Agent Failures in All-Pay Auctions

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All-pay auctions, a common mechanism for various human and agent interactions, may suffer from the possibility of players' failure to participate. We model such failures and fully characterize equilibrium for this class of games.

Auctions have been the focus of much research in economics, mathematics, and computer science, and they have received attention in both the AI and multiagent communities as a significant tool for resource and task allocation. Beyond explicit auctions as performed on the Web (for example, with eBay) and in auction houses, auctions also model various real-life situations in which people and machines interact and compete for valuable items. For example, companies advertising during the US Super Bowl football game are, in effect, bidding to be one of the few remembered by the viewer and are thus putting in tremendous amounts of money to create a memorable and unique event for the viewer that overshadows other advertisers.

A particularly suitable auction for modeling various scenarios in the real world is the *all-pay auction*, in which all participants announce their bids and all of them pay those bids, but only the highest bid wins the product. Candidates applying for a job are, in a sense, participating in such a bidding process: they put in time and effort preparing for the job interview, but only one of them will be selected for the job. This is a *max-profit* auction, as the auctioneer (the employer, in this case) receives only the top bid. In comparison, a workplace with an "employee of the month" competition is a *sum-profit* auctioneer, as it enjoys the fruits of all employees' labor, regardless of who wins the competition.

The explosion in mass usage of the Web has enabled many more all-pay auction-like interactions, including some involving an extremely large number of participants. For example, various crowdsourcing contests, such as the Netflix challenge, involve many participants putting in effort, with only the best performing one winning a prize. Similar efforts can be seen throughout the Web, such as in TopCoder.com, Amazon Mechanical Turk, Bitcoin mining, and other frameworks.

However, despite the research done on all-pay auctions in the past few years,[1–3] some basic questions about all-pay auctions remain—in a full information setting, any

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Related Work in All-Pay Auctions and Agent Failures

Initial research on all-pay auctions started in the political sciences, modeling lobbying activities,[1] but since then, much analysis (especially that dealing with the Revenue Equivalence Theorem) has been done in game-theoretic auction theory. When bidders have the same value distribution for the item, Eric Maskin and John Riley[2] showed that there's a symmetric equilibrium in auctions in which the winner is the bidder with the highest bid. A significant analysis of all-pay auctions in full information settings came from Michael Baye and his colleagues,[3] showing the equilibrium states in various cases of all-pay auctions and noting that most valuations (apart from the top two) aren't relevant to the winner's strategies.

More recent work has extended the basic model. Omer Lev and his colleagues[4] addressed issues of mergers and collusions, while several others directly addressed crowdsourcing models. Dominic DiPalantino and Milan Vojnovic[5] detailed the issues stemming from needing to choose one auction from several, and Shuchi Chawla and colleagues[6] dealt with optimal mechanisms for crowdsourcing.

The early major work on failures in auctions was by Preston McAfee and John McMillan[7] followed soon after by the work of Steven Matthews,[8] which introduced bidders who aren't certain of how many bidders there will actually be in the auction. Their analysis showed that in first-price auctions (like our all-pay auction), risk-averse bidders prefer to know the numbers, while it's in the auctioneer's best interest to hide that information. In the case of neutral bidders (such as ours), their model claimed that bidders were unaffected by the numerical knowledge. Douglas Dyer and his colleagues[9] claimed that experiments that allowed "contingent" bids (that is, someone submits several bids, depending on the number of actual participants) supported these results. Flavio Menezes and Paulo Monteiro[10] presented a model in which auction participants know the maximal number of bidders but not how many will ultimately participate. However, the decision in their case was endogenous to the bidder, and therefore a reserve price has a significant effect in their model (although ultimately without change in expected revenue, in comparison to full-knowledge models). In contrast to that, our model, which assumes a little more information is available to the bidders (they know the maximal number of bidders and the probability of failure), finds that in such a scenario, bidders are better off not having everyone show up, rather than knowing the real number of contestants appearing. Empirical work done on actual auctions[11] seems to support some of our theoretical findings (although not specifically in all-pay auction settings).

In our settings, the failure probabilities are public information and the failures are independent. Such failures have also been studied in other game-theoretic fields. Reshef Meir and colleagues[12] studied the effects of failures in congestion games and showed that, in some cases, the failures could be beneficial to social welfare. Earlier work focused on agent redundancy and agent failures in cooperative games, studying various solution concepts in such games.[13,14]

