Optimization and Stability in Games with Restricted Interactions

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Abstract. We study cooperative games where the interaction among agents forming coalitions is restricted according to a known graph, following a model first suggested by Myerson (1977). We give an efficient algorithm that computes the optimal coalition structure in simple games when the underlying interaction structure is a tree. We then consider stability in superadditive restricted games, using the minimal required subsidy (i.e., the Cost of Stability) as our measure. We put forward a strong conjecture, namely that the subsidy is always bounded by the treewidth of the interaction graph. We support this conjecture by proving some weaker results, and discuss their implications.

1 Introduction

The theory of cooperative games (also referred to as *coalitional games*) is often used to model interactions among potentially large groups of agents, focusing on coalitions that may form rather than on the particular actions of each agent. Cooperative game theory addresses stability issues in cooperative scenarios, and in particular *coalitional stability*.

In games with *transferable utility* (TU games), there is a set of agents $N = \{1, \dots, n\}$; each agent set $S \subseteq N$ can generate some revenue $v(S)$, which may be freely distributed among the agents in S. It is often assumed that, by default, agents form the coalition N, called the *grand coalition*. However, in certain situations *coalition structure* may arise, where agents may partition themselves into several disjoint coalitions, and each coalition in the partition divides its revenue among its members. For example, in a market there may be pairs (or small groups) of buyers and sellers that cooperate independently. In such cases, we are interested in recognizing the coalition structure that maximizes the social welfare, i.e., the total value of all agents.

Having formed coalitions (one or more) and generated profits, agents must decide on some reasonable manner in which to assign payoffs. An outcome is typically a division, or *imputation* of the value generated by the grand coalition N among the agents. Quite naturally, some agents may be unhappy with the resulting imputation, especially if these agents can collaborate by forming an alternative coalition whose value is greater than their total current payoff. In that case, they will be able to divide the profits from the newly formed coalition in a way that will benefit all of them. Several methods of dividing payoffs have been proposed in the literature, each with its own merits and justifications. Such methods are often referred to as *solution concepts*; a solution concept is simply a mapping from TU cooperative games to subsets of $\mathbb{R}^n$. Solution concepts are often required to implement certain desired properties, most of which relate to the *stability* of the proposed payoffs. We next describe some solution concepts that are related to our work. For a more detailed overview of cooperative games see e.g., [23].

The most prominent solution concept in cooperative games is the *core*. An imputation is in the core of a game $G = \langle N, v \rangle$ if there is no coalition $S \subseteq N$ which can get more than its current payoff. This is a very strong requirement; indeed, there are many cases where the core is empty. Several relaxations of the core and other solution concepts have been proposed, reflecting the belief that not every coalition that can gain from deviating will indeed do so. Imputations in the ε-core for example, are stable under the assumption that only agents (or coalitions) with a sufficiently large incentive will deviate; this is enforced by imposing a "penalty" of ε on deviating coalitions. Thus, for a large enough ε the ε-core will be non-empty.

Different relaxations consider constraints on coalitions that may deviate. Such constraints may be internal to the game, for example, by assuming that only coalitions that are themselves stable can deviate (a solution concept known as the *Bargaining set*). In contrast, there may be external, contextual reasons that prevent the formation of certain coalitions, e.g., due to lack of communication or trust among its members. While the contextual restrictions can be any subset of allowed coalitions, typically these constraints have some intuitive structure. For example, in some situations it is difficult to coordinate large-scale deviations, and thus it only makes sense to consider deviations of coalitions consisting of few agents. Another realistic restriction is given in the form of an interaction graph, where a coalition is possible only if its members can communicate without the help of other agents [22]. Such restrictions generally facilitate the search for the optimal coalition structure and also increase stability, as fewer coalitions can deviate from a suggested outcome (see Section 1.1).

Finally, a different approach to coalitional stability comes from mechanism design, in the form of *subsidies*. An external party can always induce stability by promising a sufficiently large subsidy to agents that form the grand coalition (or some other desired coalition structure). The *minimal* subsidy that is sufficient to stabilize a cooperative game is known as its *Cost of Stability* [3], and has been thoroughly studied in various games (again, see Section 1.1). It is important to note that the problem of finding the optimal coalition structure is closely related to the Cost of Stability, as the optimal coalition structure is also the one that is easiest to stabilize using subsidies [3].

1.1 Related Work

Restriction of the allowed coalitions using an interaction graph was first proposed by Myerson [22], and it is a special case of a more general restriction using *partition systems* (see Chapter 5 of Bilbao [6] for an overview). We refer to this graphical representation of the contextual restrictions as the *Myerson model*. A celebrated result by Demange [11], states that if the interaction graph is a *tree* and the game is superadditive, then the core is non-empty, and that there is at least one stable imputation that can be computed efficiently.

