. In the next stage we implement our algorithm into a Java computer program and test it on a real-world data set of monthly holdings in the Israeli market between the years 1995 and 2009. For the empirical compatibility test we use an input-output matrix of cash flow rights of approximately 650 listed companies - the whole population of public companies on the Tel Aviv Stock Exchange. The direct product of the analysis of these data is a comprehensive map of control relations at every time point. Moreover, we find that the corporate control relations identified by our method are richer and more accurate than those provided by different official sources (e.g. Israel Securities Authority, Banking Supervision department in the Bank of Israel, etc.).

References

[1] Berle Adolf A. and Gardiner Means. *The modern corporation and private property*. Harcourt Brace and World, New York,NY, 1932.

[2] Dahl Robert A. The concept of power. *Behavioral Science*, 20:201–215, 1957.

[3] Edmans Alex and Gustavo Manso. Governance through trading and intervention: A theory of multiple blockholders. *Review of Financial Studies, forthcoming*, 2010.

[4] Alonso-Meijide, Jos M., and Carlos Bowles. Modification of the banzhaf value for games with a coalition structure. *Annals OR*, 109(1-4):213–227, 2002.

[5] Alonso-Meijide, Jos M., and Carlos Bowles. Generating functions for coalitional power indices: An application to the imf. *Annals OR*, 137(1):21–44, 2005.

[6] Laruelle Annick and Federico Valenciano. Shapley-shubik and banzhaf indices revisited. *Mathematics of Operations Research*, 126(1):89104, 2001.

[7] Chapelle Ariane and Ariane Szafarz. Controlling firms through the majority voting rule. *Physica A*, 335:509–529, 2005.

[8] Peleg B. Coalition formation in simple games with dominant players. *International Journal of Game Theory*, 10(1):11–33, 1981.

[9] Maury C. Benjamin and Anete Pajuste. Multiple large shareholders and firm value. *Journal of Banking and Finance*, 29:1813–1834, 2005.

[10] Jensen Michael C. and William H. Meckling. Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics*, 3(4):305–360, 1976.

[11] Leech Dennis. Corporate ownership and control: A new look at the evidence of berle and means. *Oxford Economic Papers*, 39(3):534–551, 1987.

[12] Leech Dennis. The relationship between shareholding concentration and shareholder voting power in british companies: a study of the application of power indices for simple games. *Management Science*, 34(4):509–527, 1988.

[13] Leech Dennis. An empirical comparison of the performance of classical power indices. *Political Studies*, 50(1):1–22, 2002.

[14] Banzhaf John F. Weighted voting doesnt work: A mathematical analysis. *Rutgers Law Review*, 19:317343, 1965.

[15] Dan S Felsenthal and Moshe Machover. *The measurement of voting power: Theory and practice, problems and paradoxes.* Edward Elgar, Northampton, MA, 1998.

[16] Demsetz Harold and Kenneth Lehn. The structure of corporate ownership: Causes and consequences. *Journal of Political Economy*, 93(6):1155–1177, 1985.

[17] Hauswald Robert B.H.and Ulrich Hege. Ownership and control in joint ventures: Theory and evidence. *Journal of Finance*, 11:42–64, 1980.

[18] Almeida Heitor, Sang Yong Park, Marti G. Subrahmanyam, and Daniel Wolfenzon. The structure and formation of business groups: Evidence from korean chaebols. *Journal of Financial Economics*, 99:447–475, 2010.

[19] Almeida Heitor and Daniel Wolfenzon. Should business groups be dismantled? the equilibrium costs of efficient internal capital markets. *Journal of Financial Economics*, 79:99–144, 2006.

[20] Holler Manfred J. and Stefan Napel. Local monotonicity of power: Axiom or just a property. *Quality and Quantity*, 38:637–647, 2004.

[21] Glattfelder James. *Ownership networks and Corporate Control: Mapping Economic Power in a Globalized World.* PhD thesis, ETH, Zurich, Swiss, 2010.

[22] Cubbin John and Dennis Leech. The effect of shareholder dispersion on the degree of control in british companies: theory and measurement. *Economic Journal*, 93:351–369, 1983.

[23] Morck Randall K., Daniel Wolfenzon, and Bernard Yeung. Corporate governance, economic entrenchment, and growth. *Journal of Economic Literature*, 43:657–722, 2005.

