Liquids animations

Utilization of fluid mechanics to create realistic animations of dynamic liquids and gases scenes

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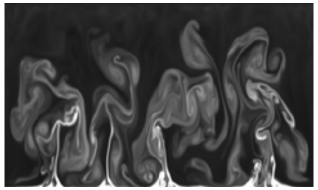
Liquids animations

- Gas behavior is described quite similarly to liquids and therefore our discussion applies to both.
- Animations of smokes and liquids are not easy to create using standard modeling software.

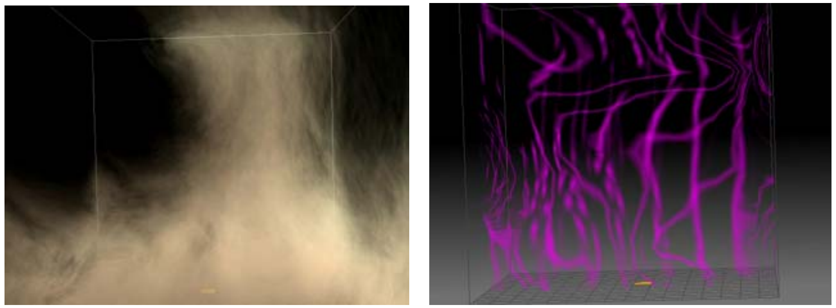


Liquids animations

- Using mathematical simulation of Newtonian mechanics applied for fluids (gases), we can achieve realistic dynamic animations of such materials.
- It is almost impossible to model such shapes with elementary geometric entities. (specially when creating dynamic scenes).
- Objects such as clouds:

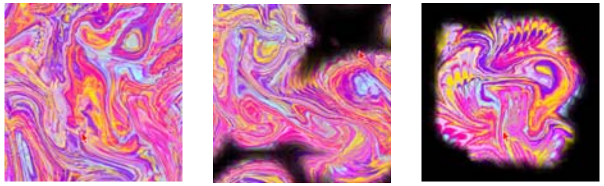
Liquids animations

- Different kinds of smoke (gases):



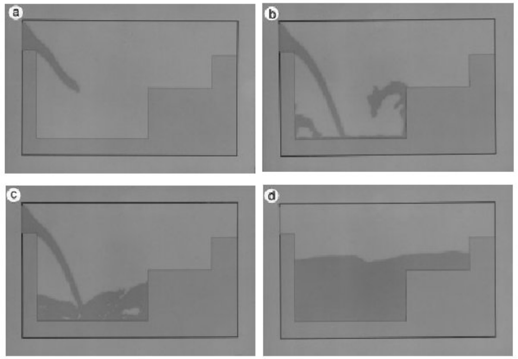
Liquids animations

- Different colored fluids mixed together:



Liquids animations

- Simulations of water colors and splashing water:



Topics discussed next

- Derivation of the mathematical model describing the fluids mechanics.
- Numerical methods for solution.
- (Computer Graphics aspects.)

Mathematical Model

- First, we begin with the model of an *ideal fluid* - a fluid who’s motion satisfies the conservation laws of mass, momentum and energy.
- Next, we add the viscous effect which arise from the molecular transport of momentum.
- Finally, we’ll end up with the **Navier-Stokes** equations for incompressible flow, by adding this property to the fluid.

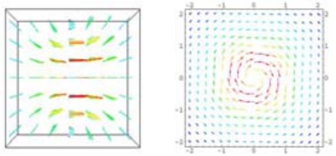
Definitions

- A real differential function defined over some subset of the Real space (plane) will be called a *Scalar Field*

$$f : D \to \Re \qquad D \subseteq \Re^{2,3}$$

- A vector differential function defined over some subset of the Real space (plane) will be called a *Vector Field*

$$f : D \to \Re^{2,3} \qquad D \subseteq \Re^{2,3}$$



Definitions

- Some essential functions used in our model:

Let $\vec{x} = (x, y, z)$ denote a point in the three dimentional space,

Let $\underline{u} : (\vec{x}, t) \to (u_x, u_y, u_z)$ denote the velocity of the particle of fluid moving through $\vec{x}$ at time t and call it the velocity field of the fluid,

Let $\rho : (\vec{x}, t) \to \Re$ denote the fluid's mass density at point $\vec{x}$ in space and at time t.

