Oblivious Rounding
and the
Integrality Gap

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Setting: A Maximization Problem \((V, X)\)

**IN GENERAL**

- \(X\) = set of feasible solutions
- \(V\) = set of linear objective functions

\(X, V\) are sets of non-negative real vectors

**EXAMPLE:** MAX-CUT in complete weighted graphs of \(n\) vertices

- \(X\) = set of cuts
- \(V\) = set of edge weight functions

\(X, V\) are sets of vectors of dimension \(\binom{n}{2}\) (vectors in \(X\) are \(\{0,1\}\)-vectors)

Given \(v \in V\), find \(x \in X\) that maximizes \(v \cdot x\)
Oblivious Rounding

Classic approach to hard maximization problem \((V, X)\):

1. Relax \((V, X)\) to \((V, Y)\) where \(X \subseteq Y\) and \(Y\) is fractional
2. Given \(v \in V\) find \(y \in Y\), guarantees \(v \cdot y\)
3. Round \(y\) to \(x \in X\)

The approximation ratio of the rounding is \(\frac{v \cdot x}{v \cdot y}\) (in the worst case)

If step 3 does not use \(v\), we call the rounding "oblivious"*

*Not to be confused with [Young’95]
Examples from the Literature

OBLIVIOUS ROUNDBING

Threshold rounding for vertex cover
  ◦ [Hochbaum’82]

Randomized rounding for set cover
  ◦ [Raghavan-Thompson’87]

Random hyperplane rounding for max-cut
  ◦ [Goemans-Williamson’95]

Welfare maximization for submodular valuations
  ◦ [Feige’09, Feige-Vondrak’10]

NON-OBLIVIOUS ROUNDBING

Rounding of SDPs for CSP
  ◦ [Raghavendra-Steurer’09]

Facility location
  ◦ [Li’13]

Welfare maximization for gross substitutes valuations
  ◦ [Nisan-Segal’06]
Main Question

For which problems and relaxations can we expect oblivious rounding to give a good approximation ratio?

A question about information
- Rounding not restricted to be in polynomial time

Answer useful for:
- Algorithm designers
- Mechanism designers (details in a few slides)
Main Result & Application
Main Result

[Informal] The approximation ratio of the best oblivious rounding scheme for a given relaxation = the integrality gap of the problem’s closure

The closure of problem \((V, X)\) is \((C(V), X)\)
- where \(C(V)\) is the convex closure of \(V\)

Corollary: If a problem is closed \((V = C(V))\) then oblivious rounding can achieve the integrality gap
Main Result Illustration

Problem \((V, X)\) \quad \text{Integrity gap} \quad \text{Relaxation} \quad \text{Integrity gap} \quad \text{Oblivious rounding} \quad (V, Y) \\
(C(V), X) \quad \text{Closure} \quad (C(V), Y) \quad \text{Relaxed closure} \quad = \quad \text{Oblivious rounding approximation ratio}
An Application: Welfare Maximization

Informally: Allocate $m$ indivisible items among buyers to maximize total value
- Each buyer $i$ has a valuation function $v_i : 2^m \rightarrow \mathbb{R}_{\geq 0}$
- Valuations belong to classes (e.g., additive, submodular, ...)

More formally:
- $v \in V = \text{the buyers' valuations, from class } V$
- $x \in X = \text{an allocation}$
- $y \in Y = \text{an allocation as if the items were divisible}$

(all sets of vectors of dimension $2^m$)
Main thm: Approximation ratio of oblivious rounding = integrality gap of problem’s closure

Welfare Max: The Relaxation

Problem: Indivisible items

Relaxation: Divisible items

\[ \text{Welfare} = 5.5 \]

\[ \text{Welfare} = 7 \]

Integrality gap
Welfare Max: The Relaxation $X \rightarrow Y$

The relaxation used in $\approx$ all welfare approximation algorithms: Configuration LP

