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COOPERATION IN A REPEATED GAME WITH RANDOM PAYMENT FUNCTION

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Abstract

A model of cooperation versus defection in a sequence of games is analysed under the assumptions that the rules of the game are randomly changed from one encounter to another, that the decisions are to be made each time anew, according to the (random) rules of the specific local game, and that the result of one such game affects the ability of a player to participate and thus, cooperate in the next game. Under plausible assumptions, it is shown that all Nash solutions of the supergame determine cooperation over a non-degenerate range of rules, determining encounters of the prisoner's dilemma type.

STOCHASTIC GAME; ESS; SUPERGAME; CONTINUOUS GAME; PRISONER'S DILEMMA

1. Introduction

In the one-shot prisoner's dilemma game described in Figure 1, the only Nash equilibrium is (D, D). This is the case also for any finite repetition of the one-shot game, a result which does not agree with experimental behavior as observed under similar circumstances (see Rapaport (1967)). However, in the infinitely repeated one-shot game, any individually rational payoff is a Nash equilibrium (folk theorem) and this result is consistent with many cases of observed cooperation in real such encounters. Yet, one never sees infinite series of encounters. Another drawback is that (D, D) is always a Nash equilibrium which leaves the problem of the evolution of cooperative behavior open.

	C	D
C	0.4, 0.4	0, 0.5
D	0.5, 0	0.1, 0.1

Figure 1

To rationalize some measure of cooperation in the finitely repeated prisoner's

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dilemma, Radner (1980) applied the ε-equilibrium concept to show that an outcome close to the cooperative one can be obtained. Kreps et al. (1982) explained the observed cooperation by incomplete information. Yet a different approach applied the concept of bounded rationality as implemented by various computing machines such as finite automata (Neyman (1985), Rubinstein (1986)) or Turing machines (Megiddo and Widgerson (1986)). Concerning the second difficulty, Axelrod and Hamilton (1981) discussed the dynamics of a process through which a cooperative behavior can be established in an initially non-cooperative population.

It seems to us, though, that one cause of the inconsistency between theory and observation may be in the discrepancy between the mathematical model of a repetition of the *same* game and the real situation, in which there might be a *continuity* of different potential future encounters between the two potential players. Thus, a crucial factor in the establishment of a tendency to cooperate may be the positive probability that a present opponent, if then alive, will cooperate in future encounters, not because he will remember and return in kind for kindness (a 'strategy' which is, by itself, disadvantageous when rare) but because with some positive probability, the realization of the encounter parameters will be such that cooperation will be in its favor. If so, an individual can increase its long-term (supergame) welfare by increasing his opponent's survival probability even by choosing a strategy which is slightly unfavorable for the short-term encounter. This is indeed so, where the encounter payment function is survival probability (or at least 'survival' as a potential player). But then, by symmetry, the range of situations (encounter parameters) for which cooperation is advantageous increases, the probability of future cooperation increases and the process perpetuates itself to a limit, as we see.

In this work we consider, more specifically, a model in which the payment function of any next encounter is a random variable of a known (multidimensional) distribution. The relevant question regarding this more general assumption is not whether to cooperate or not but under what conditions (i.e., for what realization of the payment function) to cooperate. More specifically, the (pure) strategies of the game are measurable sets of 'situation' under which the player is bound to cooperate. We assume, further, that a failure in one encounter decreases a player's chance to participate (and, therefore, to cooperate, if he is willing) in the next encounter, if occurs. Thus non-cooperation, even when locally advantageous, may be disadvantageous from a viewpoint of the supergame which is defined as a sequence of encounters of a random length.

Under plausible assumptions it is shown that, except for some singular cases (to be determined), any Nash solution of the supergame determines cooperation over a non-degenerate set of situations (i.e., realizations of the payment function) of the prisoner's dilemma type. As it appears, the widely analysed case of repetition of the *same* encounter is one singular exception.

The motivation for this work stems from a biological (or sociobiological) context, in which individuals interact repeatedly under various conditions, the payment function of an encounter is the survival probability to the next encounter and the payment function of the supergame is the survival probability to the next generation. We are, therefore, interested also in some stronger, dynamic properties of the Nash solutions, namely, in evolutionary stability. For discussion of the evolutionary aspects of the model, the reader is referred to Eshel and Weinshall (1987).

2. The model and its basic properties

We start by defining a two-player symmetric supergame as a random-length sequence of 'events' (to be defined below) to which each of the players is exposed, if not having perished in some previous event. We assume that at each moment there is a fixed probability $p > 0$ of some future event occurring, independently of past events. If the two players are alive, an event becomes a symmetric 2×2 encounter, in which each player has two alternative strategies, say 'cooperate' or 'defect'. The outcome of the encounter is the survival probabilities of the two players in that encounter, given by the matrix

	cooperate	defect
cooperate	X_1, X_1	X_3, X_2
defect	X_2, X_3	X_4, X_4

wherein the parameters $X = (X_1, \cdots, X_4)$ are random variables drawn from the four-dimensional distribution F, independently of the past.

We further assume that if only one of the players is present at the time of an event, then his probability of surviving it is equal to the one he would achieve with the lack of cooperation, say X_4, independently of the strategy. (With minor technical difficulties, though, most results of this work can be extended to the situation in which this survival probability attains any value smaller than X_1.) By choosing the term 'cooperation' for the first strategy, however, we mean that

(2.1a) $$X_1 > X_3 \quad \text{and} \quad X_2 > X_4$$

(i.e., by cooperating, a player always helps his opponent)

(2.1b) $$X_4 < X_1 > \frac{X_2 + X_3}{2}$$

(i.e., mutual cooperation is always in the Pareto's optimum of the encounter), and

(2.1c) $$X_2 > X_3$$

(i.e., if only one player defects, then his reward will be higher than that of his cooperating opponent).