Definitions

- We denote the Del partial derivatives operator by Nabla: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
- Notice that: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = grad(f)$
- The Divergence of vector field F is defined by:

$$divF = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \text{whereas} \quad F = (F_x, F_y, F_z)$$

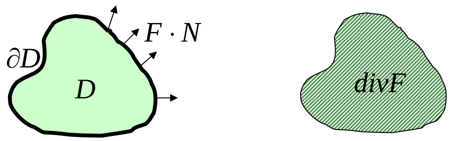
$$\nabla \cdot F = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = div(F)$$

Theorems

- The **Divergence Theorem** states that:

$$\int_{\partial D} F \cdot N \, dA = \int_{D} divF \, dV$$

- It means that the integral of some vector field multiplied with the unit outer normal of the boundary of some domain D, is equal to the integral of the divergence over that domain.



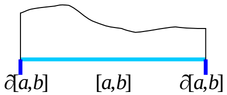
Theorems

- The Divergence Th. is a special case of the Stokes Theorem that argues:

$$\int_{\partial D} F dA = \int_{D} dF\ dV$$

- Which is actually the extension of the Fundamental theorem of the Integration calculus. The integral over the boundary is equal to the domain integral of the differential.

$$\int_a^b f(x)dx = F(b) - F(a)$$



Conservation of Mass

- Let W be a sub region of D, the total mass in W at time t is:

$$m(W,t) = \int_W \rho(\vec{x},t)dV$$

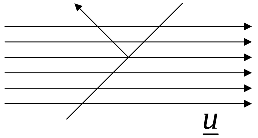
- Therefore, the rate of change of mass in W is:

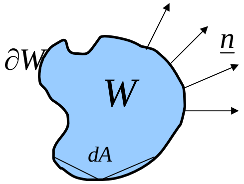
$$\frac{\partial}{\partial t} m(W,t) = \frac{\partial}{\partial t}\int_W \rho(\vec{x},t)dV = \int_W \frac{\partial}{\partial t}\rho(\vec{x},t)dV$$

Conservation of Mass

Let ∂W denote the boudary of W, let $\underline{n}$ denote the unit outward normal defined over ∂W and let dA denote the area element on ∂W. The volume flow rate across ∂W per unit area is $-\,\underline{u}\cdot\underline{n}$ and the mass flow rate per unit area is $-\,\rho\underline{u}\cdot\underline{n}$.

- A simple motivation:





Conservation of Mass

- Thus the total mass flow rate across ∂W is the following surface integral:

$$-\int_{\partial W} \rho\underline{u}\cdot\underline{n}\ dA = \frac{\text{mass}}{\text{time}}$$

- Which is the rate of change of mass crossing ∂W in the outward direction.

Conservation of Mass

- Therefore the principle of conservation of mass in W equals the rate at which mass is crossing ∂W in the inward direction with the change of mass:

$$\int_W \frac{\partial}{\partial t}\rho(\vec{x},t)dV = -\int_{\partial W} \rho\underline{u}\cdot\underline{n}\,dA$$

- Now, according the divergence theorem we get:

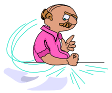
$$\int_{\partial D} F\cdot N\,dA = \int_D divF\,dV \quad \Rightarrow \quad \int_{\partial W} \rho\underline{u}\cdot\underline{n}dA = \int_W div(\rho\underline{u})dV$$

Conservation of Mass

- We will finally get the following integral form of the law of conservation of mass:

$$\int_W \frac{\partial\rho}{\partial t} + div(\rho\underline{u})\,dV = 0$$

- And since the above holds for all W subset of D, we get:



$$\boxed{\frac{\partial\rho}{\partial t} + div(\rho\underline{u}) = 0}$$

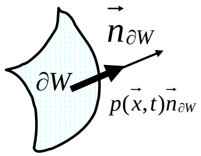
- Which is called the differential form of the law of conservation of mass, also known as the *continuity equation*.

Balance of Momentum

- Given some region W subset of D, we'll denote
- $p(\vec{x},t)$ to be the pressure (force per unit area) at point $\vec{x}$ and at time t in the direction of $\vec{n}_{\partial W}$ (which is the unit outward normal of ∂W).
- Then the total force over the surface enclosing W is:

$$-\int_{\partial W} pI\cdot ndA = -\int_W \nabla p dV$$

- The equality is followed from Stokes theorem.



Balance of Momentum

- Since the former equation holds for any W in D we may conclude that the force exerted on any fluid element by the remaining fluid is: $-\nabla(p)$ per a unit volume.
- Next we wish that every fluid particle will satisfy the second rule of the Newtonian mechanics:

F=ma

- We've just seen that **F**= $-\nabla p$, and know that **m**= ρ which will assumed to be a constant soon.

