INCOHERENT WIDEBAND COMMUNICATIONS OVER MULTIPATH CHANNELS

Elchanan Zwecher and Dana Porrat

The Hebrew University
School of Engineering and CS
Jerusalem, Israel 91904
elchanan.zwecher@mail.huji.ac.il, dana.porrat@huji.ac.il,

ABSTRACT

This paper analyzes the efficiency of spreading modulations over wideband multipath channels. We bring asymptotic wideband results, and focus on the low SNR regime. In coherent systems low SNR implies efficient communications, but in a non-coherent communications the SNR should be high enough in order to enable decoding the message even though the channel is not known. Too high SNR reduces the efficiency of the modulation. For a channel model where path gains are not known while the L path delays are known to the receiver, and for spread spectrum modulations that use the entire bandwidth W at each transmission, one can communicate efficiently in the wideband limit if $\mathrm{SNR} = \omega\left(\frac{L}{W}\right)$. If both path gains and path delays are unknown, efficient communications using spread-spectrum modulations are possible if $\mathrm{SNR} = \omega\left(\log(W/L)/W\right)$, given that the specific modulation is still efficient for such SNR. However, if $\mathrm{SNR} = o\left(\log(W/L)/W\right)$, communications are not efficient.

keywords: Information rates, Fading channels.

1. INTRODUCTION

This paper analyzes the efficiency of spreading modulations over wideband multipath channels. We bring asymptotic wideband results, and focus on the low SNR regime. We check the behavior of the achievable rate of spread spectrum modulations over two multipath channel models, both with exactly L paths: In the first channel the gains are i.i.d. and are unknown, however path delays (that are distributed uniformly) are known to the receiver. In the second channel, both path gains and path delays are unknown. The goal is to find a refined division of the low SNR into different ranges, according to their energy requirement per bit, in channels of both types.

we start by reviewing the mutual information between the transmitted signal X and the received signal $Y = \sqrt{\mathrm{SNR}}TX + Z$

$$I\left(\sqrt{\mathrm{SNR}}TX + Z; X|T\right) \tag{1}$$

where T is a $m \times m$ known channel, X is a m long random vector representing the transmitted signal, Y is the m long received signal and Z is an additive white noise. $E\left[TT^T\right] = I$, $E\left[\|X\|_2^2\right] = m$ and SNR is the signal to noise ratio. The length of X, m, is the number of channel usages per transmission. If we maintain the average power while the bandwidth increases, the number of channel usages which is the product of the bandwidth and time length of the symbol, increases linearly with the bandwidth. As a result SNR decreases, such that in the wideband limit, SNR diminishes.

A basic result regarding coherent communications over a channel represented by a $m \times m$ matrix is this given by [8], for modulations that obey

$$\lim_{\mathrm{SNR}\to 0} \frac{\|E\left[X\right]\|_2^2}{E\left[\|X\|_2^2\right]} = 0$$

states that if the SNR tends to zero, the modulation is efficient, i.e.

$$\lim_{\mathrm{SNR}\to 0} \frac{I\left(X; \sqrt{\mathrm{SNR}}TX + Z|T\right)}{m\mathrm{SNR}} = \frac{1}{2} \tag{2}$$

The mutual information measures the rates and after normalizing the SNR we have the energy per nat of information

$$\lim_{\mathrm{SNR}\to 0} \frac{1}{\frac{I\left(Y; \sqrt{\mathrm{SNR}}TX + Z|T\right)}{m\mathrm{SNR}}} = 2 \tag{3}$$

This is well known to be the minimum energy per nat [6]. Minimum energy per bit is $2/\log_2 e$.

Various modulations have different behavior in low SNR which is still a little higher than zero, or tends to zero depending on the number of path delays. Verdú [8] analyzes the behavior of the spectral efficiency

$$\mathrm{SEF}(\mathrm{SNR}) = \frac{I\left(X; \sqrt{\mathrm{SNR}}TX + Z|T\right)}{m} \tag{4}$$

in the low SNR regime using the second derivative of spectral efficiency around the point $\mathrm{SNR} = 0$. For each modulation the first derivative equals 0.5, and the second derivative, which is always negative, (since spectral efficiency is a concave function of SNR, in a known channel) is at most -0.25. For example, Gaussian signaling over the AWGN channel has a first derivative of 0.5 and second derivative of -0.25. In modulations with a small absolute value of second derivative, the spectral efficiency increases significantly in the low SNR regime. However, there are modulations where the second derivative is very negative, resulting in a very large energy requirement per bit, for SNR that is a little more than zero.