We assume a positive probability for encounters of the prisoner's dilemma type, i.e.:

(2.2) $$p(D) > 0 \qquad \text{where } D = \{X \mid X_2 > X_1 > X_4 > X_3\}.$$

We also assume, however, a positive probability for encounters in which cooperation is of immediate self reward, i.e.,

$$(2.3) \qquad p(R) > 0 \qquad \text{where } R = \{X \mid X_1 \geqq X_2; X_3 \geqq X_4\}.$$

We assume, moreover, that F has positive density f over a convex set of parameters, Ω, including at least part of the boundary $\{X \mid X_1 = X_2 > X_3 = X_4\}$ between R and D.

At each stage of the supergame each player possesses full knowledge of the present situation (i.e., about the realization $\boldsymbol{a}$ of X) as well as of the distribution F and the value p. However, we assume no memory so that a pure strategy is a measurable set G of realizations of X (game-matrices $\boldsymbol{a}$) over which the player is bound to cooperate. A mixed strategy is a measurable function $\Gamma : \Omega \to [0, 1]$, determining the probability $\Gamma(\boldsymbol{a})$ that a player will cooperate in a given realization $\boldsymbol{a} \in \Omega$.

If player i ($i = 1, 2$) chooses the strategy Γ_i, then the survival probability of player 1 during a single encounter is

$$(2.4) \qquad \begin{aligned} s_1(\Gamma_1, \Gamma_2) = \iint_\Omega \{&\Gamma_1(\boldsymbol{a})\Gamma_2(\boldsymbol{a})a_1 + \Gamma_1(\boldsymbol{a})(1 - \Gamma_2(\boldsymbol{a}))a_3 \\ &+ (1 - \Gamma_1(\boldsymbol{a}))\Gamma_2(\boldsymbol{a})a_2 + (1 - \Gamma_1(\boldsymbol{a}))(1 - \Gamma_2(\boldsymbol{a}))a_4\} dF(\boldsymbol{a}). \end{aligned}$$

The survival probability of player 2 is, by symmetry;

$$(2.5) \qquad s_2(\Gamma_1, \Gamma_2) = s_1(\Gamma_2, \Gamma_1).$$

Finally, the survival probability of any player at an event in which his opponent is missing is

$$(2.6) \qquad s_1(\Gamma_1, -) = \iint_\Omega a_4 dF = EX_4 = \lambda, \quad \text{say}$$

independently of the player strategy.

Assuming now that, given the probabilities $s_i = s_i(\Gamma_1, \Gamma_2)$, the survival of player 1 and player 2 at a given encounter are independent random variables, the probability that both players survive a single encounter is, then, $s_1 s_2$ and the probability that they both survive the entire sequence of games is, therefore,

$$(2.7) \qquad \sum_{\kappa=0}^{\infty} p^\kappa q (s_1 s_2)^\kappa = \frac{q}{1 - p s_1 s_2}.$$

(Note that $p^\kappa q$ is the probability of k encounters.) In the same way, the probability that only player 1 will survive a single encounter is $s_1(1 - s_2)$. Hence, if both players do not survive till the end of the sequence, then, using the Bayes formula, we know that there is a probability $s_1(1 - s_2)/(1 - s_1 s_2)$ that at one encounter, player 1 survives and player 2 dies. Employing the stationary property of the sequence of games, we know that the survival probability of player 1 till the end of the sequence, conditioned on his opponent's death at one stage of the supergame is

$$(2.8) \qquad \sum_{\kappa=0}^{\infty} p^\kappa q \lambda^\kappa = \frac{q}{1 - p\lambda}.$$

The unconditioned survival probability of player 1 to the end of the sequence is, therefore

$$V(\Gamma_1, \Gamma_2) = \frac{q}{1 - ps_1s_2} + \left(1 - \frac{q}{1 - ps_1s_2}\right)\frac{s_1(1 - s_2)}{1 - s_1s_2}\frac{q}{1 - p\lambda}$$

where s_1s_2 and λ are given by (2.4)–(2.6). By simple algebraic manipulation one readily gets

$$V(\Gamma_1, \Gamma_2) = \frac{q}{1 - p\lambda}(1 + p\psi) \tag{2.9}$$

where

$$\psi = \psi(\Gamma_1, \Gamma_2) = \phi(s_1, s_2) = \frac{s_1 - \lambda}{1 - ps_1s_2}. \tag{2.10}$$

Hence, the game is determined by the attempt of player 1 to choose a strategy Γ_1 that will maximize $\psi(\Gamma_1, \Gamma_2)$ against Γ_2 and of player 2 to choose a strategy Γ_2 that will maximize $\psi(\Gamma_2, \Gamma_1)$ against Γ_1. It can readily be shown (see Section 4) that when the two strategies Γ_1 and Γ_2 are sufficiently close (in the metric of F-averages) $s_1 > \lambda$ and therefore $\partial\psi/\partial s_1 > 0$ and $\partial\psi/\partial s_2 > 0$. Hence a rational behavior for both players is to seek to increase both s_1 and s_2.

3. Minimal and maximal Nash solutions of the general supergame

Let Γ be a population strategy and let Γ^* be equal to Γ on all points of Ω except for an ε-measure vicinity θ of the point $\boldsymbol{a} \in \Omega$, at which $\Gamma^* = \Gamma + \delta$ where $\varepsilon > 0$ is a small positive number and δ is either positive or negative, provided $-\Gamma(\boldsymbol{a}) < \delta < 1 - \Gamma(\boldsymbol{a})$.