Coalition structures and complexity Allowing for optimal coalition structure generation by limiting agent interaction has been extensively studied. Some classic early work on the subject includes Shehory and Kraus [25] and Deng and Papadimitriou [12] on general coalitional games. Unfortunately, even in games with very simple representations it is often computationally hard to compute the optimal coalition structure, and approximations must be applied. More recent analysis has been done by Conitzer and Sandholm [10], who show that if the number of meaningful agent interactions (i.e., those interactions that provide a strictly superadditive benefit to the agents) is limited, then computing an optimal coalition structure and core allocation is tractable. Shrot et al. [26] and Chitnis et al. [8] analyze the parameterized complexity of the optimal coalition structure in cooperative games, with the maximal coalition size as a parameter; their results are mostly negative, showing that even if coalition size is a constant, finding an optimal coalition structure is still NP-hard.

Some combinatorial representations of games use graphs or hyper-graphs to define the value function of the game. As is the case with many other combinatorial problems, having a graph with a bounded treewidth[3] guarantees that certain problems that are NP-hard in the general case become tractable. Thus, for example, Ieong and Shoham [18] show that if some particular graphical representation of the game has a bounded treewidth, then checking for core non-emptiness becomes tractable. Another result is by Bachrach et al. [4], who consider a different graphical representation, under which bounded treewidth guarantees that the optimal coalition structure can be computed efficiently. We emphasize that in both of these papers, the underlying graph is *not* the interaction graph of Myerson.

In contrast, Greco et al. [17] study the computational complexity of stability-related questions in games with restricted cooperation in the Myerson model. They provided negative results showing that checking core-emptiness and several other problems remain hard even if the graph has a treewidth of 2. However, Greco et al. did not consider the problem of computing coalitional structures.

Subsidies and the Cost of Stability The literature on subsidies in cooperative games have been mainly focused on *cost sharing games*, where agents share the *cost* of a project, rather than its profits (see, for example, [19, 13]). The maximal cost that the agents can cover by themselves (i.e., the complement of the minimal subsidy) is known as the *cost recovery ratio* of a game.

More recently, Bachrach et al. [3] introduced a similar measure for the minimal subsidy in profit sharing games, known as the *Cost of Stability* (CoS). The initial work of Bachrach et al. on bounds and computational aspects of the CoS have been continued and applied to various types of games by several researchers [24, 20, 2, 21, 16, 17]. In particular, Meir et al. [21] and Greco et al. [17] studied questions related to the CoS of

[3] The tree width of an undirected graph is a combinatorial measure of “complexity” or “cyclicity” of the graph. For example, it equals 1 if and only if the graph is acyclic. See Section 2 for the precise definition.

games with restricted cooperation in the Myerson model for graphs that are not trees, extending the result by Demange [11] (stated above). Meir et al. provided bounds on the CoS for some simple interaction graphs, and the results of Greco et al. were discussed above.

Meir et al. [21] also define the *Relative Cost of Stability* (RCoS), which is a multiplicative version of the CoS (i.e., the minimal total payoff of a stable imputation, divided by $v(N)$). While computationally there is no difference between the problems, we find the RCoS definition more convenient for stating bounds on the subsidy. For example, a result by Bachrach et al. [3] states that the RCoS of superadditive games is bounded by $\sqrt{n}$.

Coalitional stability in non-cooperative games Since coalitions are an indispensable part of most strategic interactions, they have also received much attention in the literature of non-cooperative games. Some solution concepts in fact closely correspond to the cooperative solutions mentioned above. For example, the concepts of *strong equilibrium* and ε-strong equilibrium correspond to the core and the ε-core [1]. Similarly, the idea of a *coalition-proof Nash equilibrium* is closely related to the Bargaining set [5]. *Stability scores* have recently been introduced as a measure of coalitional stability, also inspired by the idea that not all coalitions are likely to form [14]. Rather than assuming that there is an explicitly-known restriction on allowed coalitions, they aim to minimize the overall number of coalitions (of a certain size) that can deviate. While stability scores have been suggested in a non-cooperative framework, they may equally apply in cooperative games as well.

1.2 Our Contribution

We study the effects of limited interaction in the Myerson model both on coalition formation and on stability.

First, we analyze the problem of optimal coalition structure generation under a limited interaction network, ignoring incentive issues. We show that for tree interaction networks, and under some limiting assumptions on the characteristic function of the game, it is possible to find an optimal coalition structure in polynomial time. Moreover, we show that these assumptions are in some sense necessary: removing any of our restrictions makes the problem of finding a coalition structure that maximizes social welfare computationally intractable.

We then focus on the problem of bounding the cost of stability in games with restricted cooperation. Our primary contribution in this section is conceptual, in the form of the following conjecture:

The (relative) Cost of Stability of any superadditive game is bounded by the treewidth of the interaction network, and this bound is tight.

We find this conjectured connection to be interesting, as the treewidth is usually associated with computational issues, and this refers to the maximal subsidy (without regard to complexity issues).

While we have so far been unsuccessful in proving the conjecture,[4] we do show some weaker results that support it. In particular, we prove a very close bound for simple games, and a bound on general superadditive games which is within a logarithmic factor from the treewidth. We hope that our observations will promote understanding of the connections between restricted cooperation and stability, a connection that we believe to be a profound and important one.