Generally, the ability to communicate efficiently in terms of energy, in a relatively high SNR depends on the modulation and in particular on the spread of energy of the different transmitted signals over a constant spanning base, i.e. where almost every transmitted signal is projected onto a relatively small subspace of the constant base. Examples of such modulations are FSK, PPM and Time-Frequency [1]. As a result, the range of efficient SNR for such modulations is very small. The ability to communicate efficiently in terms of energy, in a relatively high SNR depends on the modulation.

Considering incoherent communications, the channel model is often structured so that its eigenfunctions, i.e. the signal that pass through it undistorted, are known ahead of the time. this is the case for block constant channelswith long blocks. The channel behaves as a linear time invariant system during each block, and its eigenfunctions are harmonic. An extension offered by [1], characterizes the eigenfunctions of underspread channels using Weyl-heisenberg set of orthonormal functions that are localized both in time and frequency.

Incoherent communications over channels with a known structure, i.e. known eigenfunctions, limits the channel uncertainty to the channel eigenvalue, that vary on the channel realization. A natural basis for the transmitted signal is formed by the channel eigenfunctions.

In non-coherent communications one can achieve minimum energy per bit for every unknown channel with pre-known eigenfunctions [7]. The incoherent fading channel is an example of such a channel, its eigenfunctions are known to be the harmonic functions and its eigenvalues depend on the realization. In these channels, one way to communicate efficiently is using for each symbol very few eigenfunctions, as in FSK over block constant channels [7]. The advantage of such modulations is that although the number of paths may be very large, one can still communicate with minimum energy per bit. The problem is that using a small subspace of the eigenfunctions for every transmission, implies a very negative second derivative of spectral efficiency as a function of SNR, so efficient communications requires extremely low SNR, which causes very slow communications. Using higher SNR requires enormous energy per bit. We get a trade-off between fast communications and efficient communications: fast communications consume much more energy per bit. Another problem with FSK and similar modulations may occur if there is a physical restriction on the energy per eigenfunction.

When can we communicate efficiently over non-coherent channels in terms of both time and energy? Zheng et al [9] investigate a model in which every unknown eigenvalue, i.e. a channel gain per single carrier, repeats many times. In such a case, the uncertainty about the channel is small, and using 'rich modulations' (i.e. modulation that are not restricted to a small subspace of eigenfunctions), scaling with an appropriate SNR, yields efficient communications. The SNR should be high enough in order to overcome the uncertainty of the channel, however not too high, in order to remain in the efficient range of the modulation. If each eigenvalue repeats s times, then SNR should be much higher than $1/s$, but still in the low SNR regime, which means that in order to communicate efficiently, s should be very high.

Concentrating on the block-constant channel with coherence time T_c, we use the term number of apparent degrees of freedom to denote the number of different values a system of bandwidth W can transmit during a single coherence time. The number of apparent DoF equals WT_c. The number of effective degrees of freedom denotes the number of parameters needed to characterize the channel realization, that are unknown to the receiver before communication starts. The number of effective degrees of freedom depends on the channel model, and on the knowledge the system has of the channel. Another channel model is the sparse fading channel: a fading channel with exactly L paths that are sparsely and uniformly spread over the duration of the impulse response. The maximal time delay of the impulse response is denoted T_d, The bandwidth is W and the coherence time of the channel, in which the channel remains constant is T_c, such that there are exactly $K_c = WT_c$ channel usages during each coherence period. We assume that

$$L << WT_d < WT_c$$

The ratio between T_d and T_c is constant, and L is sublinear in W, as W increases, such that L is sublinear in the number of channel usages [7] [3] [10].

