Compact Name-Independent Routing with Minimum Stretch

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Abstract

Given a weighted undirected network with arbitrary node names, we present a compact routing scheme, using a $\widetilde{O}(\sqrt{n})$ space routing table at each node, and routing along paths of stretch 3, that is, at most thrice as long as the shortest paths. This is optimal in a very strong sense. It is known that no compact routing using $o(n)$ space per node can route with stretch below 3. Also, it is known that any stretch below 5 requires $\Omega(\sqrt{n})$ space per node.

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1 Introduction

Consider an n-node weighted undirected graph $G = (V, E, \omega)$. Each node in V is given an arbitrary unique name in $\{1, \ldots, n\}$. In addition, for each node $u \in V$, each out going edge is given an arbitrary unique port name in $\{1, \ldots, deg(u)\}$.

A *routing scheme* is a distributed algorithm that, given a destination node's name, allows any node to route messages that will eventually arrive to the destination node. Specifically, a routing scheme can be viewed as a function on each node that maps from a given header and incoming port number to an outgoing port number and new message header. For example, the trivial solution to routing on shortest paths is for each node to store $(n-1)$ entries that contains the next hop of an all pairs shortest path algorithm. This solution requires each node to store $\Omega(n \log n)$ bits of routing information and thus does not scale well as the number of nodes in the system increases.

In order to reduce memory overhead and incur routing costs that are proportional to the actual distances between interacting parties, there are two parameters that routing schemes aim to minimize:

- *Stretch*: the maximum ratio over all pairs between the cost of the path taken by the routing scheme and the cost of a minimum cost path for the same source-destination pair.
- *Memory*: the maximum over all nodes of the number of bits stored for the routing scheme.

Routing schemes which require nodes to storing a liner number of bits are not scalable. The challenge is in providing a *compact routing scheme* that minimizes the stretch bound for any weighted graph while requiring only $o(n)$ bits of routing information per node. We refer the reader to Peleg's book [17] and to the surveys of Gavoille and Peleg [10, 12] for comprehensive background on compact routing schemes.

Gavoille and Gengler [11] show that compact routing schemes must have stretch of at least 3. Specifically they prove that there exists n node networks in which any scheme with stretch less than 3 requires a total of $\Omega(n^2)$ bits of routing information. Thorup and Zwick [20] show that any scheme with stretch less than 5 must have networks which require $\Omega(n^{3/2})$ bits of routing information. Hence these lower bounds imply that compact routing schemes must have at least stretch 3 and stretch 3 routing schemes require at least $\Omega(\sqrt{n})$ bits per node.

The problem of devising compact routing schemes has two basic variants: *labeled routing* and *name independent routing*. Awerbuch et al. were the first to distinguish in [2] between solutions that allow/disallow the designer to choose labels for nodes as part of the solution. The variant that allows the designer to name nodes with arbitrary labels is called *labeled routing*. In this model, the node's label may contain valuable topology dependent information useful for routing, usually with poly-log size labels. This tends to make the design of solutions easier. The variant that does not allow labeling of nodes in this way is called *name independent routing*. In this variant node names may be chosen arbitrarily by an adversary. This makes routing generally harder: Intuitively, the routing algorithm must first discover information about the location of the target, and only then route to it.

Indeed, optimal compact routing schemes for labeled routing are already known. Eilam et al. [7] present a stretch 5 labeled scheme with $\widetilde{O}(n^{1/2})$ memory, whereas Cowen [6] presents a stretch 3 labeled scheme with $\widetilde{O}(n^{2/3})$ memory. Later, Thorup and Zwick [21] improve to stretch 3 using

only $\widetilde{O}(n^{1/2})$ bits. They also give a generalization of their scheme and using techniques from their distance oracles [20], achieve labeled schemes with stretch $4k-5$ (and even $2k-1$ with handshaking) using $\widetilde{O}(n^{1/k})$ bit routing tables. Additionally, there exist various labeled routing schemes suitable only for certain restricted forms of graphs. For example, routing in a tree is explored, e.g., in [8, 21], achieving optimal routing. This routing requires $O(\log^2 n/\log\log n)$ bits for local tables and for headers, and this is tight [9].