2 Preliminaries

We briefly present the definitions required for our model. A *transferable utility (TU) coalitional game* is defined by specifying the collective utility that can be achieved by every coalition of agents. Formally, $G = \langle N, v \rangle$, where N is a finite set of agents $N = \{1, \ldots, n\}$, and v is a function $v : 2^N \to \mathbb{R}$. For a singleton $i \in N$, we write $v(i)$ instead of $v(\{i\})$. The function v is called the *characteristic function* of the game. We assume by convention that $v(\emptyset) = 0$. Also, we restrict our attention in this paper to non-negative, monotone games unless explicitly stated otherwise. That is, $v(S) \geq 0$ for all S, and $v(S) \geq v(S')$ for all $S' \subset S$.

A TU game is called *simple* if $v(S)$ is monotone and always equals either 0 or 1. Coalitions with $v(S) = 1$ are called *winning* coalitions. A TU game is *superadditive* if for all $S, T \in 2^N$ s.t. $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$.

When exploring computational aspects of cooperative games, one must take into account the *representation* of the characteristic function. Indeed, a naïve representation of v is simply a list of $2^n - 1$ values, which of course grows exponentially as the number of agents increases. This issue may be addressed by exploring various methods of succinctly representing v, i.e., defining a class of characteristic functions that can be encoded using a number of bits polynomial in n, and analyzing computational aspects of this class. One such succinctly representable class is the class *weighted voting games* (WVGs). A WVG is a simple game defined by a vector of weights $w_1, \ldots, w_n$ and a threshold q. A coalition S wins if and only if $w(S) \geq q$. We note that not every simple game can be represented as a WVG.

Alternatively, one may simply assume oracle access to v, i.e., there is some algorithm that computes values for v in time polynomial in n. The algorithms we present take the latter, more general approach, whereas our reductions apply a particular succinct representation, such as WVGs.

2.1 Payoffs and stability

A *payoff vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (also called a preimputation) divides the gains of the grand coalition among its members, where $\sum_{i \in N} x_i = v(N)$. We call x_i the payoff of agent i, and denote the payoff of a coalition S as $x(S) = \sum_{i \in S} x_i$. We denote the set of all preimputations in G by $\mathbb{X}(G)$.

[4] Since the submission of this paper, there have been significant developments. The reader is advised to follow the latest versions of this work, which will be available from the homepage of the first author.

A preimputation $\boldsymbol{x} \in \mathbb{X}(G)$ is *individually rational* if no agent i can gain more than x_i by itself, i.e., if $x_i \geq v(i)$ for all $i \in N$. Individually rational preimputations are called *imputations*. Similarly, a coalition $S \in 2^N$ *blocks* $\boldsymbol{x} \in \mathbb{R}^n$, if $x(S) < v(S)$. We denote by $\mathcal{S}(G) \subseteq \mathbb{R}^n$ the set of all *stable* payoff vectors, i.e., that are not blocked by any coalition. The *core* of G, denoted $\mathrm{C}(G)$, consists of all imputations that are not blocked by any coalition. Equivalently, $\mathrm{C}(G) = \mathbb{X}(G) \cap \mathcal{S}(G)$.

The *relative Cost of Stability* of a game G is the minimal total payoff that stabilizes the game, formally:

$$RCoS(G) = \max\{1, \min_{\boldsymbol{x} \in \mathcal{S}(G)} \frac{x(N)}{v(N)}\}.$$

2.2 Interaction Graphs in Cooperative Games

Myerson [22] observes that in many cooperative scenarios, not all possible coalitions can be formed. He introduces the notion of the *interaction network* as a means of limiting agent interaction. Given a cooperative game $G = \langle N, v \rangle$, an interaction network over N is simply a graph $H = \langle N, E \rangle$, whose vertices are the agents. A coalition $S \subseteq N$ may form if and only if S forms a connected subgraph of H. Simply put, given a game G and an interaction network H, we can define the game $G|_H = \langle N, v|_H \rangle$ where $v|_H(S) = v(S)$ if S forms a connected subgraph of H (i.e., there is a path between any $a, b \in S$, that does not go through vertices in $N \setminus S$), and otherwise $v|_H(S) = 0$.

2.3 Parameters for Graph complexity

There are various graph properties that can be seen as a measure of that graph's complexity. We describe two such measures that we later use in the paper.

Given an undirected graph $H = \langle N, E \rangle$, we denote by $d(i)$ the *degree* of vertex $i \in N$, and by $d(H) = \max_{i \in N} d(i)$ the degree of the graph.

A *tree decomposition* of H is a tree T. The nodes, which are called *bags* and denoted $V(T)$, are subsets of N. We require that T satisfies the following properties:

1. For any $X, Y \in V(T)$, if $i \in X$ and $i \in Y$ then for any Z on the (unique) path between X and Y we have that $i \in Z$.
2. For any edge $e = \{i, j\} \in E$, there is some bag $X \in V(T)$ such that $e \subseteq X$.

The *width* of a tree decomposition T is $tw(T) = \max_{X \in V(T)} |X| - 1$; the *tree width* of H, denoted $tw(H)$, is the minimal width of any tree decomposition of H. If a graph H has a tree width of at most k, then it is possible to find a tree decomposition of H whose width is k in $f(k)O(n)$ time, for some function f [7].