References

1. A.L. Hillman and J.G. Riley, "Politically Contestable Rents and Transfers," *Economics & Politics*, vol. 1, no. 1, 1989, pp. 17–39.
2. E. Maskin and J. Riley, "Uniqueness of Equilibrium in Sealed High-Bid Auctions," *Games and Economic Behavior*, vol. 45, no. 2, 2003, pp. 395–409.
3. M.R. Baye, D. Kovenock, and C.G. Vries, "The All-Pay Auction with Complete Information," *Economic Theory*, vol. 8, no. 2, 1996, pp. 291–305.
4. O. Lev et al., "Mergers and Collusion in All-Pay Auctions and Crowdsourcing Contests," *Proc. 12th Int'l Conf. Autonomous Agents and Multiagent Systems*, 2013, pp. 675–682.
5. D. DiPalantino and M. Vojnovic', "Crowdsourcing and All-Pay Auctions," *Proc. 10th ACM Conf. Electronic Commerce*, 2009, pp. 119–128.
6. S. Chawla, J.D. Hartline, and B. Sivan, "Optimal Crowdsourcing Contests," *Proc. 23rd Ann. ACM-SIAM Symp. Discrete Algorithms*, 2012, pp. 856–868.
7. R.P. McAfee and J. McMillan, "Auctions with a Stochastic Number of Bidders," *J. Economic Theory*, vol. 43, no. 1, 1987, pp. 1–19.
8. S. Matthews, "Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View," *Econometrica*, vol. 55, no. 3, 1987, pp. 633–646.
9. D. Dyer, J.H. Kagel, and D. Levin, "Resolving Uncertainty about the Number of Bidders in Independent Private-Value Auctions: An Experimental Analysis," *RAND J. Economics*, vol. 20, no. 2, 1989, pp. 268–279.
10. F.M. Menezes and P.K. Monteiro, "Auctions with Endogenous Participation," *Rev. Economic Design*, vol. 5, no. 1, 2000, pp. 71–89.
11. D. Lu and J. Yang, "Auction Participation and Market Uncertainty: Evidence from Canadian Treasury Auctions," *Canadian Economics Assoc. 37th Ann. Meeting*, 2003; http://economics.ca/2003/papers/0141.pdf.
12. R. Meir et al., "Congestion Games with Agent Failures," *Proc. 26th Nat'l Conf. Artificial Intelligence*, 2012, pp. 1401–1407.
13. Y. Bachrach et al., "Solving Cooperative Reliability Games," *Proc. 28th Conf. Uncertainty in Artificial Intelligence*, 2011, pp. 27–34.
14. Y. Bachrach, R. Savani, and N. Shah, "Cooperative Max Games and Agent Failures," *Proc. 13th Int'l Joint Conf. Autonomous Agents and Multiagent Systems*, 2014, pp. 29–36.

equilibrium has bidders' expected profit at 0, raising the obvious question of why bidders would participate in the first place. Several extensions to the all-pay auction model have been suggested to answer this question. Omer Lev and colleagues,[3] for example, showed that allowing bidders to collude enables the cooperating bidders to have a positive expected profit at the expense of others bidders or the auctioneer. This article addresses this question by suggesting a model in which the bidders have a positive expected profit.

Table 1. The values, in expectation, of some variables when there's no possibility of failure in an auction.

Variable	No failures
Expected bid	$\frac{1}{n}$
[Variance]	$\left[\frac{1}{2n-1}-\frac{1}{n^2}\right]$
Bidder utility	0
[Variance]	$\left[\frac{n-1}{n(2n-1)}\right]$
Sum-profit principal utility	1
[Variance]	$\left[\frac{n}{2n-1}-\frac{1}{n}\right]$
Max-profit principal utility	$\frac{n}{2n-1}$
[Variance]	$\left[\frac{n(n-1)^2}{(3n-2)(2n-1)^2}\right]$

Furthermore, in all-pay auctions, the number of participants must be known to the bidder, in order to bid according to the equilibrium.[4] We suggest a relaxation of this assumption by allowing the possibility of bidders' failure, that is, there's a probability that a bidder won't be able to participate in the auction. Therefore, we assume that the number of potential bidders and the failure probability of every bidder are common knowledge, but the exact number of participants is not. Most large-scale all-pay auction mechanisms have variable participation, and we believe this helps capture a large family of scenarios, particularly for online Web-based situations and the uncertainty they contain. We propose a symmetric equilibrium for this situation, show when it's unique, and prove its various properties. Somewhat surprisingly, allowing for failures makes the expected profit for bidders positive, justifying their participation.

Our Model

We consider an all-pay auction with a single auctioned item that's commonly valued by all participants. This is a restricted case of the model in Michael Baye and colleagues' work,[4] in which players' item valuations could be different.

Formally, we assume that each of the n bidders issues a bid of b_i, $i = 1, \ldots, n$, and all bidders value the item at 1. The highest bidders win the item and divide it among themselves, while the rest lose their bid. Thus, bidder i's utility from a combination of bids $(b_1, \ldots, b_n)$ is given by

$$\pi_i(b_1,\ldots,b_n) = \begin{cases} \dfrac{1}{\left|\arg\max_j b_j\right|} - b_i & i \in \arg\max_j b_j \\ -b_i & i \notin \arg\max_j b_j. \end{cases} \quad (1)$$

We're interested in a symmetric equilibrium, which in this case, without possibility of failure, is unique.[4,5] It's a mixed equilibrium with full support of [0, 1], so each bidder's bid is distributed in [0, 1] according to the same cumulative distribution function (CDF) F, with the probability density function (PDF) f. (Because it's nonatomic, tie-breaking isn't an issue.) As we compare this case to that of no-failures, this is a case similar to that presented in Baye and colleagues' work,[4] in which various results on the behavior of noncooperative bidders have been provided.

When we allow bidders to fail, we assume that each of them has a probability of participating $p_i \in [0, 1]$. As a matter of convenience, we order the bidders according to their probabilities, so $0 \le p_1 \le p_2 \le \ldots \le p_n \le 1$. If a bidder fails to participate, its utility is 0.

