The Communication Complexity of Coalition Formation among Autonomous Agents

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Abstract

It is self-evident that in many DAI settings, selfish agents stand to benefit from cooperating by forming coalitions. Nevertheless, negotiating a *stable* distribution of the payoff among the agents may prove challenging. The issue of coalition formation has been investigated extensively in the field of cooperative n-person game theory, but until recently little attention has been given to the complications that arise when the players are software agents. The bounded rationality of such agents has motivated researchers to study the computational complexity of the aforementioned problems.

In this paper, we examine the *communication* complexity of coalition formation, in an environment where each of the n agents knows only its own initial resources and utility function. Specifically, we give a tight $\Theta(n)$ bound on the communication complexity of the following solution concepts: Shapley value, the nucleolus and the modified nucleolus, equal excess theory, and the core.

1 Introduction

In an environment teeming with autonomous agents, it is only natural for agents to cooperate in order to achieve certain goals, even under the assumption that the agents are selfish. Cooperation is realized through the formation of *coalitions*: the members of each coalition share their resources, and ultimately divide their payoff. In the second half of the 20th century, prominent researchers in game theory have studied cooperative games; the focus of most of that research was determining which coalitions would form in a given game, and how coalitions should divide their payoff among their members. The trick is to divide the payoff in a way that keeps the coalition structure stable: the agents should be motivated to remain loyal to their respective coalitions, instead of deviating and forming new coalitions that might guarantee them a higher expected payoff. Different notions of stable solutions have been proposed. The strongest, named the *core* of the game, is sometimes empty. Other (weaker) solutions have different desirable properties.

In the past few years, a new layer has been added to the problem. The "players" in our cooperative game are software agents, which are limited in various (practical) ways. This has led researchers to study the computational complexity of different solution concepts for cooperative games [1, 2].

Another issue, that has so far received little attention in multiagent systems research, is the *communication complexity* of cooperative games in multiagent environments. In the multiparty communication complexity model, each of the players (agents[1]) holds some part of the input, and the players wish to jointly compute some function on the input. In this model, we assume the players have unlimited computational power; we are only interested in the worst-case number of bits of information they must pass among themselves. Although the study of communication complexity is not natural in the context of some problems, it seems especially appropriate in the context of multiagent systems in general[2] (see for example [3]), and cooperative games in particular.

[1]Throughout the paper, we use the terms "players" and "agents" interchangeably.

[2]As mentioned above, computational complexity issues are also important (perhaps even more so) in multiagent systems. These issues are not taken into account in the communication complexity model, but are studied separately.

Shehory and Kraus [7] analyzed the computational and communication complexity of two algorithms for payoff division in a multiagent environment. The environment introduced by Shehory and Kraus induces a partitioning of the information about the game among the agents — a fact that makes this environment an obvious candidate for a study of communication complexity.

In this paper we analyze the communication complexity of computing the payoff, in different solution concepts, for an arbitrary player. This restricted problem (as opposed to computing the payoffs for all the players) is important in its own right: it is reasonable that an agent would like to know *a priori* its expected payoff from a game, in order to decide whether to participate in the game at all.[3] It seems possible that, for some solution concepts or specific types of games, it may be sufficient for an agent to elicit information only from a small subset of the other agents in order to compute its payoff.

We focus on two categories of solution concepts:

- Singleton Solutions: Shapley value, the nucleolus (and modified nucleolus), and equal excess theory.
- Solutions that are possibly empty: the core.

For such solution concepts, one can establish hardness of suitable decision questions, such as determining whether the payoff of a certain agent is greater than 0, or determining whether the solution set is empty. Deciding these problems is clearly easier than computing an agent's payoff in a solution (*the* solution in the case of singleton solutions). Studying division schemes that do not fall into one of the two categories is less straightforward, since there are no apparent decision problems that are easier than computing the payoff in a solution. For example, one could ask whether a given payoff configuration is a solution, but even if this problem is hard, it may still be easy to generate just one solution (out of a possibly large solution set). The insistence on having such decision problems will become apparent in Section 2.

We investigate an environment in which there are n agents, each holding a constant amount of information. We show a tight bound of $\Theta(n)$ on the communication complexity of computing the payoff for an arbitrary player under all solution concepts mentioned above.

The rest of the paper is organized as follows. In Section 2 we explain the basics of cooperative games and communication complexity. In Section 3 we prove our results. In Section 4 we propose directions for future research.

2 Preliminaries

2.1 Cooperative Games

In the next few paragraphs, we follow Chapter 2 of [4].