A much simpler measure of the graph complexity is its minimal *cycle cutset width* ($ccw(H)$), which is the smallest number of vertices k s.t. by removing k vertices from H we remain with a tree. It is easy to see that $tw(H) \leq ccw(H) + 1$, since we can always add the k vertices to every bag in the tree decomposition of the remaining tree, increasing its width from 1 to $1 + k$. However typically $tw(H)$ is much smaller.

3 Finding an Optimal Coalition Structure

As previously mentioned, if v is not superadditive then forming the coalition $v(N)$ may not be the socially optimal outcome; it is then beneficial for agents to partition into disjoint coalitions, or a coalition structure. We denote by $CS^*(G)$ the optimal coalition structure of the game G. We begin by exploring how limiting interactions can allow us to find an optimal coalition structure, a problem that is known to be computationally intractable in general.

We define the decision problem OPTCS as follows: it receives as input a game $G = \langle N, v \rangle$, an interaction network H and some value $\alpha \in \mathbb{R}$; it outputs yes if and only if there is some partition $S_1, \dots, S_k$ of N such that $\sum_{j=1}^{k} v|_H(S_j) \geq \alpha$. Since we assume oracle access to v, there is some poly-size circuit representation of v, thus the input size of G is indeed polynomial in n under our assumptions.

3.1 Trees

Our first claim is that OPTCS is in P if G is a monotone simple game, and H is a tree. We introduce the following notation; given a tree $T = \langle N, E \rangle$ and some root of T $r \in N$, we set T_i to be the tree rooted in i, and $N(T_i)$ the nodes of T_i. We refer to the parent of $j \in N$ as $p(j)$ (if j is the root, then $p(j)$ is undefined) and to j's children as C_j.

Proposition 1. *Let $G = \langle N, v \rangle$ be a monotone simple game and T be a tree interaction network on N, then* OPTCS(G, T) *is in $\mathcal{P}$.*

Proof. We use the following greedy algorithm: Algorithm 1 receives as input a game

Algorithm 1: OPTCS-SIMPLEGAME(G, T, r)

```
AssignedNodes ← ∅;
WinningSetsList ← [];
for ℓ = 1 to height(T) do
    for i in height ℓ do
        if v({i} ∪ N(T_i) \ AssignedNodes) = 1 then
            WinningSetsList.append({i} ∪ N(T_i) \ AssignedNodes);
            AssignedNodes ← AssignedNodes ∪ {i} ∪ N(T_i);
return WinningSetsList;
```

$G = \langle N, v \rangle$, an interaction tree network, and an arbitrary root $r \in N$. It then proceeds by iterating over all agents in the tree network by height, starting with the leaves and ending with r. Upon reaching a node i, it checks whether i can form a winning coalition with all agents in $N(T_i)$ that were not previously assigned to a coalition; if yes, then it adds that coalition to the list of winning coalitions, marks all nodes in $N(T_i)$ as assigned, and continues. We argue that after checking node i, T_i is divided to the maximal number of winning coalitions, i.e., the coalition structure over $N(T_i)$ is optimal.

We proceed by induction on the height of T_i; if the height of T_i is 1, i.e., i is a leaf, then if $v(\{i\}) = 1$, Algorithm 1 will add $\{i\}$ to the list of winning sets and continue, if not, then the coalition $\{i\}$ is never considered again.

Now, assume by induction that this is the case in every level lower than the height of T_i. If Algorithm 1 assigns i to some winning coalition, it can increase the number of winning coalitions in T_i by at most 1. If the coalition structure that results is suboptimal, in particular this means that there is some $j \in C_i$ such that the number of winning coalitions in T_j is suboptimal; this is impossible by the induction hypothesis, so we are done.

Suppose i is not assigned, but that it is possible to restructure the coalitions in T_i so that the number of coalitions increases. Since G is monotone, this means that there is some lower point in T_i where we can get the same number of coalitions with fewer nodes - again a contradiction.

Limiting our attention to monotone simple games seems to be somewhat restrictive. However, both monotonicity and bi-values are required for tractability. Note that in both cases we show that it is hard even to distinct between the cases where $v(CS^*(G|_T)) = 1$ and $v(CS^*(G|_T)) = 0$. Thus there is no efficient approximation algorithm either.

Proposition 2. OPTCS *(G, T) is NP-complete if we allow inputs with a non-monotone G, even if we assume that the interaction network T is a tree and G is simple.*

Proof. Our reduction is from SUBSET-SUM [15]; recall that an instance of SUBSET-SUM is given by a list of integer weights $w_1, \dots, w_n$ and some quota q. It is a "yes" instance if and only if there is some subset of weights whose total weight is exactly q. Given an instance of SUBSET-SUM $\langle w_1, \dots, w_n; q\rangle$, we construct the following game on $n + 1$ players: player i is assigned a weight w_i, while player $n + 1$ has a weight of 0. The value of $v(S)$ is 1 if and only if $\sum_{i \in S} w_i = q$ (and otherwise 0). The communication network H is a star centered in player $n + 1$, with the other n players as leaves. Observe that in this game, at most one coalition containing more than one member of $\{1, \dots, n\}$ can form. To conclude, assuming that $w_i < q$ for all i, the optimal coalition structure in $G|_H$ has value of at most 1, and is 1 if and only we have a "yes" instance of SUBSET-SUM .