Denote

$$s = s_1(\Gamma, \Gamma) = s_2(\Gamma, \Gamma) = \int\int_\Omega \{a_1\Gamma^2 + (a_2 + a_3)\Gamma(1 - \Gamma) + a_4(1 - \Gamma)^2\} dF(\boldsymbol{a}). \tag{3.1}$$

Employing (2.4) one readily calculates the increment of the survival probability of player 1 at a single encounter:

$$\Delta_1 = s_1(\Gamma^*, \Gamma) - s_1(\Gamma, \Gamma) = \delta\varepsilon\{(a_1 - a_2)\Gamma(\boldsymbol{a}) + (a_3 - a_4)(1 - \Gamma(\boldsymbol{a}))\} + o(\varepsilon). \tag{3.2}$$

The corresponding increment of the survival probability of player 2 is

$$\Delta_2 = s_2(\Gamma^*, \Gamma) - s_2(\Gamma, \Gamma) = \delta\varepsilon\{(a_1 - a_3)\Gamma(\boldsymbol{a}) + (a_2 - a_4)(1 - \Gamma(\boldsymbol{a}))\} + o(\varepsilon). \tag{3.3}$$

From (3.2), (3.3) and (2.10) it follows that the increment of ϕ is

$$\Delta_1 \frac{\partial\phi}{\partial s_1} + \Delta_2 \frac{\partial\phi}{\partial s_2} + o(\varepsilon) = \delta\varepsilon(1 - ps^2)^{-2}\Delta(\boldsymbol{x}, \Gamma) + o(\varepsilon) \tag{3.4}$$

where

$$\begin{aligned}\Delta(\boldsymbol{a}, \Gamma) = (1 - \lambda ps)&\{(a_1 - a_2)\Gamma(\boldsymbol{a}) + (a_3 - a_4)(1 - \Gamma(\boldsymbol{a}))\} \\ &+ ps(s - \lambda)\{(a_1 - a_3)\Gamma(\boldsymbol{a}) + (a_2 - a_4)(1 - \Gamma(\boldsymbol{a}))\}.\end{aligned} \tag{3.5}$$

The coefficient of increment Δ can be written as

(3.6) $$\Delta(\boldsymbol{a}, \Gamma) = (1 - \Gamma(\boldsymbol{a}))\Delta^{+}(\boldsymbol{a}, \Gamma) - \Gamma(\boldsymbol{a}, \Gamma)\Delta^{-}(\boldsymbol{a}, \Gamma)$$

where

(3.7) $$\Delta^{+}(\boldsymbol{a}, \Gamma) = ps(s - \lambda)(a_2 - a_4) - (1 - \lambda ps)(a_4 - a_3)$$

and

(3.8) $$\Delta^{-}(\boldsymbol{a}, \Gamma) = (1 - \lambda ps)(a_2 - a_1) - ps(s - \lambda)(a_1 - a_3).$$

A strategy Γ is stable against expansion (of cooperative behavior) if for almost all $\boldsymbol{a}$ with $\Gamma(\boldsymbol{a}) < 1$, $\Delta(\boldsymbol{a}, \Gamma) \leqq 0$. It is stable against desertion (from cooperative behavior) if for almost all $\boldsymbol{a}$ with $\Gamma(\boldsymbol{a}) > 0$, $\Delta(a, \Gamma) \geqq 0$. Γ is a Nash solution of the game if it is stable against both expansion and desertion. If it is strictly so, then Γ is a strict Nash solution and, therefore, an ESS. As an immediate result we get the following two propositions.

Proposition 3.1. A strategy Γ is a population Nash solution of the game if and only if:
(i) $\Delta(\boldsymbol{a}, \Gamma) = \Delta^{+}(\boldsymbol{a}, \Gamma) \leqq 0$ for almost all $\boldsymbol{a}$ with $\Gamma(\boldsymbol{a}) = 0$;
(ii) $\Delta(\boldsymbol{a}, \Gamma) = 0$ for almost all $\boldsymbol{a}$ with $0 < \Gamma(\boldsymbol{a}) < 1$;
(iii) $-\Delta(\boldsymbol{a}, \Gamma) = \Delta^{-}(\boldsymbol{a}, \Gamma) \leqq 0$ for almost all $\boldsymbol{a}$ with $\Gamma(\boldsymbol{a}) = 1$.

Proposition 3.2. If Γ is a pure strategy with
(i) $\Delta^{+}(\boldsymbol{a}, \Gamma) < 0$ for almost all $\boldsymbol{a}$ with $\Gamma(\boldsymbol{a}) = 0$;
(ii) $\Delta^{-}(\boldsymbol{a}, \Gamma) < 0$ for almost all $\boldsymbol{a}$ with $\Gamma(\boldsymbol{a}) = 1$;
then Γ is an ESS.

We now prove the following result.

Proposition 3.3. Any population Nash solution Γ of the game determines full cooperation over a non-degenerate subset of the prisoner's dilemma range D.

Proof. First, if Γ is a Nash solution then $s = s_1(\Gamma, \Gamma) > \lambda$, otherwise it follows from (2.10) that $\psi(\Gamma, \Gamma) < 0$. But by unconditional defection, player 1 can guarantee $s_1 \geqq \lambda$ with $\psi \geqq 0$, in which case Γ cannot be a Nash solution.

Now, from (3.5) it follows that on the boundary set $\{\boldsymbol{a}_1 \mid a_1 = a_2 > a_3 = a_4\}$ of D

$$(\boldsymbol{a}, \Gamma) = ps(s - \lambda)(a_1 - a_3) > 0$$

and the continuity of Δ implies that $\Delta(\boldsymbol{a}, \Gamma) > 0$ for some open set in D. But from Proposition (3.1) it follows that $\Gamma(\boldsymbol{a}) = 1$ for almost all $\boldsymbol{a}$ in this set, which completes the proof.

Proposition 3.4. If Γ is stable against expansion (desertion) then there is a Nash solution $\Gamma' \leqq \Gamma$ ($\Gamma' \geqq \Gamma$).