A *cooperative n-person game in characteristic form with side payments* is a pair $(N; v)$, where $N = \{1, 2, \ldots, n\}$ is a set of players, and v is the *characteristic function*, which assigns a real number $v(S)$ to each $S \subset N$. $v(S)$ is the *value* of S: the payoff the players in S can obtain by cooperating. It always holds that $v(\emptyset) = 0$. The collection of payoffs to the players is expressed as a payoff vector: $\vec{x} = \{x_1, x_2, \ldots, x_n\}$.

A *coalition structure* is a partition of N, of the form $\phi = \{S_1, S_2, \ldots, S_r\}$, which specifies how the players in N divide themselves into coalitions. A *Payoff Configuration* is a pair

$$(\vec{x}; \phi) = \{x_1, x_2, \ldots, x_n; S_1, S_2, \ldots, S_r\},$$

where $\vec{x}$ is a payoff vector and ϕ is a coalition structure, such that:

$$\forall j \in [r] : \; x(S_j) \equiv \sum_{i \in S_j} x_i = v(S_j).$$

We shall refer to the coalition of all players as the *grand coalition.*

A game is *superadditive* if:

$$\forall S, T \subset N \; s.t. \; S \cap T = \emptyset : \; v(S \cup T) \geq v(S) + v(T).$$

[3]This becomes even more explicit when the agent has to pay so as to join the game.

Superadditivity is a reasonable assumption in many games, since the union $S \cup T$ of two coalitions may, in the worst case, act as two separate coalitions and receive the payoff $v(S) + v(T)$.

A special class of cooperative games is *weighted majority* games, in which the players are assigned weights $w_1, w_2, \ldots, w_n$, and a criterion number q is specified, such that:

$$\forall S \subset N: \; v(S) = \begin{cases} 1 & \sum_{i \in S} w_i \geq q \\ 0 & \sum_{i \in S} w_i < q \end{cases}$$

Such a game is represented by the following shorthand notation:

$$[q; w_1, w_2, \ldots, w_n].$$

Lemma 1. *Let $[q; w_1, w_2, \ldots, w_n]$ be a weighted majority game, where for all i, $w_i \in \{0, 1\}$. If $q \geq \lfloor n/2 \rfloor + 1$, then the game is superadditive.*

Proof. Let $S, T \subset N$ such that $S \cap T = \emptyset$. If $|S| < q$ and $|T| < q$, then clearly $v(S \cup T) \geq v(S) + v(T)$. Otherwise, assume without loss of generality that $|S| \geq q$. Since $q \geq \lfloor n/2 \rfloor + 1$ and $S \cap T = \emptyset$, it holds that $|T| < q$; it follows that $v(T) = 0$. Therefore,

$$v(S \cup T) = 1 = 1 + 0 \geq v(S) + v(T).$$

$\square$

2.1.1 Solution Concepts

In this part of the paper, we follow Chapters 3 and 6 of [4], as well as [6].

Over the years, many different solutions to characteristic function games with side payments have been proposed. These solutions differ in their notion of stability: given a coalition structure, the payoff division should be such that agents are not motivated to deviate, thus breaking down coalitions. For example, if $v(\{i\}) > 0$, any payoff configuration where $x_i = 0$ could not be stable, since player i would prefer to receive the payoff he could get by himself.

The Core The *core* C of a game $(N; v)$ is the following set of payoff configurations:

$$C = \{(\vec{x}, \phi) : \forall S \subset N, x(S) \geq v(S)\}.$$

Less formally, the core is the set of payoff configurations that leave no coalition in a position to improve the payoffs to all of its members.

The core is the strongest of all solution concepts; in fact, it is so strong that in some cases the core is empty. In such a case, at least one coalition will be dissatisfied with any payoff configuration.

Shapley Value The *Shapley value* is a payoff division scheme that is characterized axiomatically, and hence satisfies some important desiderata. Player i is called a *dummy* if $v(S \cup \{i\}) = v(S)$ for all coalitions S that do not include i; players i and j are *interchangeable* if $v((S - \{i\}) \cup \{j\}) = v(S)$ for every coalition S that includes i but not j. The axioms of the Shapley value are:

- *Symmetry:* If i and j are interchangeable, then $x_i = x_j$.
- *Dummies:* If i is a dummy, then $x_i = v(\{i\})$.
- *Additivity:* For any two games $(N; v)$ and $(N; w)$, x_i in $(N; v + w)$ equals x_i in $(N; v)$ plus x_i in $(N; w)$.