As for name independent routing, the situation is quite different. Initial results in [3] provide stretch 3 non-compact name independent routing with $\widetilde{O}(n^{3/2})$ total memory. However this scheme is unbalanced, $\Omega(\sqrt{n})$ nodes must store $\Omega(n)$ bits of routing information. Awerbuch and Peleg [4] were the first to show that constant-stretch is possible to achieve with $o(n)$ memory per node, albeit with a large constant. Recently, Arias et al. significantly reduced the stretch factor in [1], providing stretch 5 with $\widetilde{O}(\sqrt{n}\,)$ memory per node. However, these results leave a gap between the known lower bound of stretch 3.

1.1 Our results

We present the first optimal compact name independent routing scheme for arbitrary undirected graphs. The scheme has stretch 3, and requires $\widetilde{O}(\sqrt{n}\,)$ bits of routing information per node. Given the graph and the node names, we can construct the routing information deterministically in polynomial time. Moreover, when routing along our stretch 3 paths, each routing decision is performed in constant time.

Besides improving Arias et al. [1] stretch from 5 to 3, our results answer affirmatively the challenge of optimal name independent routing that was open since the initial statement of the problem in 1989 [3]. Surprisingly, our results show that allowing the designer to label the nodes does **not** improve the stretch factor compared to the task when node labels are predetermined by an adversary.

We note that our solution does not contain any strikingly new technique. Rather our new scheme is a non-trivial combination of simple standard techniques.

2 Preliminaries

Consider a set V of n nodes wishing to participate in a distributed routing scheme. We assume the nodes are labeled with an arbitrary permutation of the integers $\{1,\ldots,n\}$.

We assume a graph $G=(V,E,\omega)$ with nonnegative edge cost ω. For $u,v\in V$, let $d(u,v)$ denote the cost of a minimum cost path from u to v in G, were the cost of a path is the sum of the weights of along its edges.

Each node has, for each outgoing edge, a unique name from the set of integers $\{1,\ldots,n\}$. We assume the fixed-port model [8]. In this model the name of the outgoing edge is fixed *before* the adversary chooses the permutation of node labels. Thus the name of the outgoing edge has no connection to the label of the node on the other side of the edge.

When a node wants to send a message, we assume that initially the sender only knows the name of the destination node. This destination is written in the header of the message. We require *writable packet headers*, namely, we allow the routing algorithm to write a reasonable amount of information into the headers of messages as they are routed. In our case, we use $O(\log^2 n/\log\log n)$

bit headers.

We note that our use of headers aim at useful tradeoffs between current techniques used in the real world: source directed routing, where the source puts the whole path to the destination in the header, and routing based on routing tables each router knows how to forward packets to any destination. For source directed routing, the header may be very large, and for the other routing, the routing tables may become huge. In either case, we have problems with scaling. Our point here is that a small amount of information in the header can dramatically reduce the amount of information needed at the routers.

It is no coincidence that our scheme and indeed all previous name independent schemes use writable packet headers. A scheme that does not re-write packet headers must be loop free and thus must have stretch 1 on any tree. Clearly in a tree that is a star the center would have to code a permutation using $\Omega(n \log n)$ bits on the average.

Lemma 2.1 *There do not exist loop free name independent routing schemes with $o(n)$ bits for each node on every graph.*

As for lower bounds for compact routing, note that for the related problem of labeled routing, the work of [11] shows that any stretch < 3 scheme must use a total of $\Omega(n^2)$ bits. Thus it cannot be the case that all nodes use $o(n)$ bits. This bound clearly holds also for the name independent model.

Actually, a slightly stronger memory bound of $\Omega(n^2 \log n)$ bits for stretch < 3 can be proven for the name independent model. This is derived by examining the complete bipartite graph $K_{n/2,n/2}$ with uniform weights (likewise, the metric space it induces). For stretch < 3, each node must route optimally to its distance one neighbors. By counting all the permutations on names it is clear that each node must use $\Omega(n \log n)$ bits.

Lemma 2.2 *Any name independent routing scheme with $o(n \log n)$ bits per node must have stretch at least 3.*

3 The Stretch 3 Scheme

In Sections 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, we will first present some simple ingredients for our optimal stretch 3 scheme. Then, in Section 3.8, we will combine them in our optimal solution. Finally, in Section 3.9, we prove that the scheme has the optimal stretch of 3.