Proof. Assume Γ is stable against expansion. We define a decreasing set of strategies $\{\Gamma_n\}_{n=0}^{\infty}$ which is stable against expansion in the following way:
(i) $\Gamma_0 = \Gamma$;
(ii) Assume that Γ_n has been defined so that it is stable against expansion. For any $\boldsymbol{a} \in \Omega$ consider the game $\boldsymbol{a}$ in which each player attempts to maximize his gain,

multiplied by $1 - \lambda p s_n$ plus that of his opponent, multiplied by $p s_n(s_n - \lambda)$, where $s_n = s(\Gamma_n, \Gamma_n)$.

In such a 2×2 game, the expected increment of the reward of a player switching from defection to cooperation, provided his opponent cooperates in probability $\Gamma_n(\boldsymbol{a})$, is

$$(3.5)' \qquad \begin{aligned} &(1 - \lambda p s_n)\{(a_1 - a_2)\Gamma_n(\boldsymbol{a}) + (a_3 - a_4)(1 - \Gamma_n(\boldsymbol{a}))\} \\ &\quad + p s_n(s_n - \lambda)\{(a_1 - a_3)\Gamma_n(\boldsymbol{a}) + (a_2 - a_4)(1 - \Gamma_n(\boldsymbol{a}))\} = \Delta(\boldsymbol{a}, \Gamma_n) \end{aligned}$$

(see (3.5)). But since Γ_n is stable against expansion, we know that either $\Gamma_n(\boldsymbol{a}) = 1$ or $\Delta(\boldsymbol{a}, \Gamma_n) \leqq 0$. In either case it follows from (3.5)′ that a best strategy (probability of cooperation) x against $\Gamma_n(\boldsymbol{a})$ in the local game described above must be smaller than or equal to $\Gamma_n(\boldsymbol{a})$ (in a non-singular case it is either $x = \Gamma_n(\boldsymbol{a})$ or $x = 0$).

As a general property of 2×2 symmetric games we know (e.g., Eshel (1982)) that in this case there must be at least one ESS with cooperation probability smaller than or equal to $\Gamma_n(\boldsymbol{a})$. We denote the smallest of these ESSs by $\Gamma_{n+1}(\boldsymbol{a})$. From the fact that the use of Γ_{n+1} against Γ_{n+1} is at least as good as the use of Γ_n against Γ_n it follows that:

$$(3.9) \qquad \begin{aligned} &p s_n(s_n - \lambda)[(1 - \Gamma_{n+1}(\boldsymbol{a}))(a_2 - a_4) + \Gamma_{n+1}(\boldsymbol{a})(a_1 - a_3)] \\ &\quad - (1 - \lambda p s_n)[(1 - \Gamma_{n+1}(\boldsymbol{a}))(a_4 - a_3) + \Gamma_{n+1}(\boldsymbol{a})(a_2 - a_1)] \leqq 0. \end{aligned}$$

But since $\Gamma_{n+1}(X) \leqq \Gamma_n(X)$ for all $X \in \Omega$, it follows from (2.1) and (3.1) that $s_{n+1} \leqq s_n$. By replacing s_n by s_{n+1} we therefore only decrease the left side of (3.9). Employing (3.5) we get

$$(3.10) \qquad \Delta(\Gamma_{n+1}, \boldsymbol{a}) \leqq 0 \qquad \text{for all } \boldsymbol{a} \text{ with } \Gamma_{n+1}(\boldsymbol{a}) < 1.$$

Γ_{n+1} is therefore stable against expansion.

$\{\Gamma_n\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative functions, hence $\Gamma_n \xrightarrow{n \to \infty} \Gamma'$.

By continuity argument, Γ' is also stable against expansion. It is stable against desertion from its very construction, so it is a Nash solution of the game.

Definition. The set

$$G = \{\boldsymbol{a} \mid \Gamma(\boldsymbol{a}) > 0 \text{ for all Nash solutions } \Gamma\}$$

is called *the minimal range of cooperation*.

Proposition 3.5. (i) There is always a minimal Nash solution Γ^* such that $\{\boldsymbol{a} \mid \Gamma^*(\boldsymbol{a}) > 0\} = G$.

(ii) There is a positive value $x_0 > 0$ (including the possibility $x_0 = \infty$) such that

$$(3.11) \qquad G = \left\{\boldsymbol{a} \,\middle|\, \frac{a_4 - a_3}{a_2 - a_4} < x_0\right\} = G_{x_0}, \quad \text{say.}$$

Proof. (i) Let Γ_1 and Γ_2 be two Nash solutions of the supergame and let $\hat{\Gamma} = \min(\Gamma_1, \Gamma_2)$. $s = s(\hat{\Gamma}) \leqq \min(s(\Gamma_1), s(\Gamma_2))$. Let $a \in \Omega$. Without loss of generality assume $\Gamma_1(\boldsymbol{a}) \leqq \Gamma_2(\boldsymbol{a})$ hence $\hat{\Gamma}(\boldsymbol{a}) = \Gamma_1(\boldsymbol{a})$. Since $\Gamma_1(\boldsymbol{a})$ is a Nash solution we know that either $\Gamma_1(\boldsymbol{a}) = 1$ or $\Delta(\boldsymbol{a}, \Gamma_1) \leqq 0$ (Γ_1 is stable against expansion). But by differentiating the right side of (3.5) with respect to s and employing (2.1), one can easily verify that it is increasing with s, as long as $s > \lambda$. Hence, by replacing the term $s(\Gamma_1)$ in $\Delta(\boldsymbol{a}, \Gamma_1)$ by the equal or smaller term $s(\hat{\Gamma})$ one gets $\Delta(\boldsymbol{a}, \hat{\Gamma})$ and we get $\Delta(\boldsymbol{a}, \hat{\Gamma}) \leqq \Delta(\boldsymbol{a}, \Gamma_1) \leqq 0$. $\hat{\Gamma}$ is, therefore, stable against expansion in $\boldsymbol{a}$ and this is true for all $\boldsymbol{a} \in \Omega$, hence $\hat{\Gamma}$ is stable against expansion.