It is well known that the Shapley value, defined by:

$$x_i = \sum_{S \subseteq N} \frac{(|A| - |S|)!(|S| - 1)!}{|A|!} (v_S - v_{S - \{i\}}),$$

is the *unique* payoff division scheme that satisfies the axioms listed above.

The Nucleolus and the Modified Nucleolus The *excess* of a coalition S with respect to the payoff vector $\vec{x}$ is: $\mathrm{e}(S, \vec{x}) = v(S) - x(S)$. Given a payoff configuration $(\vec{x}; \phi)$, an excess $\mathrm{e}(S, \vec{x})$ can be constructed for any coalition S; there are 2^n such coalitions. Let $\theta(\vec{x})$ be a vector of length 2^n, whose components are all possible excesses, sorted in non-decreasing order:

$$\theta(\vec{x}) = \langle \theta_1(\vec{x}), \theta_2(\vec{x}), \ldots, \theta_{2^n}(\vec{x}) \rangle = \langle \mathrm{e}(S_1, \vec{x}), \mathrm{e}(S_2, \vec{x}), \ldots, \mathrm{e}(S_{2^n}, \vec{x}) \rangle,$$

where for all $i < j$: $\mathrm{e}(S_i, \vec{x}) \geq \mathrm{e}(S_j, \vec{x})$. $\theta(\vec{x})$ is said to be *lexicographically greater* than $\theta(\vec{y})$, denoted $\theta(\vec{x}) \succ \theta(\vec{y})$, if there exists an integer $q \in [2^n]$ such that $\theta_p(\vec{x}) = \theta_p(\vec{y})$ for all $p < q$, and $\theta_q(x) > \theta_q(y)$. If $\theta(\vec{x})$ is not lexicographically greater than $\theta(\vec{y})$, we write $\theta(\vec{x}) \not\succ \theta(\vec{y})$.

The *nucleolus* of a game is the set of all payoff configurations for which the sorted vector of excesses is lexicographically minimal:

$$N = \{(\vec{x}; \phi) : \theta(\vec{x}) \not\succ \theta(\vec{y}) \text{ for all } \vec{y}, \text{ given } \phi\}.$$

The *modified nucleolus* is defined identically, except that instead of a vector $\vec{\theta}$ with 2^n excesses, a sorted vector with $2^n - n - 2$ excesses is constructed, with components $e(S, \vec{x})$ for all coalitions S such that $1 < |S| < n$.

The nucleolus is unique for each coalition structure in any characteristic function game. It is also known that the modified nucleolus for the grand coalition consists of a unique payoff vector.

Equal Excess Equal excess theory yields a payoff vector that is the result of a bargaining process. At each stage, a player lodges a claim for a share of the value of each coalition of which he is a member. As a starting point, each player expects an equal share of the value of each such coalition.

Formally, the bargaining process consists of discrete *rounds*. In round r, player i has an *expectation* of the payoff he will obtain from coalition S of which he is a member; this expectation is denoted $E^r(i, S)$. Let $A^r(i, S) = \max_{T \neq S}[E^r(i, T)]$; this is player i's highest expectation from the alternative coalitions to S. Player i's expectation for round $r + 1$ is created by:

$$E^{r+1}(i, S) = A^r(i, S) + \frac{v(S) - \sum_{j \in S} A^r(j, S)}{|S|}.$$

If the bargaining process ends after m rounds, we say it is the solution generated by *m-round* equal excess. As far as we know, for all games examined, the sequence of expectations converges to an asymptotic value, but there is no proof of convergence. Assuming such a limit always exists, we say it is the solution generated by *∞-round* equal excess.

2.1.2 Environment Description

We follow the presentation of the environment in [7] (with some simplifications). An exact description of the environment is important for our purposes, since it induces a distribution of the information about the game (or input) among the agents.

Our environment consists of autonomous agents $1, 2, \ldots, n$, with tasks to fulfill. Each agent has a given amount of the resources $l_1, \ldots, l_s$, which are required in order to deal with tasks (we denote by Q the set of all possible vectors of quantities of resources). An agent receives a payoff for fulfilling tasks. These concepts can be formalized as a *payoff function* $U^i : Q \to \mathbb{R}^+$, which gives the payoff of agent i for some arbitrary resources.

We assume that resources can be traded among agents. We also assume that payoff can be transferred from one agent to another (using money, or some other divisible desirable commodity).[4] The agents may try to tackle the tasks alone, but may also prefer to form coalitions, thus pooling (and redistributing) their resources. If S is a coalition, we say that its value is $v(S)$ if $v(S) = \sum_{i \in S} U^i(\overline{q}_i)$, where $\overline{q}$ is the vector of resources after redistribution in the coalition.