3.1 Vicinity balls

For every integer $\kappa \geq 1$, and for a node $u \in V$, let the *vicinity* of u, denoted by $B_\kappa(u)$, be the set of the κ closest nodes to u, breaking ties by lexicographical order of node names. Vicinities satisfy the following Monotonicity Property:

Property 3.1 [3] *If $v \in B(u)$, and if w is on a minimum cost path from u to v, then $v \in B(w)$.*

Proof. Seeking a contradiction, suppose $v \notin B(w)$. For any $z \in B(w)$ we have $d(w,z) < d(w,v)$ (or $d(w,z) = d(w,v)$ and $z < v$). So $d(u,z) < d(u,w) + d(w,v)$ (or $d(u,z) = d(u,w) + d(w,v)$ and

$z < v$). Since w is on a minimum cost path from u to v then $d(u,z) < d(u,v)$ (or $d(u,z) = d(u,v)$ and $z < v$) hence $B(w) \subset B(u)$. But since $v \in B(u)$ we have $|B(w)| < |B(u)|$ a contradiction. □

Hereafter, the size of the vicinities is set to $\kappa = \lceil 4\alpha\sqrt{n}\log n \rceil$, where $\alpha > 2$ is some constant fixed in Section 4 and denote simply by $B(u) = B_\kappa(u)$. Let $b(u)$ denote the radius of $B(u)$, $b(u) = \max_{w \in B(u)} d(u,w)$.

As in previous compact routing schemes (see, e.g., [2, 6]), each node u will know its vicinity $B(u)$. We assume that u has a standard dictionary over the names in $B(u)$ so that in constant time it can check membership and look up associated information.

3.2 Coloring

Our construction uses a partition of nodes into sets $C_1, \ldots, C_{\sqrt{n}}$, called *color-sets*, with the following two properties:

Property 3.2

1. *Every color-set has at most* $2\sqrt{n}$ *nodes.*
2. *Every node has in its vicinity at least one node from every color-set.*

From here on, if node $u \in C_i$ we say that it has "color i", and denote $c(u) = i$. By standard Chernoff bounds Property 3.2 clearly holds w.h.p. if every node independently chooses a random color. Constructing a polynomial-time coloring satisfying Property 3.2 is discussed in Section 4.

3.3 Hashing names to colors

We shall assume a mapping h from node names to colors which is balanced in the sense that at most $O(\sqrt{n})$ names map to the same color. Each node u should be able to compute $h(w)$ for any destination w. If the names were a permutation of $\{1, \ldots, n\}$, we could just extract $\frac{1}{2}\log n$ bits from the name, but we want to deal with arbitrary names such as IP addresses. Arias et al. [1] suggest to use a $(\log n)$-universal hash function for a similar purpose. In Section 5 we will present a function h that can be computed in constant time.

3.4 Stretch 3 for metric spaces

To illustrate the use of these first three ingredients, we here observe a very simple stretch 3 scheme with $\widetilde{O}(\sqrt{n})$ bits per node for any metric space, i.e., for a distributed network in which the cost of communication between nodes is governed by a distance function.

Every node u stores the following:

1. The names of all the nodes in $B(u)$.
2. The names of all the nodes v such that $c(u) = h(v)$.

Routing from u to v is done in the following manner:

1. If $v \in B(u)$ or $c(u) = h(v)$ then u routes directly to v with stretch 1.
2. Otherwise, u forwards the packet to $w \in B(u)$ such that $c(w) = h(v)$. Then from w the packet goes directly to v. The stretch is at most 3 since $d(u,w) + d(w,v) \leq d(u,v) + 2d(u,v)$.

Note that in a general graph the main difficulty is in implementing the path from w to v.

3.5 Routing on trees

We make use of the following result concerning topology-dependent labeled routing:

Lemma 3.3 [8, 21] *There is a routing scheme for any tree T with n nodes that routes optimally between every pair of nodes. The storage per node in T and the header size are $O(\log^2 n / \log\log n)$ bits. Given the information of a node and the label of the destination, routing decisions take constant time.*

For a tree T containing a node v, we let $\mu(T,v)$ denote the routing information of node v from Lemma 3.3 and $\lambda(T,v)$ denote the destination label of v in T.

We shall apply Lemma 3.3 to several trees in the graph. Each node will participate in $\widetilde{O}(\sqrt{n}\,)$ trees, so its total tree routing information will be of size $\widetilde{O}(\sqrt{n}\,)$.