[24] Bebchuk Lucian A.and Reiner H. Kraakman and George G. Triantis. Stock pyramids, cross-ownership, and dual class equity. In Randall K. Morck, editor, *Concentrated Corporate Ownership.* University of Chicago Press, Chicago, Ill, 2000.

[25] Laeven Luc and Ross Levine. Complex ownership structures and corporate valuations. *The Review of Financial Studies*, 21(2):579–604, 2008.

[26] Faccio Mara and Larry H. P. Lang. The ultimate ownership of western european corporations. *Journal of Financial Economics*, 65:365–395, 2002.

[27] Bennedsen Martin and Daniel Wolfenzon. The balance of power in closely held corporations. *Journal of Financial Economics*, 58:113–139, 2000.

[28] Zeitlin Maurice. Corporate ownership and control: The large corporation and the capitalist class. *The American Journal of Sociology*, 79(5):1073–1119, 1974.

[29] Attig Najah, Omrane Gudhemai, and Dev Mishra. Multiple large shareholders, control contests, and implied cost of equity. *Journal of Corporate Finance*, 14:721–737, 2008.

[30] Baron David P. and John A. Ferejohn. Bargaining in legislatures. *The American Political Science Review*, 83(4):1181–1206, 1989.

[31] Dubey Pradeep and Lloyd S. Shapley. Mathematical properties of the banzhaf power index. *Mathematics of Operations Research*, 4(2):99–131, 1979.

[32] La Porta Rafael, Florencio Lopez de Silanes, and Andrei Shleifer. Law and finance. *Journal of Political Economy*, 106:1113–1155, 1998.

[33] La Porta Rafael, Florencio Lopez de Silanes, and Andrei Shleifer. Corporate ownership around the world. *Journal of Finance*, 54:471–517, 1999.

[34] Felsenthal Dan S. and Mosh Machover. A priori voting power: What is it all about? *Political Studies Review*, 2:1–23, 2004.

[35] Penrose Lionel S. The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109(1):5357, 1946.

[36] Shapley Lloyd S. and Martin Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48(3):787792, 1954.

[37] Grossman Sanford and Oliver Hart. Takeover bids, the free-rider problem, and the theory of the corporation. *Bell Journal of Economics*, 11:42–64, 1980.

[38] Edwards Jeremy S.S. and Alfons J. Weichenrieder. Control rights, pyramids, and the measurement of ownership concentration. *Journal of Economic Behavior and Organization*, 72:489–508, 2009.

[39] Claessens Stijn, Simeon Djankov, and Larry H.P. Lang. The separation of ownership and control in east asian corporations. *Journal of Financial Economics*, 58(1-2):81–112, 2000.

[40] Max Weber. *Economy and Society: An Outline of Interpretative Sociology*. Bedminster, New York, NY, 1968.

[41] Deng Xiaotie and Christos H. Papadimitriou. On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19(2):257266, 1994.

[42] Matsui Yasuko and Tomomi Matsui. Np-completeness for calculating power indices of weighted majority games. *Theoretical Computer Science*, 263(1-2):305310, 2001.

[43] Bachrach Yoram, Evangelos Markakis, Ariel D. Procaccia, Ezra Resnick, Jeffrey S. Rosenschein, and Amin Saberi. Approximating power indices: Theoretical and empirical analysis. *Journal of Autonomous Agents and Multi-Agent Systems*, 20(2):105–122, 2010.

[44] Crama Yves and Luc Leruth. Control and voting power in corporate networks: Concepts and computational aspects. *European Journal of Operational Research*, 178:879893, 2007.

A The Algorithm

We present **Algorithm 1**, which is our main algorithm for calculating control graphs. In this algorithm, we maintain a set of pairs, namely $PairsGroup$, in which every pair represents a control relation. We build this set iteratively until it stops changing, and together with the market itself, it represents the control graph in the market. In every iteration the algorithm uses the following methods sequentially (the complexity measures relate to the number of companies in the market):

In Algorithm 2 , for every company, we go over every control component we have identified so far (represented by the root of the control component) and sum its holdings in the company as follows. We sum the holdings of the mothers in the control component, and then replace those mothers by attributing their direct holdings (the summed mothers' holdings) to the root of the control component. This is done in order to convey the coalition's holdings, which are relevant when referring to the power division among the company shareholders. The complexity of this algorithm is $O(N^2)$.