It is important to recognize that such an environment can be represented using a characteristic function game, and sometimes vice versa; hence the concepts from the previous subsections can be used. However, the representation of the game proposed in this subsection is more realistic. In essence, the information about the game is distributed among the agents; each agent holds a

[4] Such a transfer of payoff is called a *side payment*. Recall that we assumed side payments are possible when we described characteristic function games in Section 2.1.

constant amount of information, namely its resources and payoff function. As an example, consider the weighted majority game $[q; w_1, w_2, \ldots, w_n]$. There is only one resource, of which each agent is given the initial quantity w_i; the utility function for all agents is:

$$U^i(z) = \begin{cases} 1 & z \geq q \\ 0 & z < q \end{cases}$$

We do not assume that the environment is superadditive (although this is implied by the above definitions), and our results also apply to environments that do not have this property. However, note that in a superadditive game, we can assume that the grand coalition forms at some stage, and concern ourselves only with the payoff division (and not the coalition structure).

2.2 Communication Complexity

In this subsection, we present the multiparty communication complexity model with which we shall deal. A recommended, more detailed overview of communication complexity theory appears in [5].

There are several ways to generalize the two-party communication complexity model (introduced by Yao in [8]) to a multiparty setting. In our model,[5] player i ($i \in [n]$) holds an input $z_i \in \{0,1\}^k$. The players wish to compute together a function $f : (\{0,1\}^k)^n \to A$. They communicate via a "public blackboard" — all the players can see any bit a player sends. We are only interested in the amount of communication among the players, and therefore we allow the players to have unlimited computational power.

A *deterministic protocol* P is a binary tree where each internal node v is labeled by a function $g_{i,v} : \{0,1\}^k \to \{0,1\}$, for a single $i \in [n]$, and each leaf is labeled with $a \in A$. The *value* of the protocol P on input $(z_1, z_2, \ldots, z_n)$ is the label of the leaf reached by starting from the root, and walking on the tree: at each internal node v labeled by $g_{i,v}$ walking left if $g_{i,v}(z_i) = 0$, and right if $g_{i,v}(z_i) = 1$. The *cost* of the Protocol P on input $(z_1, z_2, \ldots, z_n)$ is the length of the path taken on this input. The *cost* of the protocol P is the height of the tree. The *deterministic communication complexity* of a function f is the minimum cost of P, over all protocols P that compute f.

Intuitively, every internal node v labeled by $g_{i,v}$ is associated with a bit sent by player i: 0 if $g_{i,v}(z_i) = 0$, and 1 otherwise. At each point in the protocol, the current node in tree is determined by the previous bits sent by the players. In this way, players take into account the bits communicated so far.

It is also possible to consider the *nondeterminstic communication complexity* of f. An all-powerful prover is trying to convince the players that $f(z_1, z_2, \ldots, z_n) = a_0$. If it is indeed true that $f(z_1, z_2, \ldots, z_n) = a_0$, then the prover should be able to convince the players (by posting the answer and a proof to the blackboard). However, if this is not the case, the players should be able to detect the lie, regardless of what the prover says. It is obvious that a lower bound on the nondeterministic communication complexity of a problem is also a lower bound on the deterministic communication complexity of the problem. Furthermore, it is also a lower bound on the *randomized* communication complexity of the problem.

There are several techniques for obtaining lower bounds on communication complexity. The most popular technique is the *fooling set*.

Definition 2. Let $f : (\{0,1\}^k)^n \to \{0,1\}$. $H \subset (\{0,1\}^k)^n$ is called a *fooling set* (for f) if there exists a value $f_0 \in \{0,1\}$ such that:

- For every $(z_1, z_2, \ldots, z_n) \in H$, $f(z_1, z_2, \ldots, z_n) = f_0$.
- For every two distinct vectors $(z_1^1, z_2^1, \ldots, z_n^1), (z_1^2, z_2^2, \ldots, z_n^2) \in H$, there exist $r_1, r_2, \ldots, r_n \in \{1,2\}$ such that
$$f(z_1^{r_1}, z_2^{r_2}, \ldots, z_n^{r_n}) = 1 - f_0.$$

Less formally, by "mixing" the coordinates of any two vectors in the fooling set, we can obtain an input vector whose value under f is $1 - f_0$.

[5]Another interesting model is the "number on the forehead" model, where each player holds a bit that all other players can see, but he cannot.

