QUANTUM COMPUTATION - FALL 2006 - EXERCISE 1

Due: 13/11/06, noon (submit in class).

1. Bell Basis

The Bell states are:

$$\begin{aligned}\left|\Phi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ \left|\Phi^{-}\right\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ \left|\Psi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ \left|\Psi^{-}\right\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\end{aligned}$$

1.1. Show that the Bell states form an orthonormal basis to the Hilbert space of two qubits.

1.2. Choose (any) one of the Bell states, and show that it is *entangled* (i.e. it cannot be written in the form of a tensor product of two one-qubit states).

2. Order Of Measurements

2.1. Give an example of a quantum state $|\alpha\rangle$ of two qubits, such that when measuring the left qubit in the $\{|+\rangle, |-\rangle\}$ basis the probability to get $|+\rangle$ is $\frac{2}{5}$, and when measuring the right qubit the probability to get $|+\rangle$ is $\frac{3}{4}$. Express your state in both the standard basis and the $\{|+\rangle, |-\rangle\}$ basis.

2.2. Give an example of an orthonormal basis to the Hilbert space of one qubit which is neither the standard basis, nor the $\{|+\rangle, |-\rangle\}$ basis, and in which at least one of the coefficients is not real.

2.3. Show that the probability distribution over the results, which you get by measuring the left qubit of $|\alpha\rangle$ in your basis, and then the right qubit in the standard basis, is the same as when you first measure the right qubit in the standard basis, and then the left qubit in your basis.

3. Optimal Measurements

You are given one of two states, $|1\rangle$ or $|-\rangle$, with equal probabilities, $\frac{1}{2}$. You are allowed to perform a measurement, and then guess, according to the measurement outcome, which state it was. Your goal is to maximize your probability of success, by choosing the basis in which to measure the state wisely. Try to find the best basis for this purpose (do the best you can- no need to prove you found the optimal basis). What is your success probability with the basis you chose?

4. Bell and GHZ Game

(1) Use the proof in class that no deterministic strategy can win in the Bell game with more than 75%, to show the same statement but for probabilistic strategies, where the players can start by choosing a random joint string, and play according to it.
(2) Explain why this statement you have just completed the proof of, shows that no hidden variable theory can give more than 75% success in the Bell game.
(3) Consider the GHZ (Greenberger- Horne- Zeilinger) state, of three qubits:

$$\frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$$

Suppose that three parties, Alice, Bob and Charlie, are given three random bits X_A, X_B, X_C, respectively, with the constraint $X_A \oplus X_B \oplus X_C = 0$. They have to output three bits, a, b, c, such that $a \oplus b \oplus c = X_A \vee X_B \vee X_C$. Show that if Alice, Bob and Charlie share a GHZ state, then this can be done with certainty (i.e with probability 1). **Hint:** you will need the Hadamard matrix,

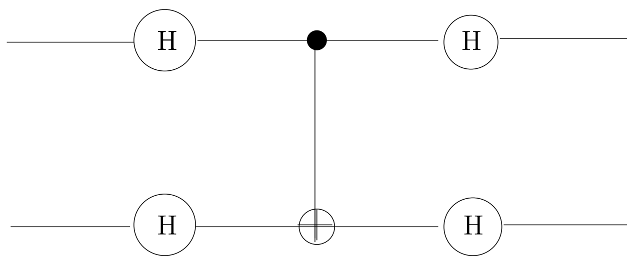
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which takes

$$\begin{aligned} |0\rangle &\rightarrow |+\rangle \\ |1\rangle &\rightarrow |-\rangle . \end{aligned}$$

5. Quantum Circuits

Calculate what the following circuit does on input standard basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$:



Can you describe the combined operation in a meaningful way?