Auctions without Failures

The expected utility of any participant with a bid b is

$$\pi(b) = (1-b)\cdot Pr(winning \mid b) + (-b)\cdot Pr(losing \mid b), \quad (2)$$

where $Pr(winning \mid b)$ and $Pr(losing \mid b)$ are the probabilities of winning or losing the item when bidding b, respectively. In a symmetric equilibrium with n players, each of the bidders chooses its bid from a single bid distribution with a probability density function $f_n(x)$ and a cumulative distribution function $F_n(x)$. A player who bids b can only win if all the other $n - 1$ players bid at most b, which occurs with probability $F_n^{n-1}(b)$. Thus, the expected utility of a player bidding b is

$$\pi(b) = (1-b)F_n^{n-1}(b) - b\left(1-F_n^{n-1}(b)\right) = F_n^{n-1}(b) - b. \quad (3)$$

The unique symmetric equilibrium is defined by the CDF $F_n(x) = x^{(1/n-1)}$.[4] This equilibrium has full support, and all points in the support yield the same expected utility to a player, $\pi(0) = \pi(x)$ for all $x \in [0, 1]$. Since $\pi(0) = 0$, this means that for all bids, $\pi(b) = 0$. Table 1 gives various properties of an auction without failures.[3]

Every Bidder with Its Own Failure Probability

If we assume that each bidder has its own probability for participating in the auction, with $0 \le p_1 \ldots \le p_n \le 1$, we can assume without loss of generality that each bidder has a positive participating probability, that is, $p_1 > 0$. If this is not the case, we can remove from the auction the bidders with zero probability of participating.

Equilibrium Properties

Before we present a symmetric Nash equilibrium, let's characterize any Nash equilibrium.

Theorem 1. In a common values all-pay auction when the item value is 1, if $p_{n-1} < 1$, then we have a unique Nash equilibrium, in which the expected profit of every participating bidder is $\Pi_{j=1}^{n-1}(1-p_j)$. Furthermore, there exists a continuous function $z:\left[0,1-\Pi_{j=1}^{n-1}(1-p_j)\right]\rightarrow[0,1]$, such that when bidder i has a positive density over an interval, it bids according to $F_i(x)=\left(z(x)+p_i-1\right)/p_i$ over that interval, and if $p_i = p_j$, then $F_i = F_j$.

As Theorem 1 applies to the case where $p_{n-1} < 1$, we now deal with the other case.

Theorem 2. In a common values all-pay auction, when the item value is 1, if $p_{n-1} = 1$, then in every Nash equilibrium the expected profit of every participating bidder is 0. At least two bidders with $p = 1$ randomize over [0, 1], with each other player i randomizing continuously over $(b_i, 1]$, $b_i > 0$ and might have an atomic point at 0. There exists a continuous function $z(x)$: $[0, 1] \rightarrow [0, 1]$ such that when bidder i has a positive density over an interval, it bids according to $F_i(x)=\left(z(x)+p_i-1\right)/p_i$ over that interval. For every i, the atomic point at 0 is equal to $F_i(0)$.

When there are at least two bidders with $p = 1$, the auction approaches the case without failures. The proof of Theorem 2 is a generalization of the case without agent failures.[4]

Symmetric Equilibrium

We're now ready to present a symmetric Nash equilibrium, so let's assume that $0 \le p_1 \ldots \le p_n \le 1$. If $p_{n-1} < 1$, from Theorem 1, it follows that the equilibrium is unique. If $p_{n-1} = 1$, the equilibrium is not unique: except for two bidders with $p = 1$, every bidder can place an arbitrary atomic point at 0. In the equilibrium that we present, every bidder has an atomic point at 0 of 0, and thus the equilibrium is symmetric.

To simplify the calculations, we add a "dummy" bidder; with index 0 and $p_0 = 0$, adding a bidder that surely won't participate in the auction doesn't affect other bidders and therefore doesn't influence the equilibrium.

We begin by defining a few helpful functions. First, we define $\lambda = \Pi_{j=1}^{n-1}(1-p_j)$ and the following expressions for all $1 \le k \le n - 1$:

$$H_k(x)=\left(\frac{\lambda + x}{\prod_{j=o}^{k-1}\left(1-p_j\right)}\right)^{\frac{1}{n-k}} \tag{4}$$

and

$$\underline{s}_k=\left(1-p_k\right)^{n-k}\prod_{j=o}^{k-1}\left(1-p_j\right)-\lambda. \tag{5}$$

For the virtual 0 index, we use $\underline{s}_0 = 1 - \lambda$. Note that because the p_is are ordered, so are the $\underline{s}_i$s: $1 \ge \underline{s}_0 \ge \underline{s}_1 \ge \ldots \ge \underline{s}_{n-1} = 0$. An equivalent definition of $\underline{s}_k$ is $\underline{s}_k=\left(1-p_k\right)^{n-k-1}\Pi_{j=o}^{k}(1-p_j)-\lambda$, so we alternate between those two definitions.