If the path delays are known to the receiver but path gains are not known, then the number of effective degrees of freedom of the channel i.e. the number of the unknown parameters of the channel is much smaller than the number of apparent DoF WT_c, similarly to the case of [9]. Porrat et al [3] show that if the number of effective DoF is L, and $\frac{WT_c}{L} = s$, then the results of [9] about the ability to overcome the uncertainty are valid, so in order to communicate efficiently SNR should be much higher than $\frac{L}{WT_c}$.

If the path delays are not known, the number of effective degrees of freedom equals WT_d (each index may be path delay or not) which is proportional to the number of channel usages WT_c, however if the ratio between the number of paths and the number of channel usages tends to zero as the bandwidth increases, the entropy of the channel does not grow linearly with the bandwidth, while the entropy of the modulation can

grow linearly with it. As a result we can communicate efficiently in terms of both time and energy. The reason is that the uncertainty of the channel is not affected exclusively by the number of degrees of freedom but also by their distributions.
In [7] it is proved that if the delays are not known and tends to infinity with bandwidth and the SNR is inversely proportional to the bandwidth as a result of maintaining the same average power per coherence period, then spread spectrum modulations achieve mutual information of $O\left(\frac{1}{W}\right)$ per transmission. In other words, if $\mathrm{SNR} = O\left(\frac{1}{W}\right)$ the spectral efficiency is $O(\mathrm{SNR})^2 = O\left(\frac{1}{W^2}\right)$, even though L is sublinear in W, as a result of using too low SNR. [3] show that if SNR is much higher than $\frac{L}{W}\log\frac{W}{L}$ one can achieve minimum per bit with spread spectrum modulations, depending on the spectral efficiency of the modulation, at the specified SNR. Our work closes the gap between the two results by showing that in the channel case, the threshold for incoherent spread spectrum communications is $\mathrm{SNR}_{\mathrm{th}} = \frac{L}{W}\log\frac{W}{L}$.

2. CHANNEL MODEL

The system bandwidth is W and the channel is a block-coherent multipath channel. We approximate the continuous channel by a real discrete model after sampling it with rate W. The model over one coherence period is given by

$$Y = \sqrt{\mathrm{SNR}}H \star X + Z \tag{5}$$

Where Y, the received signal per coherence period, is a $K_c = WT_c$ long vector. The K_c long vector X is the transmitted signal and the multipath channel is represented by the K_c long impulse response vector H. The impulse response H is composed of L paths distributed uniformly over WT_d positions, where WT_d is the maximal index for a path delay. The gains are i.i.d. random variables with zero mean and variances:

$$E\left[H_i^2\right] = \frac{1}{L} \tag{6}$$

Since the gains are i.i.d. $E\left[HH^t\right] = I$. Z is white standard Gaussian noise, and $\star$ marks convolution.

We set SNR to be the signal to noise ratio per active transmission, so there is no need to explicitly address the impulsiveness used by the system (duty cycle ratio).

The transmitted signal satisfies the energy constraint

$$E\left\|X\right\|_2^2 = K_c \tag{7}$$

The model (5) neglects side effects at the edges of of coherence period. The channel is block constant with coherence time T_c, i.e. it has i.i.d. realizations over different coherence periods. We assume $T_d << T_c$ and thus justify to an extent our loose treatment of edge effects at the beginning of each coherence period.
We consider the signal to noise ratio per active transmission, so there is no need to explicitly address the impulsiveness used by the system (duty cycle ratio). Moreover, we assume that each active transmission, we use about the same amount of energy:

$$P\left(\lim_{W\to\infty}\frac{\|X\|_2^2}{K_c\mathrm{SNR}} = 1\right) = 1 - o(1) \tag{8}$$

(7) ensures that SNR in (5) is the signal to noise ratio, per active transmission.

Spread-spectrum systems are such where almost every transmitted signal obeys:

$$\left|\left\langle \hat{X}^i, \hat{X}^j \right\rangle\right| = O\sqrt{K_c} \tag{9}$$

$i \neq j = 1, 2, ..., K_c$. The notation$\langle , \rangle$ is used for the inner product of vectors, and the notation $\hat{X}^i$ is used for a vector X that is cyclicly shifted by $i-1$ positions:

$$\hat{X}^i = \begin{pmatrix} X_{(2-i)} \\ X_{(3-i)} \\ \vdots \\ X_{(K_c+1-i)} \end{pmatrix} \tag{10}$$

where (-) indicates a mod K_c difference. Condition (9) ensures that for almost every burst of transmission, the signal is spread spectrum, i.e. it is not spanned by a small subspace of the harmonic functions. Any i.i.d. modulation (a modulation in which X_i is independent of X_j if $i \neq j$), obeys (9).