Problem:
\[
\max_{i,S} \sum x_{i,S} v_{i,S} \\
\text{s.t.} \\
\sum_S x_{i,S} \leq 1 \text{ for every buyer } i \\
\sum_{i,S:j \in S} x_{i,S} \leq 1 \text{ for every item } j \\
x_{i,S} \in \{0, 1\}
\]

Relaxation:
\[
\max_{i,S} \sum y_{i,S} v_{i,S} \\
\text{s.t.} \\
\sum_S y_{i,S} \leq 1 \text{ for every buyer } i \\
\sum_{i,S:j \in S} y_{i,S} \leq 1 \text{ for every item } j \\
y_{i,S} \geq 0
\]
Welfare Max: The Closure $V \rightarrow C(V)$

Let $V$ be unit-demand (UD) valuations:

- $v_i(S) = \max_{\text{item } j \in S} \{v(j)\}$
- Subclass of gross substitutes (GS)
- So integrality gap of configuration LP = 1

Main thm: Approximation ratio of oblivious rounding = integrality gap of problem’s closure
Main Result Applied to Welfare Max.

**Theorem:** For welfare maximization with unit-demand valuations, oblivious rounding of solutions to the configuration LP achieves $\leq 0.782$ approximation ($0.833$ for $2$ buyers)

- Despite the integrality gap of $1$ for unit-demand/GS

**Conclusion:** For welfare maximization, “ignorance is not always bliss”
- Need to know the valuations to round the fractional allocation

**Proof:** The integrality gap of the configuration LP for coverage valuations is no better than $0.782$ [cf. Feige-Vondrak’10], and coverage is the closure of unit-demand
Implications for Mechanism Design

Advantages of oblivious rounding for welfare maximization with strategic buyers:

1. Incentive compatibility
   ◦ [Duetting-Kesselheim-Tardos’15]: Can embed into a mechanism that approximately maximizes welfare in equilibrium

2. Fairness
   ◦ Treats buyers equally, approximation ratio holds per buyer

3. Communication
   ◦ Does not access exponential-sized valuations

Motivates understanding the possibilities/limitations of oblivious rounding
Sketch of Main Proof
Approx. Ratio and Integrality Gap at $y$

Fix a fractional solution $y \in Y$

\[
\text{INTEGRALITY GAP OF CLOSURE AT } y \\
\inf_{v \in C(V)} \max_{x \in X} \frac{v \cdot x}{v \cdot y}
\]

First choose worst-case $v$ from the closure
Then find the best integral solution $x$

\[
\text{APPROXIMATION RATIO OF BEST OBLIVIOUS Rounding AT } y \\
\max_{x \in C(X)} \inf_{v \in V} \frac{v \cdot x}{v \cdot y}
\]

First choose best randomized rounding
Then find the worst-case $v$ for this rounding

This is where the obliviousness comes in
Applying the Minimax Theorem

Fix a fractional solution $y \in Y$

\[ \inf_{v \in C(V)} \max_{x \in X} \frac{v \cdot x}{v \cdot y} \]
Choose minimizing mixed strategy

\[ \max_{x \in C(X)} \inf_{v \in V} \frac{v \cdot x}{v \cdot y} \]
Choose maximizing mixed strategy

Main thm: Approximation ratio of oblivious rounding = integrality gap of problem’s closure
Summary

Many commonly-used rounding schemes are oblivious
  ◦ Do not use the objective to round a fractional solution

We study when oblivious rounding suffices for good approximation
  ◦ Focus on information

Approximation ratio equals the integrality gap of a related problem – the closure

Application to welfare maximization
  ◦ Oblivious rounding does not suffice for unit-demand, gross substitutes
  ◦ Suffices for submodular

Another tool for the toolbox of algorithm and mechanism designers
Directions for Future Research

1. Use the understanding of the potential and limitations of oblivious rounding as a guide in designing rounding schemes
   ◦ for problems for which tight approximation ratios not yet known
   ◦ e.g., when best known approximation is oblivious but the problem is not closed

2. Possibilities/limitations of polynomial-time oblivious rounding

3. Other properties of combinatorial problems predicting the success/failure of rounding techniques