We're now ready to define the CDFs for our equilibrium, for every bidder $1 \le i \le n - 1$:

$$F_i(x)=\begin{cases} 1 & x \ge \underline{s}_0 \\ \dfrac{H_1(x)+p_i-1}{p_i} & x \in [\underline{s}_1, \underline{s}_0) \\ \vdots & \vdots \\ \dfrac{H_k(x)+p_i-1}{p_i} & x \in [\underline{s}_k, \underline{s}_{k-1}) \\ \vdots & \vdots \\ \dfrac{H_{n-1}(x)+p_i-1}{p_i} & x \in [\underline{s}_i, \underline{s}_{i-1}) \\ 0 & x < 0 \end{cases}. \tag{6}$$

F_n, uniquely, is very similar to F_{n-1} in its piecewise composition and has an atomic point at 0 of $1-(p_{n-1}/p_n)$, so

$$F_n(x) = \begin{cases} 1 & x \ge \underline{s}_0 \\ \dfrac{H_1(x)+p_n-1}{p_n} & x \in [\underline{s}_1, \underline{s}_0) \\ \vdots & \vdots \\ \dfrac{H_k(x)+p_n-1}{p_n} & x \in [\underline{s}_k, \underline{s}_{k-1}) \\ \vdots & \vdots \\ \dfrac{H_{n-1}(x)+p_n-1}{p_n} & x \in [\underline{s}_{n-1}, \underline{s}_{n-2}) \\ 1-\dfrac{p_{n-1}}{p_n} & x = 0 \\ 0 & x < 0 \end{cases}. \tag{7}$$

Note that all CDFs are continuous and piecewise differentiable, and when $p_i = p_j$ it follows that $F_i = F_j$, making this a symmetric equilibrium. Note also that when $\Pi_{j=o}^{k-1}(1-p_j) = 0$ and H_k is undefined for some k, there's no range for which H_k is used. The intuition behind this equilibrium is that bidders with low probability of participating, will participate rarely and usually bid high, while those that frequently participate in auctions with less competition would more commonly bid low.

Theorem 3. The strategy profile $F_1, \ldots, F_n$ defined in Equations 6 and 7 is an equilibrium in which the expected profit of the bidders, if they haven't failed, is λ.

Proof. In the course of proving this is indeed an equilibrium, we calculate the expected utility of the bidders when they participate. When bidder i bids according to this distribution, that is, $x \in [\underline{s}_k, \underline{s}_{k-1})$ for $1 \le k \le i$,

$$
\begin{aligned}
\pi_i(x) &= (1-x)\prod_{j=1;j\neq i}^{n}\left(p_jF_j(x)+1-p_j\right) \\
&\quad - x\left(1-\prod_{j=k;j\neq i}^{n}\left(p_jF_j(x)+1-p_j\right)\right) \\
&= \prod_{j=1;j\neq i}^{n}\left(p_jF_j(x)+1-p_j\right)-x \\
&= \prod_{j=1}^{k-1}\left(p_jF_j(x)+1-p_j\right) \\
&\quad \prod_{j=k;j\neq i}^{n}\left(p_jF_j(x)+1-p_j\right)-x \\
&= \prod_{j=1}^{k-1}\left(1-p_j\right)\prod_{j=k;j\neq i}^{n}H_k(x)-x \\
&= \left(\prod_{j=1}^{k-1}\left(1-p_j\right)\right)\left(H_k(x)^{n-k}\right)-x \\
&= \prod_{j=1}^{k-1}\left(1-p_j\right)\left(\frac{\lambda+x}{\prod_{j=0}^{k-1}\left(1-p_j\right)}\right)-x \\
&= \lambda.
\end{aligned} \tag{8}
$$

If bidder i bids outside its support, that is, $x \in [\underline{s}_k, \underline{s}_{k-1})$ for $i+1 \le k \le n-1$, the same equation becomes

$$
\begin{aligned}
\pi_i(x) &= \prod_{j=1;j\neq i}^{k-1}\left(1-p_j\right)\prod_{j=k}^{n}H_k(x)-x \\
&= \left(\prod_{j=1;j\neq i}^{k-1}\left(1-p_j\right)\right)\left(\frac{\lambda+x}{\prod_{j=1}^{k-1}\left(1-p_j\right)}\right)^{\frac{n-k+1}{n-k}}-x. \\
&= \frac{\lambda+x}{1-p_i}\left(\frac{\lambda+x}{\prod_{j=1}^{k-1}\left(1-p_j\right)}\right)^{\frac{1}{n-k}}-x
\end{aligned} \tag{9}
$$

Now,

$$
x < \underline{s}_{k-1} = \left(1-p_{k-1}\right)^{n-k}\prod_{j=1}^{k-1}\left(1-p_j\right)-\lambda, \tag{10}
$$

and hence $\lambda + x < \left(1-p_{k-1}\right)^{n-k}\Pi_{j=1}^{k-1}(1-p_j)$. Plugging it all together,

$$
\begin{aligned}
\pi_i(x) &< \frac{\lambda+x}{1-p_i}\left(\frac{\left(1-p_{k-1}\right)^{n-k}\prod_{j=1}^{k-1}\left(1-p_j\right)}{\prod_{j=1}^{k-1}\left(1-p_j\right)}\right)^{\frac{1}{n-k}}-x \\
&= \frac{\lambda+x}{1-p_i}\left(1-p_{k-1}\right)-x \\
&= (\lambda+x)\frac{p_i-p_{k-1}}{1-p_i}+\lambda.
\end{aligned} \tag{11}
$$

Finally, $i+1 \le k$, $p_i \le p_{k-1}$; hence $p_i - p_{k-1} \le 0$, and therefore, $\pi_i(x) < \lambda$.