From Proposition (3.4) it therefore follows that there is a Nash solution $\Gamma_3 \leqq \hat{\Gamma} = \min(\Gamma_1, \Gamma_2)$. Now, for all $\boldsymbol{x} \in \Omega$, denote $\Gamma^*(\boldsymbol{x}) = \text{Inf}\{\Gamma(\boldsymbol{a}) \mid \Gamma \text{ is a Nash solution}\}$. Indeed, $\tilde{\Gamma}^* = G$. Let $\Gamma^*(\boldsymbol{a}) < 1$. From the definition of Γ^* it is implied that there is a sequence $\{\Gamma_n\}$ of Nash solutions such that $\Gamma_n(\boldsymbol{a}) \rightarrow \Gamma^*(\boldsymbol{a})$ as $n \rightarrow \infty$. We therefore conclude that there exists a decreasing sequence of Nash solutions $\{\Gamma_n^*\}$ so that $\Gamma_1^* = \Gamma_1$ and, for all $n = 1, 2, \cdots, \Gamma_{n+1}^* \leqq \min\{\Gamma_n^*, \Gamma_{n+1}\}$.

Since $\{\Gamma_n^*\}$ is a decreasing sequence of functions with $\Gamma_n^* \geqq \Gamma^*$, there exists a limit function

$$\Gamma = \lim_{n \rightarrow \infty} \Gamma_n^*$$

with $\Gamma \geqq \Gamma^*$. We also know

$$\Gamma(\boldsymbol{a}) = \lim_{n \rightarrow \infty} \Gamma_n^*(\boldsymbol{a}) = \Gamma^*(\boldsymbol{a}).$$

But for all n, Γ_n^* is a Nash solution and, therefore, stable against expansion. As a special case, it is stable against expansion at the point $\boldsymbol{a}$. $\Gamma^*(\boldsymbol{a}) < 1$ and, therefore, for large enough n. $\Gamma_n^*(\boldsymbol{a}) < 1$ and we know that $\Delta(\boldsymbol{a}, \Gamma_n^*) \leqq 0$. By continuity argument $\Delta(\boldsymbol{a}, \Gamma) \leqq 0$. But $\Gamma(\boldsymbol{a}) = \Gamma^*(\boldsymbol{a})$ hence, by employing (3.5) we get

$$\begin{aligned} 0 \geqq \Delta(\boldsymbol{a}, \Gamma) = (1 - \lambda ps)\{(a_1 - a_2)\Gamma^*(\boldsymbol{a}) + (a_3 - a_4)(1 - \Gamma^*(\boldsymbol{a}))\} \\ + ps(s - \lambda)\{(a_1 - a_3)\Gamma^*(a) + (a_2 - a_4)(1 - \Gamma^*(\boldsymbol{a}))\} \end{aligned} \tag{3.12}$$

where $s = s(\Gamma)$. But $\Gamma^* \leqq \Gamma$ and, therefore, as we have seen, $s^* = s(\Gamma^*) \leqq s(\Gamma) = s$ and by replacing s by s^* in the right side of (3.12) we obtain $\Delta(\boldsymbol{a}, \Gamma^*) \leqq 0$.

This is true for all $\boldsymbol{a} \in \Omega$ with $\Gamma^*(\boldsymbol{a}) < 1$ and Γ^* is, thus, stable against expansion. From Proposition (3.4) we now deduce that there exists a Nash solution $\Gamma^{**} \leqq \Gamma^*$, but, from the definition of Γ^*, $\Gamma^* \leqq \Gamma^{**}$, hence $\Gamma^{**} = \Gamma^*$ is a Nash solution, and this completes the proof of the first part of the proposition. Namely, there is a minimal Nash solution Γ with $\{\boldsymbol{a} \mid \Gamma(\boldsymbol{a}) > 0\} = G$.

(ii) Suppose there is no value x for which $G = G_{x_0}$, then there are two points $\boldsymbol{a} \in G$, $\boldsymbol{b} \in G$ such that

$$\frac{a_4 - a_3}{a_2 - a_4} > \frac{b_4 - b_3}{b_2 - b_4}.$$

$\Gamma(\boldsymbol{b}) = 0$ (where Γ is the minimal Nash solution). Hence it is implied from Proposition 3.1 that

$$ps(s - \lambda)(b_2 - b_4) - (1 - \lambda ps)(b_4 - b_3) = \Delta^+(\boldsymbol{a}, \Gamma) \leqq 0.$$

This, in turn, implies

$$\frac{a_4 - a_3}{a_2 - a_4} > \frac{b_4 - b_3}{b_2 - b_4} > \frac{ps(s - \lambda)}{1 - \lambda ps}. \tag{3.13}$$