It is known that the existence of a fooling set of size m for f entails a lower bound of log$m$ on the nondeterministic communication complexity of f.

In the context of cooperative games, f is the function which, given the resources and utility function for all the players and a distinguished player, outputs that player's payoff in some solution, according to a fixed solution concept. However, when proving our lower bounds, we will deal with boolean functions[6] (so that we can use the fooling set technique).

3 Results

This section is devoted to proving a tight bound of $\Theta(n)$ on the communication complexity of computing a player's expected payoff in the following solution concepts (in this order): Shapley value, the nucleolus and modified nucleolus, m-round equal excess and ∞-round equal excess, and the core.

3.1 Upper Bound

Obtaining an upper bound on the communication complexity of any solution concept in our environment is immediate:

Proposition 3. *The deterministic communication complexity of any solution concept (even when computing the payoff of all players) is* $O(n)$.

Proof. Recall that each agent holds a constant amount of information. Therefore, all n agents can communicate their entire part of the input (resources and utility function), and then compute a solution. □

3.2 Lower Bounds

The following lemma will be essential in the proof of Lemma 6.

Lemma 4. $log\binom{n}{\lfloor \frac{n}{2} \rfloor+1} = \Omega(n)$.

Proof. Without loss of generality, assume n is even (for an odd n small changes are required, but the proof is similar). Observe that

$$\binom{n}{\frac{n}{2}+1} = \frac{n!}{(\frac{n}{2}+1)!(\frac{n}{2}-1)!} = \frac{n \cdot (n-1) \cdots (n/2+2)}{(n/2-1) \cdot (n/2-2) \cdots 1}.$$

We associate each factor in the denominator which is greater than or equal to $n/4+1$, with a factor in the numerator that is exactly twice as large. For example, $n/4+1$ is associated with $n/2+2$. We have $n/4-1$ such pairs, each with a ratio of 2. The other $n/4$ factors in the numerator are all greater than the other $n/4$ factors in the denominator. Therefore, we have: $\binom{n}{\frac{n}{2}+1} \geq 2^{n/4-1}$, and hence:

$$\log\binom{n}{\frac{n}{2}+1} \geq \log\left(2^{n/4-1}\right) = n/4 - 1 = \Omega(n).$$

□

The next definition and lemma are a part of the proof of Lemma 7. They yield a somewhat roundabout proof for Lemma 7, which is meant to provide intuition for the correctness of some of the main propositions.

Definition 5. The *majority function*, denoted by maj, returns 1 if at least $\lfloor \frac{n}{2} \rfloor + 1$ players have the bit 1, and 0 otherwise.

Let E be the set of input vectors such that the number of ones is at most $\lfloor \frac{n}{2} \rfloor + 1$, and denote by $maj_{|_E}$ the majority function restricted to the games in E.

[6]Determining whether a given player's payoff is greater than 0, or determining whether the solution set is nonempty. This strengthens our results, since these decision problems are weaker than computing payoff.

Lemma 6. *The nondeterministic communication complexity of $maj_{|E}$ is $\Omega(n)$.*

Proof. We exhibit a fooling set of size $\binom{n}{n'}$, where $n' = \lfloor \frac{n}{2} \rfloor + 1$; the result follows from Lemma 4. The fooling set consists of all binary vectors of length n with exactly n' ones. Clearly, for any vector there is a majority of ones, and thus it remains to show that for any two vectors $\vec{w}^1$ and $\vec{w}^2$ in the fooling set, we can create a vector $\vec{w}$ where $w_i = w_i^1$ or $w_i = w_i^2$ for all $i \in [n]$, in such a way that $maj(\vec{w}) = 0$. Indeed, for any two such vectors, there must be $i_0 \in [n]$ such that $w_{i_0}^1 = 1$ but $w_{i_0}^2 = 0$. Let $w_i = w_i^1$ for all $i \neq i_0$, and $w_{i_0} = w_{i_0}^2 = 0$. $\vec{w}$ has exactly $n' - 1$ ones, as required. □

Lemma 7. *Let G be the set of weighted-majority games $[q; w_1, w_2, \dots, w_n]$ where $q = \lfloor \frac{n-1}{2} \rfloor + 2$, such that $w_i \in \{0, 1\}$ for all i, $w_1 = 1$, and $\#\{i : i \geq 2 \wedge w_i = 1\} \leq \lfloor \frac{n-1}{2} \rfloor + 1$. Assume that some singleton solution concept satisfies the following property:*

$$\forall g \in G,\ maj_{|E}(w_2, w_3, \dots, w_n) = 1 \Longrightarrow x_1(g) > 0 \textit{ for the grand coalition.} \tag{1}$$

Then the nondeterministic communication complexity of computing the payoff of an arbitrary agent in this solution concept is $\Omega(n)$ (even to decide whether the payoff of a given agent is greater than 0).