3.6 Landmarks

Borrowing from [3, 6], we designate one color to be special, e.g., we set color 1 as the special color and call its color *red*. We use the red nodes as routing *landmarks*. Let L denote the set of red nodes. We have $|L| \leq 2\sqrt{n}$. We have already shown how to assign colors such that for every $v \in V$, the vicinity $B(v)$ contains one red node from L. For a node $v \in V$, let ℓ_v denote the closest red node in $B(v)$. For any red node $\ell \in L$, denote by $T(\ell)$ a single-source minimum-cost-path tree rooted at ℓ. Note that there are only $\widetilde{O}(\sqrt{n}\,)$ red trees.

3.7 Partial shortest path trees

For any node u let $T(u)$ denote the single-source minimum-cost-path tree rooted at u. In a partial shortest path tree, every node v maintains $\mu(T(u),v)$ if and only if $u \in B(v)$. Notice that the set of nodes that maintain $\mu(T(u),\cdot)$ is a subtree of $T(u)$ that contains u.

Lemma 3.4 *If $x \in B(y)$ then given the label $\lambda(T(x),y)$, node x can route to node y along a minimum cost path.*

Proof. By Property 3.1 for any node w on the minimum cost path of $T(x)$ between x and y we have $x \in B(w)$. Thus every node w on this path maintains $\mu(T(x),w)$. □

3.8 The stretch 3 scheme

Every node u stores the following:

1. The names of all the nodes in the vicinity $B(u)$ and what link to use to reach them.
2. Routing information $\mu(T(\ell), u)$ of the tree $T(\ell)$ for every red node $\ell \in L$.
3. Routing information $\mu(T(x), u)$ of the tree $T(x)$ for every node $x \in B(u)$.
4. The names of all the nodes v such that $c(u) = h(v)$. For every node v such that $c(u) = h(v)$, store one of the following two options that produces the minimum cost path out of the two:
   (a) Store the labels $\langle\, \lambda(T(\ell_v), \ell_v),\ \lambda(T(\ell_v), v)\,\rangle$. The routing path in this case would be from u to $\ell_v \in B(v)$ using $\lambda(T(\ell_v), \ell_v)$ on the tree $T(\ell_v)$, and from ℓ_v to v using $\lambda(T(\ell_v), v)$ on the same tree $T(\ell_v)$.
   (b) Let $P(u, w, v)$ be a path from u to v composed of a minimum cost path from u to w, and of a minimum cost path from w to v with the following properties: $u \in B(w)$, and there exists an edge (x, y) along the minimum cost path from w to v such that $x \in B(w)$ and $y \in B(v)$. Among all these paths choose the lowest cost path $P(u, w, v)$ and store the labels $\langle\, \lambda(T(u), w),\ x,\ (x \to y),\ \lambda(T(y), v)\,\rangle$.
   The routing path in this case would be from u to w on $T(u)$ using $\lambda(T(u), w)$. This part is possible by Lemma 3.4 on $u \in B(w)$. Then from w to y since $x \in B(w)$ and the port number $(x \to y)$ is stored. Finally from y to v on $T(y)$ using $\lambda(T(y), v)$. This part is possible by Lemma 3.4 on $y \in B(v)$.

Routing from u to v is done in the following manner:

1. If $v \in B(u)$ or $v \in L$ (v is a red node) or $c(u) = h(v)$ then u routes to v using its own information.
2. Otherwise, u forwards the packet to $w \in B(u)$ such that $c(w) = h(v)$. Then from w the packet goes to v using w's routing information.

3.9 Analysis

Theorem 3.5 *Let $s, t \in V$ be any two nodes. The route of the above scheme from s to t has stretch at most 3.*

Proof. There are three cases to consider (see Fig. 1):

1. If $t \in B(s)$ or $t \in L$ then s routes on a minimum cost path directly to t.

Otherwise, denote $d = d(s, t)$, let z be a node such that $z \in B(s)$ and $c(z) = h(t)$. For the case $c(s) = h(t)$, we set $z = s$. Let $p(z, t)$ be the cost of the path chosen by z as the lowest cost path from z to t among options 4a and 4b.

2. If on every minimum cost path from s to t there is a node y such that $y \notin B(s)$ and $y \notin B(t)$ then $b(s) + b(t) \leq d(s, t)$.
   By examining option 4a the cost $d(s, z) + p(z, t)$ of the path taken by our routing scheme is bounded by the cost of the path $s \rightsquigarrow z \rightsquigarrow \ell_t \rightsquigarrow t$. Thus $d(s, z) + p(z, t) \leq d(s, z) + d(z, \ell_t) + d(\ell_t, t) \leq b(s) + [b(s) + d + b(t)] + b(t) \leq 3d$.