(Note that from the definition of cooperation we have assumed (2.1), so that always $X_2 > X_4$, hence the denominators are positive.) Denote by Γ' the strategy obtained from Γ by determining $\Gamma' \equiv 0$ over an ε-measure vicinity of $\boldsymbol{a}$ and $\Gamma' \equiv \Gamma$ elsewhere. For $\varepsilon > 0$ sufficiently small, the value $s' = s(\Gamma')$ is sufficiently close to s so that it follows from (3.13) that

$$\frac{x_4 - x_3}{x_2 - x_4} > \frac{ps'(s' - \lambda)}{1 - \lambda ps'} \tag{3.14}$$

for all $\boldsymbol{x}$ at the ε-vicinity of $\boldsymbol{a}$. Moreover, since $\Gamma' \leqq \Gamma$ with strict inequality on a positive measure set then $s' = s(\Gamma') < s(\Gamma) = s$ and (3.14) is indeed true for all $\boldsymbol{x} \notin G$. Assume now $\Gamma'(\boldsymbol{x}) > 0$ we know that either $\Gamma'(\boldsymbol{x}) = \Gamma(\boldsymbol{x}) = 1$ or else (since $\Gamma(\boldsymbol{x}) \geqq \Gamma'(\boldsymbol{x}) > 0$ and Γ is a Nash solution) $\Delta(\boldsymbol{x}, \Gamma) \leqq 0$. But since $s' \leqq s$, $\Delta(\boldsymbol{x}, \Gamma') \leqq \Delta(\boldsymbol{x}, \Gamma)$, hence $\Delta(\boldsymbol{x}, \Gamma') \leqq 0$ and Γ' is stable against expansion. It, therefore, follows from Proposition 3.4 that a Nash solution $\Gamma'' \leqq \Gamma'$ exists with $\Gamma''(\boldsymbol{a}) = 0$ while $\Gamma(\boldsymbol{a}) > 0$ contradiction to the assumption that Γ is a minimal Nash solution. We, thus, proved $G = G_{x_0}$.

Finally, since $G_0 = \{\boldsymbol{a} \mid a_4 \leqq a_3\}$ contains no point of the prisoner's dilemma type, it follows from Proposition 3.3 that $x_0 > 0$ and this completes the proof.

In a similar way, one can prove the following dual proposition.

Proposition 3.5′. There is a maximal Nash solution Γ and a positive value $y_0 > 0$ such that

$$\{\boldsymbol{a} \mid \Gamma(\boldsymbol{a}) = 1\} = \left\{\boldsymbol{a} \,\middle|\, \frac{a_2 - a_1}{a_1 - a_3} < y_0\right\} = H_{y_0}, \quad \text{say}, \tag{3.15}$$

(the maximal range of full cooperation).

As it follows from Proposition 3.3 $p(H_{y_0} \cap D) > 0$, i.e., H_{y_0} includes a non-degenerate subset of the prisoner's dilemma range. As we see, however, it is possible that $H_{y_0} = \Omega$ and full cooperation is a Nash solution.

4. The model of positive association

In many classes of human (and, maybe, animal) conflicts, the temptation to defect is higher when one's opponent defects, i.e.

$$p\{X_4 - X_3 > X_2 - X_1\} = 1. \tag{4.1}$$

We refer to (4.1) as the assumption of *positive association*. By simple algebraic manipulations, (4.1) can be written equivalently as,

$$(4.2)\qquad \begin{aligned} &\frac{X_4 - X_3}{X_2 - X_4} > \frac{X_2 - X_1}{X_1 - X_3} \quad \text{if } X_4 \geqq X_3 \\ &\frac{X_4 - X_3}{X_1 - X_3} > \frac{X_2 - X_1}{X_2 - X_4} \quad \text{if } X_4 < X_3 \text{ (and } X_2 < X_1) \end{aligned}$$

with probability 1.

Henceforth, for any measurable set $G \subseteq \Omega$ we denote by G the pure strategy Γ, $\Gamma(x) = 1_G$.

Proposition 4.1. In a model of positive association, the minimal Nash solution is the pure strategy G_{x_0} itself, namely: cooperate if and only if $(a_4 - a_3)/(a_2 - a_4) < x_0$.

Proof. Let Γ be the minimal Nash solution. From Proposition 3.5 we know that $\{\boldsymbol{a} \mid \Gamma(\boldsymbol{a}) > 0\} = G_{x_0}$. From the minimal property of Γ it also follows that

$$(4.3)\qquad \frac{a_4 - a_3}{a_2 - a_4} \leqq \frac{ps(s-\lambda)}{1 - \lambda ps}$$

for almost all $\boldsymbol{a} \in G_{x_0}$, otherwise we can build (in a similar way as in the proof of Proposition 3.5) a strategy Γ' which is equal to Γ everywhere except on an ε-measure vicinity of the point $\boldsymbol{a} \in G_{x_0}$ which does not obey (4.3). We then define $\Gamma'(\boldsymbol{x}) = 0$ over this vicinity and show that for a small enough $\varepsilon > 0$, Γ' is stable against expansion and therefore there is a Nash solution $\Gamma'' \leqq \Gamma' < \Gamma$ contrary to the minimal property of Γ.

Moreover, for any $\boldsymbol{a} \in G$ it follows from (4.2) that either

$$(4.4a)\qquad \frac{a_2 - a_1}{a_1 - a_3} < \frac{a_4 - a_3}{a_2 - a_4} \leqq \frac{ps(s-\lambda)}{1 - \lambda ps}$$

or

$$(4.4b)\qquad a_2 < a_1 \quad \text{and} \quad a_4 < a_3.$$

In both cases it follows from (3.8) that $\Delta^-(\boldsymbol{a}, \Gamma) < 0$. From (4.3) and (3.7) it is implied that (for almost all $\boldsymbol{a} \in G_{x_0}$) $\Delta^+(\boldsymbol{a}, \Gamma) \geqq 0$, hence (from 3.6)

$$(4.5)\qquad \Delta(\boldsymbol{a}, \Gamma) > 0 \qquad \text{for almost all } \boldsymbol{a} \in G_{x_0}$$

$$\text{(since } \Gamma(a) > 0 \text{ on } G_{x_0}).$$

From Proposition 3.1 we, thus, infer that $\Gamma(\boldsymbol{a}) = 1$ for all $\boldsymbol{a} \in G_{x_0}$ and the pure strategy G_{x_0} is, therefore, the minimal Nash solution.

