The Fourier Transform

Given a continuous function \( f(x) \), its Fourier transform \( F(u) \) is defined as:

\[
F(u) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi ux} dx
\]

The inverse transform is:

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(u) e^{i2\pi ux} du
\]

In general the transform is complex:

\[
F(u) = R(u) + iI(u)
\]

Amplitude:

\[
|F(u)| = \sqrt{R^2(u) + I^2(u)}
\]

Phase:

\[
\phi(u) = \tan^{-1}\left( \frac{I(u)}{R(u)} \right)
\]
The Discrete FT (DFT)

- Defined as:
  \[ F(u) = \sum_{x=0}^{N-1} f(x) \exp\left[-i2\pi ux / N\right] \]

- The inverse transform is:
  \[ f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) \exp[i2\pi ux / N] \]

The Convolution Theorem

- The convolution of two functions \( f(x) \) and \( g(x) \), denoted by \( f(x) * g(x) \), is defined by the integral:
  \[ f(x) * g(x) = \int_{-\infty}^{+\infty} f(a) g(x-a) da \]

- The Fourier transform of the convolution \( f(x) * g(x) \) is the pointwise product of the two Fourier transforms \( F(u)G(u) \), and vice versa:
  \[ f(x) * g(x) \iff F(u)G(u) \]
  \[ f(x)g(x) \iff F(u) * G(u) \]
The Sampling Theorem

- A bandlimited signal $f(x)$, with cutoff frequency $w$, can be reconstructed exactly from uniform samples taken at frequency at least $2w$.

- Reconstruction is performed by convolving the discrete signal with the (properly scaled) sinc function:

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$
Sampling and Reconstruction

Box filter

Hat filter

Gaussian filter
**Proper Sampling**

- Continuos input
- Prefilter
- Bandlimited continuous signal
- Sample
- Discrete output