Profits

When bidders actually participate, their expected utility is λ, and therefore the overall expected utility for bidder i is $p_i\lambda$ (which, naturally, decreases with n). Notice that, as expected, a bidder's profit rises as fellow bidder reliability or participation drops. However, the most reliable bidders don't affect the profits of the rest. If a bidder can set its own participation rate, if there's no bidder with $p_j = 1$, that is the best strategy; otherwise, the optimal probability should be 1/2, as that maximizes $p_i\left(1-p_i\right)\Pi_{j=1;j\neq i}^{n-1}(1-p_j)$.

Expected bid. To calculate each bidder's expected bid, we need to calculate the bidders' *equilibrium* PDF for $1 \le i \le n-1$:

$$
f_i(x) = \begin{cases}
0 & x \ge \underline{s}_0 \\
\dfrac{(\lambda+x)^{\frac{2-n}{n-1}}}{p_i(n-1)} & x \in [\underline{s}_1, \underline{s}_0) \\
\vdots & \vdots \\
\dfrac{(\lambda+x)^{\frac{k+1-n}{n-k}}}{p_i(n-k)\prod_{j=0}^{i-1}\left(1-p_j\right)^{\frac{1}{n-k}}} & x \in [\underline{s}_k, \underline{s}_{k-1}) \\
\vdots & \vdots \\
\dfrac{(\lambda+x)^{\frac{i+1-n}{n-l}}}{p_i(n-k)\prod_{j=0}^{i-1}\left(1-p_j\right)^{\frac{1}{n-i}}} & x \in [\underline{s}_i, \underline{s}_{i-1}) \\
0 & x < \underline{s}_i
\end{cases} \tag{12}
$$

and $f_n(x) = (p_{n-1}/p_n)f_{n-1}(x)$. In the equilibrium, the expected bid of bidder i for $1 \le i \le n-1$ is

$$
\mathbb{E}\left[bid_i\right] = \sum_{k=1}^{i}\int_{\underline{s}_k}^{\underline{s}_{k-1}} xf_i(x)\mathrm{d}x. \tag{13}
$$

Theorem 4. For every $1 \le i \le n-1$,

$$
\mathbb{E}\left[bid_i\right] = \frac{1}{p_i}\left(\frac{1}{n}+\sum_{k=1}^{i}\frac{\left(1-p_k\right)^{n-k}\prod_{j=1}^{k}\left(1-p_j\right)}{(n-k)(n-k+1)} - \frac{\left(1-p_i\right)^{n-i}\prod_{j=1}^{i}\left(1-p_j\right)}{n-i}-p_i\lambda\right) \tag{14}
$$

and

$$
\mathbb{E}\left[bid_n\right] = \frac{p_{n-1}}{p_n}\mathbb{E}\left[bid_{n-1}\right]. \tag{15}
$$

The expected bid decreases with n, indicating, as in the no-failure model, that as more bidders participate, the chance of losing increases, causing bidders to lower their exposure.

Auctioneer's sum-profit model. In the equilibrium, the auctioneer's expected profit in the sum-profit model is

$$\mathbb{E}[AP] = \sum_{i=1}^{n} \mathbb{E}[bid_i]. \tag{16}$$

When summing over all bidders, we receive a much simpler expression.

Theorem 5. The sum-profit auctioneer's equilibrium profits are

$$\sum_{i=1}^{n} p_i \mathbb{E}[bid_i] = 1 - \lambda\left(1 + \sum_{i=1}^{n} p_i\right). \tag{17}$$

In this case, growth with n is monotonically increasing, and hence, any addition to n is a net positive for the sum-profit auctioneer.

Auctioneer's max-profit model. To calculate a max-profit auctioneer's profits, we need to first define the max-profit auctioneer's profits equilibrium CDF:

$$G(x) = \prod_{i=1}^{n} \left(p_i F_i(x) + 1 - p_i\right), \tag{18}$$

that is,

$$G(x) = \begin{cases} 1 & x \geq \underline{s}_0 \\ (\lambda + x)^{\frac{n}{n-1}} & x \in [\underline{s}_1, \underline{s}_0) \\ \vdots & \vdots \\ \dfrac{(\lambda + x)^{\frac{n-k+1}{n-k}}}{\prod_{j=0}^{k-1} (1 - p_j)^{\frac{1}{n-k}}} & x \in [\underline{s}_k, \underline{s}_{k-1}) \\ \vdots & \vdots \\ \dfrac{(\lambda + x)^2}{\prod_{j=0}^{n-2} (1 - p_j)} & x \in [\underline{s}_{n-1}, \underline{s}_{n-2}) \\ 0 & x < 0 \end{cases}. \tag{19}$$

This is differentiable, and hence we can find $g(x) = (\partial/\partial x) G(x)$ and the max-profit auctioneer's expected profit.