If there is uncertainty about path gains, however path delays are known to the receiver, the channel is denoted by H_{gains}. If both path gains and path delays are unknown, we denote the channel H_{paths}.
We assume that the number of paths L diverges as the bandwidth increases in a sublinear manner, $L_{W\to\infty} \to \infty$ and $L = o(W)$ [5] [4].

3. RESULTS

The mutual information between transmitted and received signal in incoherent setting is broken in two:

$$I\left(\sqrt{\mathrm{SNR}}H \star X + Z; X\right) \tag{11}$$

$$= I\left(\sqrt{\mathrm{SNR}}H \star X + Z; H, X\right) \tag{12}$$

$$- I\left(\sqrt{\mathrm{SNR}}H \star X + Z; H|X\right) \tag{13}$$

We know that the coherent rate is upper bounded

$$I\left(\sqrt{\mathrm{SNR}}H\star X+Z;H,X\right)$$
$$\leq\frac{1}{2}K_c\mathrm{SNR}\ \mathrm{nats} \tag{14}$$

Since the mutual information in nats cannot exceed the channel capacity, i.e. half of the received energy.

From (13) and (14) we get that

$$I\left(\sqrt{\mathrm{SNR}}H\star X+Z;X\right)\leq\frac{1}{2}K_c\mathrm{SNR}$$
$$-I\left(\sqrt{\mathrm{SNR}}H\star X+Z;H|X\right) \tag{15}$$

So one way to upper bound the mutual information in non-coherent communications, is to lower bound the penalty term (15). This is the base of Theorem (1) and theorem (3).

$$I\left(\sqrt{\mathrm{SNR}}H\star X+Z;H|X\right) \tag{16}$$

To lower bound the mutual information (11), one should note that:

$$I\left(\sqrt{\mathrm{SNR}}H\star X+Z;H,X\right)=$$
$$I\left(\sqrt{\mathrm{SNR}}H\star X+Z;H\right)$$
$$+I\left(\sqrt{\mathrm{SNR}}H\star X+Z;X|H\right)$$
$$\geq I\left(\sqrt{\mathrm{SNR}}H\star X+Z;X|H\right) \tag{17}$$

Replacing (12) by the lower bound (17) yields a lower bound on the requested term (11).

$$I\left(\sqrt{\mathrm{SNR}}H\star X+Z;X\right)$$
$$\geq I\left(\sqrt{\mathrm{SNR}}H\star X+Z;X|H\right) \tag{18}$$
$$-I\left(\sqrt{\mathrm{SNR}}H\star X+Z;H|X\right) \tag{19}$$

By computing the mutual information over a known channel (18), and an upper bound (19), we can achieve a lower bound on the incoherent mutual information. This is ate base of Theorem (2) and theorem (4)

Using (17) we can lower bound the mutual information by upper bounding the penalty term (19), and we do so using the low number of degrees of freedom (in the H_{gains} case, Theorem (2)) which decreases the uncertainty, or by proving that the statistics of the sparse fading channel reduce its entropy (in the H_{paths} case, Theorem (4)). If L is sublinear in W, the entropy of effective DoF is much smaller, so incoherent communications are possible **??**).

In order to upper bound the penalty term in (19), we use the connection between mutual information and the minimum mean square error (I-mmse connection) [2], adapted to the fading channel:

$$I\left(H;\sqrt{\mathrm{SNR}}\star X+Z|X\right)$$
$$=\frac{1}{2}\int_0^\infty E_X\left[\mathrm{mmse}(\mathrm{SNR})\right]d\mathrm{SNR} \tag{20}$$

When the mmse is given by

$$E_X\left[\mathrm{mmse}(\mathrm{SNR})\right]$$
$$=E_{XHZ}\left[\left\|X\star H-X\star\hat{H}(Y;\mathrm{SNR})\right\|_2^2\right] \tag{21}$$

And $\hat{H}$ is the mmse estimate of H given X and Y.