OPTCS is NP-complete for monotone non-simple games as well.

Proposition 3. OPTCS *(G, T) is NP-complete if we allow inputs with a non-simple G, even if the interaction network T is a tree, and that v is allowed only three different values.*

Proof. Our reduction is from the SET-COVER [15] problem. Recall that an instance of SET-COVER is given by a finite set of elements M, a set $\mathcal{F} = \{S_1, \dots, S_m\} \subseteq 2^M$ and a parameter k. It is a "yes" instance if and only there is some $\mathcal{F}' \subseteq \mathcal{F}$ of size $\leq k$ such that $\bigcup_{S \in \mathcal{F}'} S = M$. We define the characteristic function as follows: there is an agent i_j corresponding to each $S_j \in \mathcal{F}$, plus one dummy agent d. The value of a coalition $C \subset N$ is 0 if it is empty, n if $\{S_j\}_{i_j \in C}$ cover M, and 1 otherwise. Our interaction network H is a star with d in the center, and with all i_j as leaves. Thus, only one coalition that covers M may form. Clearly, in an optimal coalition structure a coalition

C^* that covers M will form, with the addition of as many singletons as possible. The value of the optimal coalition structure is more than $n + (n + 1 - k) = 2n + 1 - k$ if and only if $|C^*| \leq k$, which concludes the proof.

3.2 Bounded Treewidth

While Proposition 1 holds when the communication network is a tree, i.e., has a treewidth of 1, if the communication network H has $tw(H) \geq 2$ then OPTCS (G, H) is NP-hard.

Proposition 4. OPTCS *(G, H) is NP-hard if $tw(H) \geq 2$, even if G is simple and $tw(H) = 2$.*

Proof. Our reduction is from an instance of the SET-COVER [15] problem. Recall that an instance of SET-COVER is given by a finite set C, list of sets $\mathcal{S} = (S_1, \ldots, S_n)$ and an integer M; it is a "yes" instance if and only if there is a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $\mathcal{S}'$ *covers* C, i.e. $\bigcup_{S \in \mathcal{S}'} S = C$, and $|\mathcal{S}'| \leq M$. Given an instance of SET-COVER $(C, \mathcal{S}, M)$, as described above, we define the player set to be $\{1, \ldots, n, x, y\}$. We define the characteristic function as follows: for any $S \subseteq \{1, \ldots, n\}$, $v(S \cup \{x\}) = 1$ if and only if the set $\{S_i\}_{i \in S}$ covers C; $v(S \cup \{y\}) = 1$ if and only if $|S| \geq M$. Our interaction network H over the player set is defined as follows: there are edges (i, x) and (i, y) for all $1 \leq i \leq n$; observe that $tw(H) = 2$. One can easily verify that an optimal coalition structure over $G|_H$ has a value of 2 if and only if $(C, \mathcal{S}, M)$ is a "yes" instance of SET-COVER .

A similar reduction from the PARTITION [15] shows that OPTCS is still hard under the conditions of Proposition 4, even if we limit G to be a weighted voting game.

4 Bounding the Cost of Stability

Our primary conjecture can be stated as follows.

Conjecture 1. Let G be a superadditive game over an interaction network H. Then $\text{RCoS}(G|_H) \leq tw(H)$.

4.1 Implications of the Main Conjecture

Should Conjecture 1 hold, it would immediately imply the results by Demange [11], which is simply a special case for $k = tw(H) = 1$.

Second, consider the proposition by Meir et al. [21] stating that if H contains a single cycle then $\text{RCoS}(G|_H) \leq 2$. Since cycles have treewidth of 2, this is another special case of our conjecture.

Meir et al. [21] proposed a different conjecture, that $\text{RCoS}(G|_H) \leq d(H)$, where $d(H)$ is the maximal degree of a node in H. They also stated that this bound is tight, using the projective plane of dimension $q = k - 1$ as an example. We note that the example given by Meir et al. [21] is incorrect; the projective plane of dimension q has a degree that is at least $2q = 2k-2$, even if we remove all redundant edges. Further, while the conjecture itself holds for $d = 2$ (since all graphs with degree 2 are a collection of disjoint cycles and degenerated trees), it no longer holds for higher values of d.

Proposition 5. *For any $k \in \mathbb{N}$, there is a simple superadditive game with RCoS$(G) \geq k$ over an interaction network H with $d(H) = 6$.*

Proof. We show that *any* superadditive simple game can be embedded in a 3-dimensional grid network $H = \langle N', E \rangle$, if N' is sufficiently large.

For this, consider first a 3-dimensional grid drawing W of the complete graph K_n. This is an embedding of n vertices in a grid, s.t. every edge (i, j) is replaced by a path, and pa the edges—if drawn as straight lines—do not intersect. Such a drawing always exists using a grid of $O(n) \times O(n) \times O(n)$ (see e.g., [9]). However, W itself is not a grid graph, but just another representation of K_n.