Theorem 6. In the equilibrium, the max-profit auctioneer's profits are

$$\mathbb{E}[AP] = \int_{\underline{s}_{n-1}}^{\underline{s}_0} x g(x) \mathrm{d}x = \frac{n}{2n-1} - \lambda + \sum_{k=1}^{n-1} \frac{(1 - p_k)^{2n-2k-1} \prod_{j=1}^{k} (1 - p_j)^2}{4(n-k)^2 - 1}. \tag{20}$$

From Theorem 6, we can see that the max-profit auctioneer would prefer to minimize λ, have two reliable bidders ($p_n = p_{n-1} = 1$), and have the other $n - 2$ bidders be as unreliable as possible.

Example 1. Consider how four bidders interact. Our bidders have participation probability of $p_1 = (1/3)$, $p_2 = (1/2)$, $p_3 = (3/4)$, and $p_4 = 1$. Let's look at each bidder's equilibrium CDFs:

$$F_1(x) = \begin{cases} 1 & x \geq \frac{11}{12} \\ 3\left(\frac{1}{12} + x\right)^{\frac{1}{3}} - 2 & x \in \left[\frac{23}{108}, \frac{11}{12}\right) \\ 0 & x < \frac{23}{108} \end{cases}$$

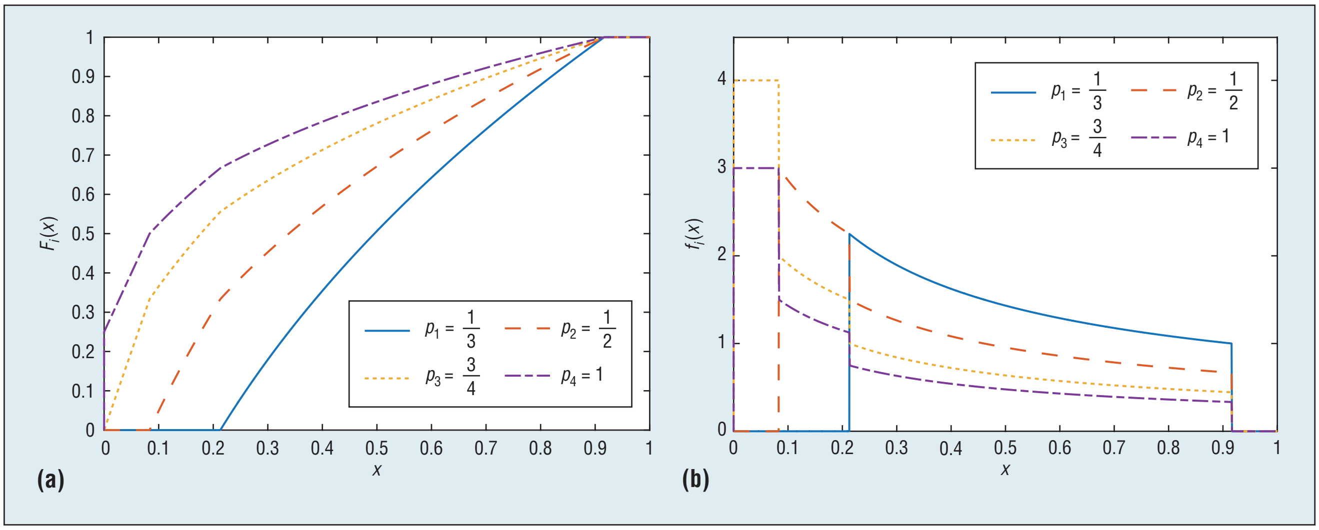
$$F_2(x) = \begin{cases} 1 & x \geq \frac{11}{12} \\ 2\left(\frac{1}{12} + x\right)^{\frac{1}{3}} - 1 & x \in \left[\frac{23}{108}, \frac{11}{12}\right) \\ 2\left(\dfrac{3\left(\frac{1}{12} + x\right)}{2}\right)^{\frac{1}{2}} - 1 & x \in \left[\frac{1}{12}, \frac{23}{108}\right) \\ 0 & x < \frac{1}{12} \end{cases}$$

$$F_3(x) = \begin{cases} 1 & x \geq \frac{11}{12} \\ \frac{4}{3}\left(\frac{1}{12} + x\right)^{\frac{1}{3}} - \frac{1}{3} & x \in \left[\frac{23}{108}, \frac{11}{12}\right) \\ \frac{4}{3}\left(\dfrac{3\left(\frac{1}{12} + x\right)}{2}\right)^{\frac{1}{2}} - \frac{1}{3} & x \in \left[\frac{1}{12}, \frac{23}{108}\right) \\ 4\left(\frac{1}{12} + x\right) - \frac{1}{3} & x \in \left[0, \frac{1}{12}\right) \\ 0 & x < 0 \end{cases}$$

$$F_4(x) = \begin{cases} 1 & x \geq \frac{11}{12} \\ \left(\frac{1}{12} + x\right)^{\frac{1}{3}} & x \in \left[\frac{23}{108}, \frac{11}{12}\right) \\ \left(\dfrac{3\left(\frac{1}{12} + x\right)}{2}\right)^{\frac{1}{2}} & x \in \left[\frac{1}{12}, \frac{23}{108}\right) \\ 3\left(\frac{1}{12} + x\right) & x \in \left[0, \frac{1}{12}\right) \\ \frac{1}{4} & x = 0 \\ 0 & x < 0 \end{cases} \tag{21}$$

A graphical illustration of the bidders' CDFs and PDFs can be found in Figure 1. The expected utility for bidder 1 is 0.027 for the expected bid of 0.518; for bidder 2, 0.041 for the expected bid of 0.394; for bidder 3, 0.0625 for the expected bid of 0.277; and for the last bidder, 0.083 for the expected bid of 0.207.