The mmse in (21) quantifies the inability to recover $H\star X$. The mean square error of increases the integral in (20), so the penalty term (19) increases. If X is a narrow band modulation and the transmitted signal is known, all the energy of $H\star X$ is concentrated in a small subspace of the harmonic functions where the signal X is projected, so we can estimate $H\star X$ well. However, if X is a spread spectrum modulation, we cannot overcome the additive noise of the system if SNR is too low.

3.1. The H_{gains} case

In the H_{gains} channel model the threshold for efficient communications using spread spectrum modulations is

$$\mathrm{SNR}=\frac{L}{W} \tag{22}$$

Theorem 1 *In the H_{gains} channel model, if $\mathrm{SNR}=o\left(\frac{L}{W}\right)$ and the modulation obeys (9)*

$$\lim_{W\to\infty}\frac{I\left(\sqrt{\mathrm{SNR}}H\star X+Z;X\right)}{K_c\mathrm{SNR}}=0 \tag{23}$$

Proof 1 *see [10].*

3.1.1. Discussion of Theorem 1

(23) means that as the bandwidth diverges, we need infinite energy per bit if the SNR is much lower than the threshold (22). This result is valid for any spread-spectrum modulation, so communications with a small SNR are possible only by narrow-band modulations.

Theorem 2 *In the H_{gains} channel model, spread-spectrum modulations with $\mathrm{SNR}=c=\omega\left(\frac{L}{W}\right)$, that obey (9) and achieve minimum energy per bit over an AWGN channel for $\mathrm{SNR}=c$, also achieve minimum energy per bit over the H_{paths} channel, and $\mathrm{SNR}=c$.*

Proof 2 *see [3].*

3.1.2. Discussion of Theorem 2

We see that non-coherent communications are possible with SNR that is much higher than the threshold (1), but efficiency depends on the modulation. Since in that range of SNR the penalty term (19) is negilible, narrow-band modulations are not preferable to spread-spectrum modulations. Moreover, narrow-band modulations lose their efficiency in a small SNR, so spread-spectrum modulations may be better.

3.2. The H_{paths} case

In the H_{paths} channel model the threshold for efficient communications using spread spectrum modulations is

$$\mathrm{SNR} = \frac{L}{W} \log \frac{W}{L} \tag{24}$$

If SNR is much lower than (24) communications are impossible.

Theorem 3 *In the H_{paths} channel model, if $\mathrm{SNR} = o\left(\frac{L}{W}\log\frac{W}{L}\right)$ and the modulation obeys (9) then*

$$\lim_{W\to\infty} \frac{I\left(\sqrt{\mathrm{SNR}}H \star X + Z; X\right)}{K_c \mathrm{SNR}} = 0 \tag{25}$$

Proof 3 *see [10].*

3.2.1. Discussion of Theorem 3

If the SNR is much lower than the threshold (24), the channel is not recoverable, and we cannot find the path delays, so the minimum mean square error estimate of H is close to the zeros vector. As a result the square error (21) almost equals $H \star X$, and by the I-mmse connection we know that the penalty term (16) equals $\frac{1}{2}K_c\mathrm{SNR}$ and therefore prevents efficient communications.

If the SNR is much higher than the threshold (24) then efficient communications are possible:

Theorem 4 *In the H_{paths} case, spread spectrum modulations with $\mathrm{SNR} = \omega\left(\frac{L}{W}\log\frac{W}{L}\right)$ that obey (9) and achieve minimum energy per bit over an AWGN channel, also achieve minimum energy per bit over the H_{paths} channel [3].*

Proof 4 *see [3].*

3.2.2. Discussion of Theorem 4

We see that non-coherent communications are possible with SNR that is much bigger than the threshold (24), but efficiency depends on the the modulation. Since in that range of SNR the penalty term (19) is negligible, narrow-band modulations are not preferable to spread-spectrum modulations. Moreover, narrow-band modulations lose their efficiency in a small SNR, so spread-spectrum modulations may be better.

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