The graph $H' = \langle N', E' \rangle$ that we will use is a 3-dimensional grid that is attained by replacing every vertex in the grid underlying W, with a grid of $n \times n \times n$ (thus $|N'| = O(n^6)$). In particular, every original vertex $i \in N$ is replaced with a cube $A_i \subseteq N'$ of n^3 vertices. Next, for every $(i, j) \in E$ (assume $i < j$), we identify a path $P(i, j) \subseteq N'$, s.t. $P(i, j)$ connects A_i and A_j; and no two paths intersect. Since the projection of W on H' is extremely sparse, it is very easy to refrain from path intersections.

We next use G to define the embedded game $G' = \langle N', v' \rangle$, with the following winning coalitions. For every winning coalition $S \subseteq N$ of G, we set $v'(S') = 1$, where $S' = \bigcup_{i \in S} A_i \cup \bigcup_{i,j \in S} P(i, j)$. Since S is connected in K_n, then S' is connected in H'. Moreover, since G is superadditive, every two winning coalitions S_1, S_2 intersect at some $i \in N$. Thus S'_1, S'_2 also intersect (in all vertices of A_i), which entails that G' is also superadditive.

Finally, we argue that $\text{RCoS}(G'|_{H'}) = \text{RCoS}(G') \geq \text{RCoS}(G)$. Indeed, since every winning S' is connected, the first equality applies. Then, assume that there is some payoff vector $\mathbf{x}' \in \mathcal{S}(G')$ that stabilizes G'. We define a payoff vector $\mathbf{x}$ for G, where $x_i = x'(A_i) + \sum_{j \in N} x'(P(i, j))$. Clearly $x(N) \leq x'(N') = \text{RCoS}(G')$. Moreover, for every winning $S \subseteq N$, $x(S) = x'(S') \geq v'(S') = 1$, thus $\mathbf{x}$ stabilizes G.

For any k, there is a simple superadditive game G_k whose RCoS is at least k (e.g., the game defined by the projective plane of order k. See [3]). As shown above, G_k can be embedded (like any other game) in a grid H' of degree 6. □

These observations highlight the conclusion that the relevant parameter of the interaction network (w.r.t. stability) is the treewidth, rather than the degree. In the remainder of this section, we prove some weaker variations of this conjecture, and provide other supporting evidence and directions.

4.2 Simple games

First, we show that if G is simple then we can show a bound close to the one stated in Conjecture 1. First, we make a few simple, yet important, observations.

Lemma 1. *If G is a simple game, then G is superadditive if and only if for any two winning coalitions $S, T \subseteq N$ we have that $S \cap T \neq \emptyset$.*

Proof. If G is superadditive, then for any two disjoint coalitions $S, T \subseteq N$ we have that $v(S) + v(T) \leq v(S \cup T)$. If $v(T) = 1$, then $v(S \cup T) = 1$, thus it must be the case that $v(S) = 0$. The other direction follows from a similar argument. □

Let $H = \langle N, E \rangle$, and let T be a tree decomposition of H. For every subset $S \subseteq N$, we denote by T_S the set of bags of T that intersect with S.

Lemma 2. *Let T be a tree decomposition of some graph $H = \langle N, E \rangle$. If $S \subseteq N$ forms a connected component in H, then T_S is a connected component in T.*

Proof. For any $i \in S$, let P_i be the path of bags in T containing i. Note that $T_S = \bigcup_{i \in S} P_i$. Let A and B be two bags of T_S, and let $a \in A \cap S, b \in B \cap S$. Since T is a tree decomposition, there is a path $a = a_0, a_1, \ldots, a_t = b$ s.t. for all $i \leq t$, $a_i \in S$ and $(a_{i-1}, a_i) \in E$. Thus P_{i-1} and P_i intersect at some bag A_i of T. Since each P_i is connected, there is a path in T from any bag in P_0 to any bag in P_t. In particular, there is a path from $A \in P_0$ to $B \in P_t$. Therefore T_S is connected. □

Using Lemmas 1 and 2, we obtain the following result.

Theorem 1. *Let $G = \langle N, v \rangle$ be a simple superadditive game. If $H = \langle N, E \rangle$ is its interaction graph, then RCoS$(G|_H) \leq tw(H) + 1$.*

Proof. Let T be a tree-decomposition of H with treewidth k; T is a tree where every bag contains at most $k + 1$ elements corresponding to agents in N.

For any winning coalition $S \subseteq N$, T_S is connected (in T) by Lemma 2.

Let $S, S' \subseteq N$ be winning coalitions. By Proposition 1 we know that $S \cap S' \neq \emptyset$, so T_S and $T_{S'}$ must intersect at some bag.

Next, we argue that there is a bag $A \in T$, s.t. $A \in T_S$ for *any* winning coalition S. Indeed, since any winning S, S' that intersect, $T_S, T_{S'}$ must also intersect. A set of subtrees that are pairwise intersecting must intersect at a particular bag A.