Proof. Notice that all games in G are superadditive by Lemma 1, and thus we can assume the grand coalition forms. There is an obvious reduction from $maj_{|E}$ with $n-1$ players to games in G: given an input $\langle w_2, w_3, \dots, w_n \rangle$ for $maj_{|E}$, complete the vector of weights with $w_1 = 1$, and set $q = n' + 1$, where $n' = \lfloor \frac{n-1}{2} \rfloor + 1$. Assuming property (1) holds, then $maj(w_2, w_3, \dots, w_n) = 1$ if and only if $x_1(g) > 0$: "only if" follows from the fact that if $maj(w_2, w_3, \dots, w_n) = 0$, then $\sum_{i \geq 2} w_i < n'$, and so the value of all coalitions is 0.

We have that if the non-deterministic communication complexity of deciding whether $x_1 > 0$ is $f(n)$ for some function f, then the non-deterministic communication complexity of $maj_{|E}$ with $n-1$ players is at most $f(n)$. From Lemma 6, $f(n) = \Omega(n-1) = \Omega(n)$. □

We now prove our lower bounds. We start with the solution concepts that correspond to a singelton set of stable payoff configurations. These proofs rely on Lemma 7.

Proposition 8. *The nondeterministic communication complexity of computing the payoff of an arbitrary agent according to the Shapley value is $\Omega(n)$ (even to decide whether the payoff of a given agent is greater than 0).*

Proof. Let G be the set of weighted-majority games $[q; w_1, w_2, \dots, w_n]$ where $q = \lfloor \frac{n-1}{2} \rfloor + 2$, such that $w_i \in \{0, 1\}$ for all i, $w_1 = 1$, and $\#\{i : i \geq 2 \wedge w_i = 1\} \leq \lfloor \frac{n-1}{2} \rfloor + 1$. We show that the Shapley value has property (1); this is sufficient to complete the proof by Lemma 7.

Fix a game from G, and let $S^* = \{i : w_i = 1\}$ (this coalition includes player 1). If we assume that $maj_{|E}(w_2, w_3, \dots, w_n) = 1$, then $|S^*| = q$; it follows that $x_1 > 0$, since:

$$\begin{aligned} x_1 &= \sum_{S \subseteq A} \frac{(|A| - |S|)!(|S| - 1)!}{|A|!}(v_S - v_{S - \{1\}}) \\ &\geq \frac{(|A| - |S^*|)!(|S^*| - 1)!}{|A|!}(v_{S^*} - v_{S^* - \{1\}}) \\ &= \frac{(n-q)!(q-1)!}{n!} \cdot (1 - 0) \\ &> 0. \end{aligned}$$

In fact, since all q players with $w_i = 1$ are interchangeable, and the rest are null players, we have from the axioms that characterize the Shapley value that $x_1 = \frac{1}{q}$. □

Proposition 9. *The nondeterministic communication complexity of computing the payoff of an arbitrary agent according to the nucleolus*[7] *and the modified nucleolus is $\Omega(n)$ (even to decide whether the payoff of a given agent is greater than 0).*

[7] Recall that the nucleolus is a singleton for a given coalition structure. Here we investigate the nucleolus of the grand coalition.

Proof. Let G be the set of weighted-majority games $[q; w_1, w_2, \dots, w_n]$ where $q = \lfloor \frac{n-1}{2} \rfloor + 2$, such that $w_i \in \{0, 1\}$ for all i, $w_1 = 1$, and $\#\{i : i \geq 2 \wedge w_i = 1\} \leq \lfloor \frac{n-1}{2} \rfloor + 1$. We show that the nucleolus has property (1); this is sufficient to complete the proof by Lemma 7.

Fix a game from G, and let $\vec{x}$ be the nucleolus of the grand coalition. We assume that $maj(w_2, \dots, w_n) = 1$; it follows that $\#\{i : w_i = 1\} = q$. We wish to show that $x_1 > 0$. Consider the payoff division $\vec{x^*}$, where $x_i^* = \frac{1}{q}$ for all i such that $w_i = 1$, and $x_i^* = 0$ otherwise. $\mathrm{e}(R, \vec{x^*}) = 0$ for coalitions R that contain all players with $w_i = 1$, and coalitions that do not contain any of these players. Moreover, for any other coalition, $\mathrm{e}(R, \vec{x^*}) < 0$: if the coalition contains $k < q$ players with $w_i = 1$, then $\mathrm{e}(R, \vec{x^*}) = -\frac{k}{q}$.