In Algorithm 3 , we simply go over each company, and change the holdings of its current mothers according to the scores given to them by the chosen test. In tests that require an exponential number of calculations, we use an approximation algorithm that calculates the voting power over $\frac{1}{\epsilon^2}log(\frac{1}{\delta})$ samples, where ϵ is the required accuracy, and δ is the required confidence level. We denote this number of samples as k. Therefore, the complexity of this algorithm is $O(Nk)$.

In Algorithm 4 , for each company, we leave only dominating mothers, i.e., we delete the holdings of mothers that are not dominant according to the given test and threshold. We note that there cannot be more than one dominant mother, and therefore if two mothers' holdings are above the control threshold and are the same, both of them will be deleted. The complexity of this algorithm is $O(N^2)$.

In Algorithm 5 , we deal with the special case of control cycles. We do this by going over every root (which inherently cannot be a part of a cycle), and then recursively going through the graph (DFS - Depth-first search), while keeping track of the route throughout the search in order to know when a cycle is reached. We then check if it is a true control cycle, and then unify the new cycle with the other correlated ones. The complexity of this algorithm is $O(N^2)$.

In Algorithm 6 , we extract the new control relations from the process, and create a set of pairs $NextPairGroup$. The complexity of this algorithm is $O(N^2)$.

Eventually, due to the fact that the paths of control, represented through $PairsGroup$, constantly grow until the algorithm stops; we conclude that the main loop in the algorithm runs

$O(N)$ time. Therefore, the complexity of our main algorithm is $O(kN^2)$ for markets with large breadth (i.e. many companies hold every company in the market), and $O(N^3)$ for those with small breadth.

Algorithm 1 Attaches the controlling root to each company

procedure CREATEDOMINATIONTREES($Market$)
    $NextPairsGroup \leftarrow ()$ ▷ Holds pairs of companies and their attached roots so far
    while $NextPairsGroup \neq PairsGroup$ **do**
        $PairsGroup \leftarrow NextPairsGroup$
        $NewMarket \leftarrow ValidCoalitionJoining(Market, PairsGroup)$
        $NewMarket \leftarrow RunTest(NewMarket, TestType, Threshold, Majority)$
        $NewMarket \leftarrow DirectControlRelation(newMarket, Threshold)$
        $NewMarket \leftarrow DealWithCycles(newMarket, PairsGroup)$
        $NextPairsGroup \leftarrow SingleControllerThroughPathes(newMarket)$
    end while
    Return $PairsGroup, Market$
end procedure

Algorithm 2 For every company, removes mothers from the same coalition, and puts an edge from the root with the sum of holdings

```
procedure ValidCoalitionJoining(Market, PairsGroup)
    for comp in companies(NewMarket) do
        Pcomp←Mothers(comp)
        PcompPairsGroup←AttachRoot(Pcomp, PairsGroup)
        for root in Roots(PcompPairsGroup) do
            if root≠comp then
                PcompRoot←sons(PcompPairsGroup, root)    ▷ comp's parents ∩ root's descendants
                sum←Sum the direct holdings of PcompRoot in comp
                in Market, Delete all holdings of companies from PcompRoot in comp
                in Market, Add holdings for root in comp, that equal to sum
            end if
        end for
    end for
    Return Market
end procedure

procedure AttachRoot(CompanySet, PairsGroup)
    NewPairsGroup←()
    for p in CompanySet do
        if there exists Z s.t. (z, p)∈PairsGroup then
            Add (z, p) to NewPairsGroups
        else
            Add (p, p) to NewPairsGroups
        end if
    end for
    return NewPairsGroup
end procedure
```

Algorithm 3 Runs the required voting test on every mother set, and updates their holdings according to their power index

```
procedure RunTest(Market, TestType, Threshold, Majority)
    for company in companies(Market) do
        weights ← MotherHoldings(company)
        votingPower ← test results for running TestType, with weights, Threshold and Majority as input
        in Market, change that holdings of the mothers in company to be their voting power
    end for
    return Market
end procedure
```

Algorithm 4 Deletes relations that do not represent control relations

```
procedure DirectControlRelation(Market, Threshold)
    for company in Companies(Market) do
        for mother in Mothers(company) do
            if Holdings(mother, company) < Threshold then
                in Market, DeleteHoldings(mother, company)    ▷ mother is not controlling company
            end if
        end for
        if NumberOfMothers(company) > 1 then
            maximalMother ← Mothers(company)    ▷ null if no unique maximum exists
            in Market, DeleteMothers(company)
            in Market, returnMother(maximalMother, company)
        end if
    end for
    Return Market
end procedure
```