We pay $v(N) = 1$ to every $i \in A$; for every winning coalition S we have $A \in T_S$, and thus $p(S) = |A \cap S| \geq 1 = v(S)$. The total payoff is $p(N) = p(A) = |A| \leq k + 1 = (k + 1)v(N)$, thus RCoS$(G|_H) \leq k + 1$. □

4.3 Beyond simple games

Since $tw(H) \leq ccw(H) + 1$, Conjecture 1 implies that the RCoS is always bounded by $ccw(H) + 1$. Our next result shows that the latter bound indeed holds, i.e., the cycle-cutset width bounds the RCoS.

Proposition 6. *Given a superadditive game G with an interaction network H, RCoS$(G|_H) \leq ccw(H) + 1$.*

Proof. Given a cutset $C \subseteq N$ of size $\leq k$, we pay each $i \in C$ a value of $v(N)$; thus, if $S \cap C \neq \emptyset$ then S cannot profitably deviate. The nodes in $N \setminus C$ form a tree T; according to Demange [11], $G|_T$ has a non-empty core. Thus, if we pay each agent in T according to some core allocation $\boldsymbol{x}$, then every $S \subseteq N$ such that $S \cap C = \emptyset$ cannot profitably deviate. The total payoff is bounded by $|C|v(N) + x(N \setminus C) \leq (k+1)v(N)$. □

Unfortunately, there may be a gap of $\Theta(n)$ between $ccw(H)$ and $tw(H)$, and in fact this is the case for most graphs with a low tree width. However, using a simple technique, we can get a bound that depends on the actual tree width, up to a logarithmic factor in n. Note that while this bound is still far from our conjecture, it is a significant improvement over the bound for the general (unrestricted) case, which is $\sqrt{n}$.

Proposition 7. *Given a superadditive game G with an interaction network H, RCoS$(G|_H) \leq (tw(H)+1)(\log_2(n)+2)$.*

Proof. Let T be a tree-decomposition of H of width k s.t. $|V(T)| \leq 4n$ (such a decomposition always exists. See e.g., [7]). We prove by induction on the size of T, that there is a stable payoff vector whose total cost is at most $(k+1)\log_2(|V(T)|)v(N)$. Clearly for $|V(T)| = 1$, we can just pay $v(N)$ to each of the $k+1$ agents, and thus $x(N) \leq (k+1)v(N)$.

Now given some tree T, let bag $M \in V(T)$ be a *median node* of T. Let $T' = T \setminus \{M\}$; we remove all agents in M from T' and set $A' \in V(T')$ to be the bag of T' corresponding to $A \in V(T)$, such that $A' = A \setminus M$. Since M is a bag of T, T' is a forest $T' = (T'_1, \dots, T'_m)$; T_i is the subtree of T corresponding to T'_i. For any $A_i \in V(T_i), A_j \in V(T_j)$, we have that $A_i \cap A_j \subseteq M$, by the tree decomposition property; thus $A'_i \cap A'_j = \emptyset$. Let N_i be the set of agents that occur in T'_i, then $N = M \uplus N_1 \uplus \cdots \uplus N_m$; that is, $M, N_1, \dots, N_m$ form a partition of N. Moreover, since M is the median node, it holds that $|V(T_i)| < |V(T)|/2$; finally, $tw(T'_i) \leq tw(T)$ for all i.

By our induction hypothesis, there is a payoff vector $\boldsymbol{x}_i \in \mathbb{R}^{|N_i|}$, such that:

1. $\boldsymbol{x}_i$ stabilizes $G_i = \langle N_i, v\rangle$, i.e., for all $S \subseteq N_i$, $x_i(S) \geq v(S)$.
2. $x_i(N_i) \leq (k+1)\log_2(|N_i|)v(N_i)$.

We define $\boldsymbol{x}$ by setting $x_j = x_i(j)$ for any $j \in N_i$. We complete the vector by setting $x_j = v(N)$ for each $j \in M$. For the stability of of $\boldsymbol{x}$, note that any $S \subseteq N$ either intersects with M (in which case it is paid at least $v(N) \geq v(S)$), or is contained in some N_i. In the latter case we have that $x(S) = x_i(S) \geq v(S)$ by induction hypothesis.

In total, we distribute a payoff of

$$\begin{aligned}
x(N) = x(M) + \sum_{i=1}^{m} x(N_i) &\leq (k+1)v(N) + \sum_{i=1}^{m}(k+1)\log_2(|V(T_i)|)v(N_i) \\
&< (k+1)v(N) + \sum_{i=1}^{m}(k+1)(\log_2(|V(T)|/2))v(T_i) \\
&= (k+1)\left(v(N) + (\log_2(|V(T)|)-1)\sum_{i\leq m} v(N_i)\right) \\
&\leq (k+1)\left(v(N) + (\log_2(|V(T)|)-1)v(N)\right) \qquad \text{(by superadditivity)} \\
&= (k+1)\log_2(|V(T)|)v(N),
\end{aligned}$$

which completes the induction step.

Finally, since $|T| \leq 4n$, we have that

$$\text{RCoS}(G|_H) \leq (k+1)\log_2(|V(T)|) \leq (k+1)\log_2(4n) = (k+1)(\log_2(n)+2). \quad \square$$

4.4 Tightness

We argue that the bound of k conjectured above is tight, in the sense that for every k, there is a (simple) superadditive game G and a network H of treewidth k, s.t. RCoS$(G|_H) \geq k$. We provide an explicit construction for $k = 2$ and $k = 3$. Note that it is sufficient to describe a particular decomposition T rather than a graph H.