A sum-profit auctioneer will see an expected profit of 0.0784, while a max-profit one is expected to get 0.490.



Figue 1. The cumulative distribution functions (CDFs) (a) and the probability density function (PDFs) (b) when $p_1 = (1/3)$, $p_2 = (1/2)$, $p_3 = (3/4)$, and $p_4 = 1$.

As a comparison, in the case where we don't allow failures, the CDF of the bidders is $x^{1/3}$ with an expected bid of 1/4 and expected utility of 0. The expected profit of the sum-profit auctioneer is 1, while the expected profit of the max-profit auctioneer is 4/7.

False Identity and Sabotage

Suppose our bidder can influence others' perceptions and create a false sense of its participation probability. What would its best strategy be, and how should the participation probability be altered? Any bid beyond $1 - \lambda$ is sure to win, but as that would give a profit of less than λ, which is less than the expected profit for nonmanipulators, it isn't worthwhile. Therefore, our bidder will bid in its support, with the expected profit being λ. However, our bidder might increase its expected profit by trying to portray its participation probability as being as low as possible, thus lulling other bidders into a false sense of security. Of course, this reduces the payment to auctioneers of any type, and therefore the auctioneers would try to expose such manipulation.

Also interesting is the possibility of a player changing another player's participation probability by using sabotage; thus, our bidder would be the only bidder knowing the real participation probability. Our bidder i sabotages bidder r with a perceived participation probability of p_r, changing its real participation probability to p'_r. Bidder i's expected profit with bid x is

$$\pi_i(x) = \left(p'_r F_r(x) + 1 - p'_r\right) \prod_{j=1;j\neq i,r}^{n} \left(p_j F_j(x) + 1 - p_j\right). \quad (22)$$

The values of this function change according to the relation among r, i, and x. To find the optimal strategy for a bidder, we must examine all the options.

Theorem 7. Let $p_1, \ldots, p_n$ be the announced participation probabilities, and let $p'_r < p_r$ be bidder r's real participation probability. For every $i \neq r$, Algorithm 1 finds the optimal bid for bidder i.

From Theorem 7 it follows that bidder i's best interest is to bid in the intersection of its support and bidder r's support.

Uniform Failure Probabilities

If we allow our bidders to have the same probability of failure (such as when failures stem from weather conditions), many of the calculations become more tractable, and we can further understand the scenario.

Bids

As this case is a particular instance of the general case presented earlier, we can calculate the expected equilibrium bid of every bidder and its variance.

Theorem 8. The expected equilibrium bid of every bidder is

$$\mathbb{E}[bid] = \frac{1}{np}\left(1 - \lambda\left(1 + p(n-1)\right)\right) \quad (23)$$

and the variance of the bid is

$$\mathrm{Var}[bid] = \frac{1-(1-p)^{2n-1}}{(2n-1)p} - \frac{\left(1-(1-p)^n\right)^2}{n^2p^2}. \quad (24)$$

The expected bid and the variance are neither monotonic in n nor in p.

Profits

We're now ready to examine the profits of the bidder and the auctioneer, both in the sum- and max-profit models.

Bidder. From the general case, we can deduce that the expected equilibrium profit of every bidder is $p(1-p)^{n-1}$. Note the profit decreases as n increases and is maximized when $p = (1/n)$. We can now compute the variance of bidder profit.

> **Theorem 9.** The variance of the bidder equilibrium is

$$\mathrm{Var}[BP] = \frac{n-1}{n(2n-1)} - \frac{(1-p)^n}{n} + \left(p + \frac{1}{2n-1}\right)(1-p)^{2n-1}, \tag{25}$$

and the variance is monotonically increasing in p.

Auctioneer's sum-profit model. The expected bid of every bidder is $(1/pn)(1-(1-p)^{n-1}(1+p(n-1)))$, therefore, the expected profit of the sum-profit auctioneer, in the equilibrium, is

$$\mathbb{E}[AP] = np \cdot \mathbb{E}[bid] = 1-(1-p)^{n-1}\left(1+p(n-1)\right), \tag{26}$$

which increases with p and n. Therefore, the auctioneer's best interest is to have as many bidders as possible. Note that as n grows, the auctioneer's expected revenue approaches that of the no-failure case. From Theorem 9, we get the variance of the auctioneer equilibrium profit in the sum-profit model:

Input: $i,\ r,\ p_1,\ \ldots\ ,\ p_n$ and p'_r

Output: The optimal bid for bidder i

1. let $\lambda = \prod_{j=1}^{n-1}(1-p_j)$
2. for $k = 1,\ \ldots,\ \min\{i, r\}$
    - 2.1. if $\frac{1}{n-k} \in [p_{k-1},\ p_k]$ then
        - 2.1.1 let $x_{i,k} = \left(1-\frac{1}{n-k}\right)^{n-k} \prod_{j=1}^{k-1}(1-p_j) - \lambda$
        - 2.1.2 let $\pi_{i,k} = \frac{1}{n-k}\left(1-\frac{1}{n-k}\right)^{n-k-1} \prod_{j=1}^{k-1}(1-p_j)\frac{p_r - p'_r}{p_r} + \lambda$
    - 2.2. else if $\frac{1}{n-k} < p_{k-1}$ then
        - 2.2.1 let $x_{i,k} = \underline{S}_{k-1}$
        - 2.2.2 let $\pi_{i,k} = p_{k-1}(1-p_{k-1})^{n-k-1} \prod_{j=1}^{k-1}(1-p_j)\frac{p_r - p'_r}{p_r} + \lambda$
    - 2.3. else if $\frac{1}{n-k} > p_k$ then
        - 2.3.1 let $x_{i,k} = \underline{S}_{k-1}$
        - 2.3.2 let $\pi_{i,k} = p_{k-1}(1-p_{k-1})^{n-k-1} \prod_{j=1}^{k-1}(1-p_j)\frac{p_r - p'_r}{p_r} + \lambda$
3. let $j \in \mathrm{argmax}_k \pi_{i,k}$
4. return $x_{i,j}$

Algorithm 1. Optimal bid.

$$\mathrm{Var}[AP] = np^2\mathrm{Var}[bid] = \frac{n \cdot p\left(1-(1-p)^{2n-1}\right)}{2n-1} - \frac{\left(1-(1-p)^n\right)^2}{n}. \tag{27}$$

Auctioneer's max-profit model. For the max-profit auctioneer, the expected profit in equilibrium is

$$\frac{n}{2n-1} + \frac{n-1}{2n-1}(1-p)^{2n-1} - (1-p)^{n-1}, \tag{28}$$

which is monotonically increasing in p and n (for $n \geq 1$); when n is large enough, it approaches the expected revenue in the no-failure case.

> **Theorem 10.** The variance of the auctioneer equilibrium profit in the max-profit model is

$$\begin{aligned}\mathrm{Var}[AP] &= (1-p)^{2n-2} - \frac{2n(1-p)^{n-1}}{2n-1} + \frac{n}{3n-2} - \frac{2(n-1)^2(1-p)^{3n-2}}{(3n-2)(2n-1)} \\ &\quad - \left(\frac{n}{2n-1} + \frac{n-1}{2n-1}(1-p)^{2n-1} - (1-p)^{n-1}\right)^2.\end{aligned} \tag{29}$$

Bidders failing to participate in auctions are common: people choose to apply to one job but not to another, or to participate in the Netflix challenge but not in a similar

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challenge offered by a competitor. Examining these scenarios helps us understand certain fundamental issues in all-pay auctions. In the complete reliability (classic) versions, each bidder has an expected revenue of 0. In contrast, in a limited reliability scenario, such as the one we dealt with here, bidders have positive expected revenue and are incentivized to participate. Auctioneers, on the other hand, mostly lose their strong control of the auction and no longer pocket almost all the revenues involved. However, by influencing participation probabilities, max-profit auctioneers can effectively increase their revenue in comparison to the no-failure model.

The idea of the equilibrium we explored was that frequent participants could allow themselves to bid lower, as there would be plenty of contests in which they would be one of the few participants and hence win with smaller bids. Infrequent bidders, on the other hand, would wish to maximize the few times they participate and therefore bid fairly high bids. As exists in the no-failure case as well, as more participants join, there's a concentration of bids at lower price points, because bidders are more afraid of fierce competition. Hence, it's fairly easy to see in all of our results that as n approached larger numbers, the various variables were closer to their no-failure brethren.

There's still much left to explore in these models—not only more techniques of manipulation by bidders and potential incentives by auctioneers but further enrichment of the model as well. Currently, participation rates aren't influenced by other bidders' probability of participation, but, obviously, many scenarios in real life effectively have a feedback loop in this regard. We assumed that the item is commonly valued by all the bidders and the cost of effort is common, which isn't always the case. Future research could examine a more realistic model with heterogeneous costs or valuations. In our model, the failure happened before the bidder placed a bid, but in other models, the failure could happen after bidders place their bids and before the auctioneer collects them. Finding a suitable model for such interactions, while an ambitious goal, might help us gain even further insight into these types of interactions. ■

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References

1. D. DiPalantino and M. Vojnovic', "Crowdsourcing and All-Pay Auctions," *Proc. 10th ACM Conf. Electronic Commerce*, 2009, pp. 119–128.
2. S. Chawla, J.D. Hartline, and B. Sivan, "Optimal Crowdsourcing Contests," *Proc. 23rd Ann. ACM-SIAM Symp. Discrete Algorithms*, 2012, pp. 856–868.
3. O. Lev et al., "Mergers and Collusion in All-Pay Auctions and Crowdsourcing Contests," *Proc. 12th Int'l Conf. Autonomous Agents and Multiagent Systems*, 2013, pp. 675–682.
4. M.R. Baye, D. Kovenock, and C.G. Vries, "The All-Pay Auction with Complete Information," *Economic Theory*, vol. 8, no. 2, 1996, pp. 291–305.
5. E. Maskin and J. Riley, "Uniqueness of Equilibrium in Sealed High-Bid Auctions," *Games and Economic Behavior*, vol. 45, no. 2, 2003, pp. 395–409.