Now, assume $x_1 = 0$. If there exists i such that $w_i = 0$ but $x_i > 0$, then the coalition $S^* = \{i : w_i = 1\}$ must have $\mathrm{e}(S^*, \vec{x}) > 0$, since $v(S^*) = 1$ but $x(S^*) < 1$. But this means $\vec{\theta}(x) \succ \vec{\theta}(x^*)$, and thus we can assume the payoff is distributed only between the players with $w_i = 1$. $\mathrm{e}(R, \vec{x}) = 0$ for coalitions R that contain all players with $w_i = 1$, and all coalitions that do not contain any of these players. Additionally, it holds that $\mathrm{e}(\{1\}, \vec{x}) = 0$. The number of zeros in $\vec{\theta}(\vec{x})$ is greater than the number of zeros in $\vec{\theta}(\vec{x^*})$, and thus $\vec{x} \succ \vec{x^*}$; this is a contradiction to the assumption that $\vec{x}$ is the nucleolus.

For the modified nucleolus we have to slightly change the end of the proof, because the excess of coalitions of size 1 is no longer considered. However, observe that if $x_1 = 0$ (and we still have that $x_i = 0$ for all players with weight 0), there must be a player i_0 with $w_i = 1$ and $x_i \leq \frac{1}{q-1}$; hence, $\mathrm{e}(\{1, i_0\}, \vec{x}) \geq -\frac{1}{q-1}$. It holds that the number of 0's in $\vec{\theta}(\vec{x})$ and $\vec{\theta}(\vec{x^*})$ is equal, but the next smallest coordinate in $\vec{\theta}(\vec{x})$ is at least $-\frac{1}{q-1}$, while in $\vec{\theta}(\vec{x^*})$ it is $-\frac{2}{q}$. Hence, $\vec{x} \succ \vec{x^*}$ — a contradiction to our assumption that $\vec{x}$ is the modified nucleolus of the grand coalition. □

Proposition 10. *The nondeterministic communication complexity of computing the payoff of an arbitrary agent according to m-round and ∞-round equal excess is $\Omega(n)$ (even to decide whether the payoff of a given agent is greater than 0).*

Proof. Let G be the set of weighted-majority games $[q; w_1, w_2, \dots, w_n]$ where $q = \lfloor \frac{n-1}{2} \rfloor + 2$, such that $w_i \in \{0, 1\}$ for all i, $w_1 = 1$, and $\#\{i : i \geq 2 \wedge w_i = 1\} \leq \lfloor \frac{n-1}{2} \rfloor + 1$. We show that m-round equal excess has property (1); this is sufficient to complete the proof by Lemma 7.

Fix a game from G, and let $S^* = \{i : w_i = 1\}$; we assume that $|S^*| = q$. Observe that for all rounds r and players $i \in S^*$: $E^r(i, S^*) = 1/q$, by the symmetry of all players in S^*. Moreover, clearly it holds (again, by the symmetry of the players in S^*) that for all rounds r, $i \in S^*$ and $T \subset N$: $E^r(i, T) \leq 1/q$. Therefore,

$$\forall r, \forall i \in S^*, \forall T \subset N \ s.t.\ S^* \subsetneq T: \ A^r(i, N) = 1/q. \tag{2}$$

For all players $i \notin S^*$ it holds that:

$$\forall T \subset N: \ E^0(i, T) \leq E^0(i, T^*) = \frac{1}{q+1}, \tag{3}$$

where $T^* = S^* \cup \{i\}$. In subsequent rounds r, we claim that:

$$\forall i \notin S^*, \forall T \subset N: \ E^r(i, T) \leq \frac{1}{q+1}. \tag{4}$$

This can be proven by induction: the base is given by equation (3). For the induction step, we have that $A^r(i, T) \leq \frac{1}{q+1}$ from the assumption. Any coalition T with non-zero value of which player i is a member also contains S^*. For such coalitions:

$$\begin{aligned}
E^{r+1}(i, T) &= A^r(i, T) + \frac{v(s) - \sum_{j \in T} A^r(j, T)}{|T|} \\
&\leq \frac{1}{q+1} + \frac{1 - \sum_{j \in S^*} A^r(j, T) - \sum_{j \in T - S^*} A^r(j, T)}{|T|} \\
&\overset{(2)}{=} \frac{1}{q+1} - \frac{\sum_{j \in T - S^*} A^r(j, T)}{|T|} \\
&\leq \frac{1}{q+1}
\end{aligned}$$