Algorithm 5 Deals with cycles

```
procedure DealWithCycles(Market, PairsGroup)
    Undominated←getUndominatedCompanies(Market, PairrsGroup)
    for company in Undominated do
        RouteList←()
        recursiveFindCycle(Market, PairsGroup, company, RouteList)
    end for
    CalculateDominationsInCycles(Market)
    Return Market
end procedure

procedure recursiceFindCycle(Market, PairsGroup, company.RouteList)
    if company∈RouteList then                                  ▷ If we closed a cycle
        newCycle←extractCycleFromRoute(List, company)
        if everyone in the cycle has the same root then     ▷ Meaning its a control cycle
            AddToCycles(Market, newCycle)
        end if
    else
        Add company to RouteList
        for daughter in daughters(company) do
            recursiveFindCycle(Market, PairsGroup, daughter, RouteList)
        end for
    end if
end procedure

procedure AddToCycles(Market, newCycle)
    for cycle in cycles(Market) do
        if cycle∩newCycle≠∅ then
            Remove cycle from cycles(Market)
            newCycle = newCycle∪cycle
        end if
    end for
    Add newCycle to cycles(Market)
end procedure
```

Algorithm 6 Update the pairs of control in the market

```
procedure SingleControllerThroughPathes(Market)
    NextPairsGroup ← ()
    for company in companies(Market) do
        if company has no mothers (in Market) then
            for descendant in getAllDescendants(Market, company) do
                Add (company, descendant) to NextPairsGroup
            end for
        end if
    end for
    Return NextPairsGroup
end procedure
```

B Proofs

Let the network of shareholding relationships among a set V of n firms and shareholders be described by the $n \times n$ adjacency matrix $H \in [0,1]^{n \times n}$ as defined in Section 4.1 where $n < \infty$. Fix a majority quota $q \in (0,1]$ and power index threshold $\theta \in (0.5, 1]$. Further, denote by L the maximal path length among all the holdings paths in the network H, i.e. $L := max\{|P(y,x)| : P(y,x) \subseteq H$ is a directed simple path of holdings from y to x$\}$. Where $|P(y,x)|$ is the length of $P(y,x)$ defined as the number of edges in the path, and we always have $L \leq n-1$.

1. Notice that for any adjusted matrix of holdings resulting from each iteration (after joining concerts and replacing them by the root) the maximal length of holdings path cannot be larger than L - the maximal length of holdings path in the original matrix, H. Thus, the algorithm, by construction, converge after at most L steps to a solution, i.e. the solution will not change any more for any iteration $\geq L$. The finiteness of L implies that the process terminates.

2. We show correctness (exclusive controllers) by induction over L. For $L = 1$, a power index test (Shapley-Shubik or Banzhaf) with $\theta > 0.5$ implies the uniqueness of the controller and thus it leads to a correct solution. Assume that the statement holds for $L = k-1$. For $L = k$, since by the induction hypothesis the matrix of holdings is fixed and correct after $k-1$ steps, we are in fact in the same case as $L = 1$. The last non tested nodes (the leaf in level $L = k$) will yield a correct solution after the power index test since it is uniquely determined with $\theta > 0.5$. Intuitively, assume that there was a mistake during some iteration $1 \leq m \leq L$, i.e. a firm is identified as controlled by some shareholder, that does not actually control it by the data in H. However, in each iteration (in our case iteration m), the power index test uniquely and correctly identifies controllers from a given updated matrix of holdings H_{m-1} (which is H adjusted for concerts after iteration $m-1$). Then, after adjusting for concerts and applying test in iteration m, it cannot be the case that the mistake was in iteration m, i.e. the mistake must have occurred before. By the same logic we will trace back to iteration 1, i.e. to the original matrix of holdings, H. But since the power index test is unique and correct for $\theta > 0.5$, it cannot be the case that the mistake was in iteration 1. A contradiction.

3. We perform a similar induction over L as in above. For $L = 1$ all possible control relations are identified by the construction of the power index test with $\theta > 0.5$ and are thus similar to the control relations in $\Psi(H, q, \theta) = F$. Assume that the statement holds for $L = k-1$. For $L = k$ we have an updated matrix of holdings derived from all possible (and correct) control relations in level $L = k-1$ by the induction hypothesis. Thus, the last test is similar to the case of $L = 1$, identifying all relations in level $L = k$. This completes the proof.