QUANTUM COMPUTATION-EXERCISE 1

1. Bell Basis

The Bell states are:

$$\begin{aligned}
\left|\Phi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
\left|\Phi^{-}\right\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
\left|\Psi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
\left|\Psi^{-}\right\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}$$

1.1. Show that the Bell states form an orthonormal basis to the Hilbert space of two qubits.

1.2. Choose (any) one of the Bell states, and show that it is *entangled* (i.e. it cannot be written in the form of a tensor product of two one-qubit states).

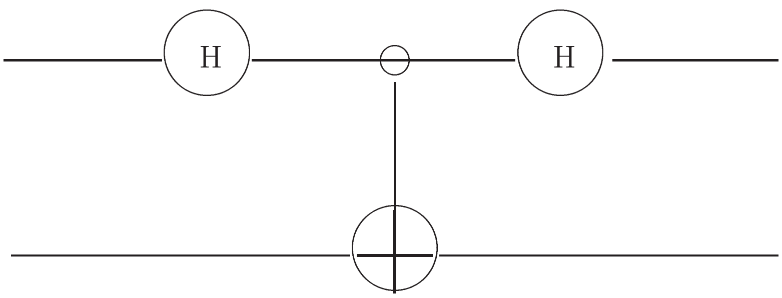
1.3. Find a unitary matrix U, that permutes the Bell states as follows:

$$\left|\Phi^{+}\right\rangle \to \left|\Phi^{-}\right\rangle \to \left|\Psi^{+}\right\rangle \to \left|\Psi^{-}\right\rangle \to \left|\Phi^{+}\right\rangle$$

(i.e. $U\left|\Phi^{+}\right\rangle = \left|\Phi^{-}\right\rangle$, etc.).

2. Quantum Circuits

Calculate what the following circuit does on input standard basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$:



Can you describe the combined operation in a meaningful way?

Date: 4.11.2004 - 18.11.2004. This exercise should be submitted individually.

3. Optimal Measurements

You are given one of two states, $|0\rangle$ or $|+\rangle$, with equal probabilities, $\frac{1}{2}$. You are allowed to perform a measurement, and then guess, according to the measurement outcome, which state it was. Your goal is to maximize your probability of success, by choosing the basis in which to measure the state wisely. Try to find the best basis for this purpose (do the best you can- no need to prove you found the optimal basis). What is your success probability with the basis you chose?

4. GHZ Game (bonus question)

Consider the GHZ (Greenberger- Horne- Zeilinger) state, of three qubits:

$$\frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$$

Suppose that three parties, Alice, Bob and Charlie, are given three random bits X_A, X_B, X_C, respectively, with the constraint $X_A \oplus X_B \oplus X_C = 0$. They have to output three bits, a, b, c, such that $a \oplus b \oplus c = X_A \vee X_B \vee X_C$. Show that if Alice, Bob and Charlie share a GHZ state, then this can be done with certainty (i.e with probability 1). **Hint:** you will need the Hadamard matrix,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which takes

$$\begin{aligned} |0\rangle &\rightarrow |+\rangle \\ |1\rangle &\rightarrow |-\rangle . \end{aligned}$$