Example 1 ($k = 2$). Consider a simple game G whose decomposition is $R = \{a, b, c\}$ with three children $W = \{a, w, b\}$; $Y = \{b, y, c\}$; $Z = \{c, z, a\}$. The minimal winning coalitions are $(a, b, c), (w, a, z), (y, b, w), (z, c, y)$. Note that all winning coalitions intersect, and thus G is superadditive. We leave it for the reader to check that indeed RCoS$(G) =$ RCoS$(G|_H) = 2$.

Example 2 ($k = 3$). We construct the tree decomposition T as follows. The root is $R = \{a, b, c, d\}$. For every $i \in R$, we add two variables z_i, y_i, and two bags $Z_i = (R \setminus \{i\}) \cup \{z_i\}$, and $Y_i = (R \setminus \{i\}) \cup \{y_i\}$. All 8 new bags are children of R.

For every $i \in R$, we define two winning coalitions that contain i. Note that for each i there are exactly 6 elements from $\{z_j, y_j\}_{j \in R}$ whose subscript is *not* i. We partition them so that E_i contains i and 3 elements with distinct subscripts, and F_i contains i and the other 3 elements. It is possible to partition so that for all $j \neq j'$, both $E_j \cap E_{j'} = \{z_i\}$ and $E_j \cap F_{j'} = \{z_{i'}\}$. In fact, the intersection points are z_i and $z_{i'}$ s.t. $\{j, j', i, i'\} = R$. By symmetry, $F_j \cap F_{j'} = \{y_i\}$ and $F_j \cap E_{j'} = \{y_{i'}\}$. Thus all 8 winning coalitions intersect, and all of them intersect the 9th coalition R. We enumerate all minimal winning coalitions of one possible construction explicitly:

$$E_a = \{a, z_b, z_c, z_d\} \; E_b = \{b, y_a, y_c, z_d\} \; E_c = \{c, y_a, y_b, z_d\} \; E_d = \{d, y_a, y_b, z_c\}$$
$$F_a = \{a, y_b, y_c, y_d\} \; F_b = \{b, z_a, z_c, y_d\} \; F_c = \{c, z_a, z_b, y_d\} \; F_d = \{d, z_a, z_b, y_c\}$$

What is the minimal payoff? there are two types of agents: R and $N \setminus R$. W.l.o.g. we pay α to the first type and β to the second. Since R is winning, $4\alpha \geq 1$. Since E_j, F_j are winning, $\alpha + 3\beta \geq 1$. Thus $3x(N) = 3(4\alpha + 8\beta) = 4\alpha + 8(\alpha + 3\beta) \geq 1 + 8 = 9$, i.e., CoS$(G|_H) = x(N) \geq 3 = k$.

5 Discussion

The main objective of our work is to provide deeper insights as to the effects limited interaction has on the outcome of cooperative games. In this context, it is crucial to obtain a meaningful measure of the complexity of agent interaction.

We chose to focus on the *treewidth* of the interaction graph in the Myerson model as our measure, and analyzed its effect both on stability and on tractability of optimization problems. At least the optimization question is quite natural, as in many combinatorial problems bounded treewidth guarantees the existence of efficient algorithms (including computational problems in the area of cooperative games, see Section 1.1).

Our results in Section 3 show that if the interaction graph is *acyclic* (i.e. has a treewidth of 1), then efficient algorithms indeed exist for finding the optimal coalition structure in simple games. However, this is not the case for graphs with bounded treewidth (even bounded by 2). These results are in line with the results of Greco et al. [17], which reached similar conclusions w.r.t. other computational problems. It therefore seems reasonable to conclude that the treewidth of Myerson's interaction graph is largely irrelevant to the computational complexity of related questions (except for the special case of 1). We believe that other parameters should be examined if one aims to find efficient algorithms.

In contrast, we prove a close connection between the treewidth of the interaction graph and the *stability* of cooperative games, and we conjecture this connection to be even tighter. This is while the *degree* of the graph seems to be an almost irrelevant factor (see Proposition 5).

A relatively natural extension of Conjecture 1 is a relaxation of the superadditivity requirement, which results in bound on the effort required to stabilize the *optimal coalition structure*. Some results in this direction have been obtained w.r.t. previous bounds on the CoS [11, 20, 17].

6 Conclusions and Future Work

We provided various evidence that the treewidth (of the interaction graph) is the "correct" parameter when considering bounds on stability, but less so when studying computational complexity.

Further research is in order. First, Conjecture 1 remains open (however see Footnote 4); it would appear that greedy approaches similar to the one adopted by Demange [11] do not provide the desired bound. Second, it would be useful to find other meaningful parameters (whether properties of the interaction graph, or of other parts of the game description), that leads to good bounds on the CoS.

Finally, the study of interaction graphs in non-cooperative games could contribute much to a better understanding of coalitional stability in a broad sense.

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