Consequently, for all rounds r:

$$\begin{aligned}
E^{r+1}(1,N) &= A^r(1,N) + \frac{v(N) - \sum_{j\in N} A^r(j,N)}{n} \\
&\overset{(2)}{=} \frac{1}{q} + \frac{1 - \sum_{j\in S^*} A^r(j,N) - \sum_{j\in N-S^*} A^r(j,N)}{n} \\
&\overset{(2)}{=} \frac{1}{q} - \frac{\sum_{j\in N-S^*} A^r(j,N)}{n} \\
&\overset{(4)}{\geq} \frac{1}{q} - \frac{n-q}{n}\frac{1}{q+1} \\
&> 0.
\end{aligned}$$

For m-round equal excess it holds that $x_1 = E^m(1,N)$; this completes the proof. Observe that we have from the proof that even for $m = \infty$, $E^\infty(1,N) > 0$. □

We now wish to analyze the communication complexity of payoff division according to the core. The core is not necessarily a singleton, but may be empty. Clearly, determining whether the core is empty is easier than computing the payoff of a player in some payoff configuration that is in the core.

Proposition 11. *The nondeterministic communication complexity of computing the payoff of an arbitrary agent according to the core is $\Omega(n)$ (even to decide whether the core is empty).*

Proof. We exhibit a fooling set of size $\binom{n}{n'}$, where $n' = \lfloor \frac{n}{2} \rfloor + 1$; the result follows from Lemma 4. All the inputs in the fooling set correspond to weighted majority games with $q = n' - 1 = \lfloor \frac{n}{2} \rfloor$; thus, an input in the fooling set can be fully represented by the vector of the agents' weights. The vectors $\langle w_1, w_2, \ldots, w_n \rangle$ in the fooling set have 1 in exactly n' coordinates, and 0 in the other $n - n'$ coordinates. There are $\binom{n}{n'}$ such vectors. Fix a game $\vec{w}$ in the set; we wish to show the core is empty. Since the game is supperadditive,[8] it is sufficient to show that there is no stable payoff division for the grand coalition. Indeed, let $\vec{x}$ be a payoff vector, and let $i_0 = \text{argmax}_i\{x_i : w_i = 1\}$. The coalition S of q players i with $w_i = 1$ and $i \neq i_0$ benefits by deviating, since $v(S) = 1 = x(S) + x(N-S) > x(S)$. It follows that $\vec{x}$ cannot be in the core.

We still need to show that for any two vectors $\vec{w^1}$ and $\vec{w^2}$ in the fooling set, we can create a vector $\vec{w}$ where $w_i = w_i^1$ or $w_i = w_i^2$ for all $i \in [n]$, in such a way that the core is non-empty. For any two such vectors, there must be $i_0 \in [n]$ such that $w_{i_0}^1 = 1$ but $w_{i_0}^2 = 0$. Let $w_i = w_i^1$ for all $i \neq i_0$, and $w_{i_0} = w_{i_0}^2 = 0$. $\vec{w}$ has 1 in exactly q coordinates. It is clear that the following payoff distribution is in the core: $x_i = 1/q$ for all i such that $w_i = 1$, and $x_i = 0$ for all other players. □

4 Discussion and Future Research

As our upper bound is trivial, clearly the significance of the results lies in the lower bounds. Fortunately, a communication complexity lower bound of $\Omega(n)$ is usually not an obstacle. Nevertheless, when interpreted negatively, our results show that solving cooperative games may be infeasible in scenarios where the communication is severely restricted, or the number of agents is very large.

There are several appealing directions in which this research can be extended. An important issue is to determine whether there exist reasonable singleton solution concepts with communication complexity of $o(n)$ (*small O*). It may also be the case that the communication complexity may be lowered for certain solutions, but only in specific games (where other solutions are still hard), or particular environments. For example, *veto* games contain a distinguished player which is a necessary member of any coalition with nonzero value. Knowing the identity of the veto player provides much information about the structure of the game; thus, in some veto games less communication is required in order to calculate a solution.

In addition, our methods of obtaining lower bounds have prevented us from investigating such important solutions as the kernel and the bargaining set (see [4]). These solutions should also be examined in the future.

[8] The proof of superadditivity is similar to Lemma 1, and relies on the fact that if $v(T) = 1$, T contains at least q players with $w_i = 1$, and so $N - T$ contains at most one such player.

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