We shall see that G_{x_0} is also a strict Nash solution and an ESS. We prove more than this. Let us extend the definition of G_{x_0} so as to denote, for any $x \geqq 0$:

$$(4.6)\qquad G_x = \left\{ \boldsymbol{a} \,\middle|\, \frac{a_4 - a_3}{a_2 - a_4} < x \right\}.$$

Denote also

$$(4.7)\qquad s_x = s(G_x, G_x)$$

and

$$g(x) = \frac{ps_x(s_x - \lambda)}{1 - \lambda ps_x}. \tag{4.8}$$

Since G_{x_0} is a Nash solution, it follows from (3.7) and Proposition 3.1 that

$$\frac{a_4 - a_3}{a_2 - a_4} \geqq \frac{ps_{x_0}(s_{x_0} - \lambda)}{1 - \lambda ps_{x_0}} = g(x_0)$$

for almost all $\boldsymbol{a} \notin G_{x_0}$. (4.3) can be written as $(a_4 - a_2)/(a_2 - a_4) \leqq g(x_0)$ for almost all $\boldsymbol{a} \in G_{x_0}$. From this and the definition of (4.6) it follows immediately that

$$g(x_0) = x_0. \tag{4.9}$$

Proposition 4.2. If $g(x) = x$ then G_x is an ESS.

Proof. From the definition of G_x it follows that for almost all $\boldsymbol{a} \notin G_x$

$$\frac{a_4 - a_3}{a_2 - a_4} > x = g(x) = \frac{ps_x(s_x - \lambda)}{1 - \lambda ps_x}$$

hence from (3.8) $\Delta^+(\boldsymbol{a}, G_x) < 0$.

On the other hand, for all $\boldsymbol{a} \in G_x$

$$\frac{a_2 - a_1}{a_1 - a_3} < \frac{a_4 - a_3}{a_2 - a_4} < x = \frac{ps_x(s_x - \lambda)}{1 - \lambda ps_x}$$

or

$$\frac{a_2 - a_1}{a_1 - a_2}, \frac{a_4 - a_3}{a_2 - a_4} < 0 \leqq x = \frac{ps_x(s_x - \lambda)}{1 - \lambda ps_x}$$

and (employing (3.8)) $\Delta^-(\boldsymbol{a}, G_x) < 0$. From Proposition 3.2 it, therefore, follows that G_x is an ESS.

As a special case we know that the minimal Nash solution G_{x_0} is an ESS. We also conclude that x_0 is the smallest positive solution of $g(x) = x$.

We shall see now that the condition $g(x) = x$ is sufficient but not necessary for G_x to be an ESS.

Proposition 4.3. The pure strategy G_x is stable against expansion if and only if

$$g(x) \leqq x. \tag{4.10}$$

Proof. G_x is stable against expansion if and only if

$$\frac{a_4 - a_3}{a_2 - a_4} \geqq \frac{ps_x(s_x - \lambda)}{1 - \lambda ps_x} = g(x)$$

for almost all $\boldsymbol{a} \notin G_x$, i.e., for almost all $\boldsymbol{a}$ with $(a_4 - a_3)/(a_2 - a_4) \geqq x$. This is true if and only if $g(x) \leqq x$.

We use this result in order to demonstrate a partial set of all evolutionary stable strategies. Note that $G_0 = \{X \mid X_4 \leqq X_3\} \supseteq R = \{X \mid X_4 \leqq X_3; X_2 \leqq X_1\}$ hence $p(G_0) \geqq p(R) > 0$ and $S_0 = \lambda + \iint_{G_0}(a_1 - a_4)dF(\boldsymbol{a}) > \lambda$, since

$$S_0 = \iint_{G_0} a_1 dF(\boldsymbol{a}) + \iint_{\Omega - G_0} a_4 dF(\boldsymbol{a}) = \iint_{G_0} a_1 dF(\boldsymbol{a}) + \lambda - \iint_{G_0} a_4 dF(\boldsymbol{a}).$$

This implies

$$g(0) = \frac{pS_0(s_0 - \lambda)}{1 - \lambda p S_0} > 0. \tag{4.11}$$

Denote

$$\sup_{\boldsymbol{a} \in \Omega} \text{ess} \frac{a_4 - a_3}{a_2 - a_4} = x^* \tag{4.12}$$

(including the case $x^* = \infty$).

Employing the continuity of the distribution $F(\boldsymbol{a})$, we know that $g(x)$ is continuous, hence there are two possibilities:

(i) $g(x) \geqq x$ for all $0 < x \leqq x^*$ with, perhaps, an equality on a set of isolated points,

(ii) $g(x) < x$ over some interval $[x_1, x_2]$, $x_1 < x < x_2$.

Proposition 4.4. In a model of positive association:

(i) If $g(x) > x$ for some range $c \leqq x < x^*$, $(c < x^*)$, then Ω (full cooperation) is an ESS.

(ii) If $g(x) > x$ for all $0 < x < x^*$ then Ω is essentially the only ESS and also the only Nash solution of the game (i.e., for any other Nash solution Γ, $p(\boldsymbol{a}; \Gamma(\boldsymbol{a}) < 1) = 0$).

(iii) If $g(x) < x$ for some value $0 < x < x^*$, then there is a range (x_1, x_2) (including, perhaps, the case $x_2 = \infty$) such that G_ξ is an ESS for all $x_1 \leqq \xi < x_2$.