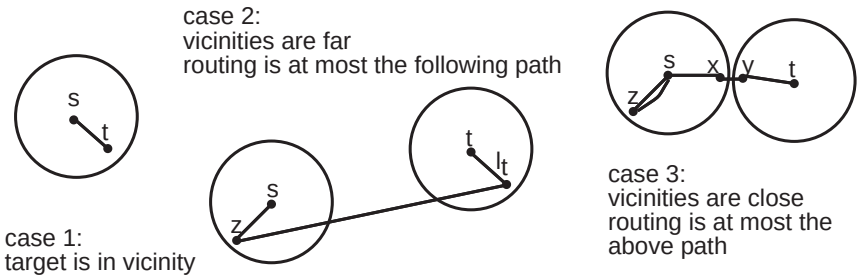

Figure 1: The three cases

3. If there exists a minimum cost path, in which every node is in $B(s) \cup B(t)$ then let (x, y) be an edge of this path such that $x \in B(s)$ and $y \in B(t)$.

   By examining the best choice in option 4b the cost $d(s, z) + p(z, t)$ of the path taken by our routing scheme is bounded by the cost of the path $s \rightsquigarrow z \rightsquigarrow s \rightsquigarrow x \rightarrow y \rightsquigarrow t$. Thus $d(s, z) + p(z, t) \leq d(s, z) + d(z, s) + d(s, t) \leq b(s) + b(s) + d \leq 3d$.

□

4 On Polynomial Time Coloring

In this section, we discuss how to "derandomize" the coloring discussed in Section 3.2. We shortly describe how the coloring of vertices can be done deterministically in polynomial time. Let us first consider our coloring problem in a slightly more abstract setting.

Given are m subsets of $\{1, \ldots, n\}$, $B_1, \ldots, B_m$, where $|B_i| \geq \alpha k \log n$, for all i where k is a parameter and α is some constant large enough. Our task is to color the n items with k colors, $c : \{1, \ldots, n\} \rightarrow \{1, \ldots, k\}$, such that the following properties hold: (a) Each color appears at most $(\alpha n \log n)/k$ times; (b) Each color appears in each set B_i. (In our application the n items are the vertices, $m = n$, the m sets are the vicinities of the vertices, and $k = \sqrt{n}$.)

Clearly a random coloring will do the trick, as long as m is polynomial in n. Very coarse calculations reveal that the probability that any fixed condition of type (a) or type (b) is not satisfied is bounded by $n^{-\alpha}$, for any value of k. Thus the probability that some condition is not satisfied is bounded by $(mk + k)n^{-\alpha}$.

How can we derandomize this construction and obtain an explicit construction? While $O(\sqrt{n} \log n)$-wise independence suffices for the argument above, this still does not provide a deterministic construction. It turns out that this can be done using the pseudo-random generators for space bounded computation of [16]. Each of the conditions can be computed in Logspace and in [15] it is shown how the seed of the pseudo-random generator can be incrementally chosen as to pass a Logspace test.

Specifically, we can look at the output string of the generator as being $y_1, \ldots, y_n$, where each y_i is of length $\log k$ bits and denotes the color of item i. Now, a Logspace machine with a one-way access to this stream, can verify by simple counting, for any fixed color j, that color j appears at most $(\alpha n \log n)/k$ times. Similarly, such a machine can verify, for any fixed B_i and any fixed color

j, that B_i includes an element of color j. It follows that the output of the generator will also almost surely pass each one of these tests.

Since the generator of [16] accepts $O(\log^2 n)$ bits as input, this is still not a complete derandomization. However, in [15] it is shown how the input bits of the generator can be incrementally chosen in polynomial time such that the output passes a given Logspace test. As described in detail in [18], the same algorithm with the same argument actually allows choosing the input bits such that the output passes a given *polynomial size family* of Logspace tests. As mentioned above, this is exactly the case here.

5 On Hashing in Constant Time

In this section, we will implement a constant time hashing function h from n arbitrary names to $\sqrt{n}$ colors, as assumed in Section 3.3. The constant time assumes that we can perform standard arithmetic operations on names in constant time, hence that each name is stored in a constant number of words. For example, these names could be IP addresses. The representation of our hash function will take $O(\sqrt{n})$ space. We can store such a representation with each node without violating our space bounds.

