QUANTUM COMPUTATION - EXERCISE 2

(1) **Locality**
  (a) Let $|\alpha\rangle$ be an arbitrary state in a Hilbert space of dimension N, and let $V = \{|v_1\rangle, \dots, |v_N\rangle\}$ be an orthonormal basis to that space. Show that there exists a unitary matrix U, such that measuring $|\alpha\rangle$ in the basis V is equivalent to measuring $U|\alpha\rangle$ in the *standard* basis.
  (b) Consider a bipartite quantum state $|\psi\rangle_{A,B}$. Show that if Alice performs an arbitrary unitary operation on her part of the system, and then Bob measures his qubits in the standard basis, then the measurement outcome is independent of Alice's actions.
  (c) Conclude that there is no measurement that Bob can perform to tell which unitary operation Alice performed on her qubits.

(2) **Teleportation Of More Than One Qubit** Suppose Alice and Bob share two Bell states. Show how Alice can use them to teleport an arbitrary (even entangled) state of two qubits to Bob. How many bits of classical information should she send to him?

(3) **Algorithms** We have seen in class how we can apply on the state

$$|\Psi_X\rangle = \frac{1}{\sqrt{2^n}} \sum_{Y \in \{0,1\}^N} (-1)^{X \cdot Y} |Y\rangle$$

$H^{\otimes n}$ and measure, and that this gives us X with certainty. Now suppose that instead of $|\Psi_X\rangle$ we have the state

$$|\Psi_{X,k}\rangle = \frac{1}{\sqrt{M_k}} \sum_{Y : |Y| \leq k} (-1)^{X \cdot Y} |Y\rangle$$

for some $k \leq n$, where $M_k = \sum_{j=0}^{k} \binom{n}{j}$, and $|Y|$ is the number of $1'$s in the string Y, namely, the Hamming weight of Y. Note that if $k = n$ we are in the first case.
  (a) Show that, by applying on $|\Psi_{X,k}\rangle$ the same procedure: $H^{\otimes n}$ and measure, we can determine the value of X with a probability of success

$$P_{success}(n,k) = |\langle \Psi_X \,|\, \Psi_{X,k}\rangle|^2 .$$

  (b) Compute the value $P_{success}(n,k)$.
  (c) Suppose that $k = (\frac{1}{2} + \epsilon)n$ for some constant ϵ. Show that the probability of success is exponentially close to 1. (Hint: you can use the Chernoff bound.)

(4) **Computation With Real Coefficients** Prove that we can assume w.l.o.g that all amplitudes in a quantum computation are real numbers. The following guidance should be helpful: We know (though we haven't seen the proof) that the set of two

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gates, Hadamard and controlled-i:

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; controlled - i : \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & i \end{pmatrix}$$

is universal. Now show that any quantum circuit on n qubits that uses t gates from this set, can be simulated exactly by a quantum circuit on $n+1$ qubits, that uses at most $4t$ gates from the set Hadamard and $T =$Tofolli. (Hint: use the extra qubit to decide whether we are in the imaginary or real part of the world.) (Hint number two: After a little thought, you will find that you can use one Toffoli to simulate one controlled-i, except that the phase of one of the states is wrong. Now recall that $HXH = Z$; Using this intuition, show how using three more gates from the set Toffoli and Hadamard, the phase issue can be fixed.)