Proof. (i) If $g(x) > x$ for all $c \leqq x < x^*$ then for all $\boldsymbol{a} \in \Omega$

$$\frac{a_2 - a_1}{a_1 - a_3} < \frac{a_4 - a_3}{a_2 - a_4} \leqq x^*$$

or

$$\frac{a_2 - a_1}{a_1 - a_3}, \frac{a_4 - a_3}{a_2 - a_4} < 0 \leqq x^*$$

where

$$x^* \leqq \lim_{x \to \infty} g(x) = \frac{ps_{x^*}(s_{x^*} - \lambda)}{1 - \lambda p s_{x^*}}.$$

Employing (3.7) and Proposition 3.2 we, therefore, know that Ω is an ESS. (Since $\Delta^+(\boldsymbol{a}, \Omega) \geqq 0$ and $\Delta^-(\boldsymbol{a}, \Omega) < 0$ for all $\boldsymbol{a} \in \Omega$.)

(ii) If $g(x) > x$ for all $0 \leqq x < x^*$ then we know (Part (i)) that Ω is an ESS. Also, from Proposition 4.3, we know that no strategy $G_x \neq \Omega$ is stable against expansion and therefore Ω is the minimal Nash solution, hence unique.

(iii) Assume now $g(x) < x$ for some $0 < x < x^*$, then there is an interval (x_1, x_3) so that $g(x_1) = x_1$ and $g(\xi) < \xi$ for all $x_1 < \xi < x_3$.

For any $\boldsymbol{a}$ on the boundary of G_{x_1}

$$\frac{a_2 - a_1}{a_1 - a_3} < \frac{a_4 - a_3}{a_2 - a_4} = x_1 = g(x_1).$$

But the boundary of G_{x_1} is a compact set, hence, for $\xi = x_1$

$$\max_{\text{bound}(G_\xi)} \frac{a_2 - a_1}{a_1 - a_3} < g(\xi). \tag{4.13}$$

Both sides of (4.12) are continuous functions of ξ, hence there is a value x_2, $x_3 \leqq x_2 > x_1$, such that (4.12) holds for all $x_1 \leqq \xi < x_2$ and it follows from (3.8) that $\Delta^-(\boldsymbol{a}, G_\xi) < 0$ for all $\boldsymbol{a} \in G_\xi$. But since $x_1 \leqq \xi \leqq x_3$, $g(\xi) < \xi$ and we also know that $\Delta^+(a, G_\xi) < 0$ for all $a \notin G_\xi$ (Proposition 4.3). From Proposition 3.2 it, therefore, follows that G_ξ is an ESS.

Remarks. (i) Note that although the value x^* itself is highly sensitive to minor changes in 'tail probabilities' of F, the basic result is essentially robust in the sense that if full cooperation is the only ESS with the distribution F and if by minor tail-changes of F to $\tilde{F}$ one increases the right side of (4.11) from x^* to $\tilde{x}^*$ (which may be much larger) then, by following the arguments of the proof one can readily verify that with $\tilde{F}$, any possible ESS (and there exists at least one) determines full cooperation except, maybe, to the event $x^* - \varepsilon < x < \tilde{x}^*$, which is rare even in terms of $\tilde{F}$.

(ii) Except for a singular case we know that either $g(x) > x$ for all $0 \leqq x < x^*$ or $g(x) < x$ for some $0 < x < x^*$.

In the singular case where $g(x) \geqq x$ for all $0 \leqq x < x^*$ with equality holding on a discrete set of points x_i, we know from Proposition 4.2 that G_{x_i} is indeed an ESS (and even a strict Nash solution). Yet it is not the *continuously stable* property that, as we have suggested elsewhere (see Eshel (1983), Eshel and Motro (1981)) is more appropriate for population games with a non-discrete set of pure strategies. More specifically we know that if a large enough majority of the population plays exactly the strategy G_{x_i} then it will be better off than any sufficiently small minority playing any other alternative strategy. However, for any $\varepsilon > 0$ there are infinitely many pure strategies G which differ from G_{x_i} on a set of measure less than ε, so that if a large enough majority in the population will choose to play G, then it will be individually advantageous for any player to choose a strategy further off G_{x_i} (i.e., different from G_{x_i} on a larger set). For example, for any $x_i < x < x_i + \delta$ with $\delta > 0$ sufficiently small we know that $g(x) > x$ and, therefore, G_x is unstable with respect to expansion (it is stable with respect to desertion).

We may thus conclude the main findings of this section as follows.

Theorem. In a model of positive association, Ω (i.e., unconditional cooperation) is either the only ESS (or at least, in a singular case, the only ESS which is continuously stable) or there is a continuity of ESSs G_ξ, $x_1 \leqq \xi < x_2$, all of which being continuously neutral to each other, and indeed, all determining full cooperation on a non-degenerated subset $G_{x_0} \cap D$ of encounters of the prisoner's dilemma type.

Remarks. (i) If the ratio $(a_4 - a_3)/(a_2 - a_4)$ is not bounded over the support of F, then $x^* = \infty$ and since

$$\lim_{x\to\infty} g(x) = \frac{pEX_1E(X_1 - X_4)}{1 - pEX_1EX_4} < \infty,$$

Ω is never an ESS. Hence, a continuity of ESSs $\{g_x\}_{x_1 \leqq x \leqq x_2}$ always exists.

(ii) By the dual arguments mentioned in Proposition 3.5′ one can readily show that if Ω is not an ESS, then in addition to the class $\{G_x\}_{x_1 \leqq x < x_2}$ there is always another continuous class of ESSs $\{H_y\}_{y_1 < y \leqq y_2}$ where

$$H_y = \left\{ \boldsymbol{a} \,\middle|\, \frac{a_2 - a_1}{a_1 - a_3} \leqq y \right\}.$$

Moreover, there is always a maximal Nash solution which is a pure ESS of the form H_{y_0}, $G_{x_0} \subseteq H_{y_0}$.

G_{x_0} can be interpreted as the strategy toward which the population is bound to evolve from absolute selfishness. H_{y_0} can be interpreted as the strategy toward which the population is bound to collapse, starting from full (or unstably too high) cooperation.

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