In previous work of Arias et al. [1] they used a $(\log n)$-universal hash function, but with current implementations via degree-$(\log n)$ polynomials, the evaluation of this function takes more than constant time. With the constant time hash function we propose, each routing decission is made in constant time.

We will now give a randomized construction of the hash function which works with high probability. For simplicity we assume $\sqrt{n}$ is a power of two so that $(\log n)/2$ is an integer. First we use a standard universal hash function h_0 mapping names into $[n^{2.5}]$ in constant time. With high probability this mapping is collision free (see, e.g., [14, §8.4.1]).

Set $q = (\log n)/2$. We are now dealing with n distinct reduced names of $5q$ bits, and we want to get down to colors of q bits. We will use an idea of Tarjan and Yao [19]. For $i = 1, \ldots, 4$, let T_i be a random table mapping q bits into $(5-i)q$ bits. Note that each table has $2^q = \sqrt{n}$ entries. We then hash a $(5q)$-bit reduced name x as follows. For $i = 1, \ldots, 4$, let y be the q least significant bits of x, and set $x = T_i[y] \oplus (x \gg q)$. At the end, x has only $q = (\log n)/2$ bits which we return as the color.

The above computation of colors from reduced names takes constant time. We will now bound the number of reduced names mapping to each color.

Lemma 5.1 *W.h.p., there are $O(\sqrt{n})$ reduced names mapping to each of the $\sqrt{n}$ colors.*

Proof. We start with n disjoint $5q$-bit reduced names. In each of 4 iterations, the names get further reduced by q bits. In this process, some names collide. Let's consider an iteration starting with $(i+1)q$ bits, and reducing to iq bits. For any iq-bit name x, let n_x be the number of original names reduced to x. Also, let m_i be the maximal number of original names reduced to the same iq-bit name. We will show that $m_1 \leq 3\sqrt{n}$ w.h.p.

Now consider an arbitrary iq bit name x. By symmetry we expect the same number of original names mapping to each x, and the mean number μ_i is

$$\mu_i = E[n_x] = n/2^{iq} = n^{1-i/2}.$$

Using Chernoff bounds, we will argue that n_x is unlikely to deviate a lot from its mean. Let zy be an $(i+1)q$-bit name with y the q least significant bits. Now zy is reduced to z if and only if $x = T_{i+1}[y] \oplus z$. It follows that

$$n_x = \sum_y n_{x \oplus T_{i+1}[y]}.$$

Each of these terms is independent since the different $T_{i+1}[y]$ are independent. Moreover, each term is bounded by m_{i+1}. Using standard Chernoff bounds (see, e.g., [14, eq. (4.10)]), it follows that

$$\Pr[n_x > k\mu_i] \leq (e/k)^{k\mu_i/m_{i+1}}.$$

When we start with $i = 4$, w.h.p., we may assume that $m_5 = 1$. Moreover, we have $\mu_4 = n^{1-4/2} = 1/n$. Set $k = en$. We get that $\Pr[n_x > e] < (1/n)^e$. There are n^2 possible values of x, so the probability that $n_x < e$ for all of them is at most $n^{2-e} = o(1)$. Thus, w.h.p., we may assume that $m_4 \leq e$. Similarly, for $i = 3$, we can prove that $m_3 \leq 4e$ w.h.p.

Now, for $i = 2$, we have $\mu_4 = n^{1-4/2} = 1/n$. We set $k = \log n$. We get that $\Pr[n_x > \log n] < (e/\log n)^{(\log n)/4e} = 1/n^{\omega(1)}$, so we conclude that $m_2 \leq (\log n)$ w.h.p.

Finally, for $i = 1$, we have $\mu_4 = \sqrt{n}$. We set $k = 3$. We get that $\Pr[n_x > 3\sqrt{n}] < (e/3)^{3\sqrt{n}/\log n} = 1/\exp(\Omega(n))$. Thus we conclude that $m_1 \leq 3\sqrt{n}$ w.h.p. □

Thus it follows that we expect a maximum of $O(\sqrt{n})$ reduced names to map to the same color. Moreover the mapping took constant time, and its representation took $O(\sqrt{n})$ space. This completes our randomized construction of the desired hash function. The construction uses the same ingredients as are used for the deterministic dictionaries of Hagerup et al. [13]. Using the techniques from [13], we can derandomize our construction to run in $O(n \log n)$ time.

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