Balance of Momentum

- We are left to evaluate **a** the acceleration parameter of the particle.
- Denoting the path of a specific particle (volume element) as: $\vec{x} = (x(t), y(t), z(t))$ We get that : $\underline{v}(t) = \underline{u}(x(t), y(t), z(t), t) = (x`(t), y`(t), z`(t))$
- According the chain rule we get:

$$a(t) = \frac{\partial \underline{v}}{\partial t} = \frac{\partial \underline{u}}{\partial t}(x(t), y(t), z(t), t) = \frac{\partial \underline{u}}{\partial t} + \frac{\partial \underline{u}}{\partial x}x` + \frac{\partial \underline{u}}{\partial y}y` + \frac{\partial \underline{u}}{\partial z}z` =$$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} \text{ whereas } \underline{u} \cdot \nabla = x`\frac{\partial}{\partial x} + y`\frac{\partial}{\partial y} + z`\frac{\partial}{\partial z}$$

Balance of Momentum

- And therefore the expression **F=ma** is:

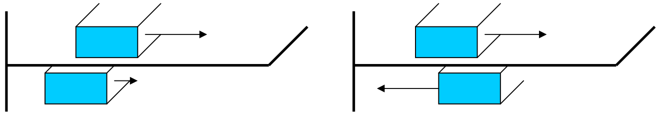
$$\rho(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla)\underline{u} = -\nabla p \Leftrightarrow (\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla)\underline{u} + \frac{1}{\rho}\nabla p = 0$$

- Adding some external force F (gravity for example) we get:

$$\boxed{\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} + \frac{1}{\rho}\nabla(p) = F}$$

Viscous Effects

- So far we have described the mechanics of an ideal fluid. Where no tangential interaction between adjacent elements was taken into account.



- Consider the case illustrated above, where two close fluid particles (in this case we can think of molecules) move in the same direction but in a different speed (or in the opposite directions).

Viscous Effects

- We'd expect that in such cases the different particles will exchange momentum (velocity).
- This interaction is called the *Viscous Effect* and rises only when neighboring fluid elements move relative to each other.
- According to experimental data the viscous force on a fluid element at any point is proportional to the velocity's gradient at that point $\propto \nabla \underline{u}$.

Viscous Effects

- Using the divergence theorem such as before:

$$F_{vis}(W) = \int_{\partial W} \nabla \underline{u} \cdot N \, dA = \int_{W} div \nabla \underline{u} \, dV$$

- Which applies for all W subset of D thus the force on every fluid particle is:

$$div \nabla \underline{u} = \frac{\partial^2 \underline{u}}{\partial x^2} + \frac{\partial^2 \underline{u}}{\partial y^2} + \frac{\partial^2 \underline{u}}{\partial z^2} = \nabla^2 \underline{u} = \Delta \underline{u}$$

- This is the Laplacian of u, usually multiplied by scalar ν know as the *kinematic viscousity.* Giving: $\boxed{F_{vis} = \nu \Delta \underline{u}}$

The Navier-Stokes Eqs.

- So far we derived the following three equations:
- Mass cons. - $\boxed{\frac{\partial \rho}{\partial t} + div(\rho \underline{u}) = 0}$
- Momentum cons. - $\boxed{\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} + \frac{1}{\rho}\nabla(p) = F}$
- Viscous Effect - $\boxed{F_{vis} = \nu \Delta \underline{u}}$

The Navier-Stokes Eqs.

- Assuming that the fluid is incompressible we get that: $\rho = \rho_0 \Rightarrow \frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \underline{u} = div(\underline{u}) = 0$
- Plugging all together we finally get:

$$\boxed{\begin{gathered} \nabla \cdot \underline{u} = 0 \\ \frac{\partial \underline{u}}{\partial t} = -(\underline{u} \cdot \nabla)\underline{u} - \frac{1}{\rho}\nabla(p) + \nu \Delta \underline{u} + F \end{gathered}}$$

- Which is widely known as the ***Navier-Stokes eqs. for incompressible fluids.***

The Navier-Stokes Eqs.

- The explicit (scalar) form of those eqations is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} = -\frac{\partial u^2}{\partial x} - \frac{\partial uv}{\partial y} - \frac{\partial uw}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + F_x$$

$$\frac{\partial v}{\partial t} = -\frac{\partial vu}{\partial x} - \frac{\partial v^2}{\partial y} - \frac{\partial vw}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) + F_y$$

$$\frac{\partial w}{\partial t} = -\frac{\partial wu}{\partial x} - \frac{\partial wv}{\partial y} - \frac{\partial w^2}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) + F_z$$

- Whereas $\underline{u} = (u, v, w)$ And $F = (F_x, F_y, F_z)$

The Navier-Stokes Eqs.

- These are 4 partial differential equations with four unknowns: u, v, w, p
- These equations must be supplemented by boundary conditions.
- The conditions in our case are $\underline{u} = \vec{0}$ on the boundary of the fluid space.
- Together with these conditions the equations are well-posed, i.e. a unique solution exists and depends continuously on the initial data.

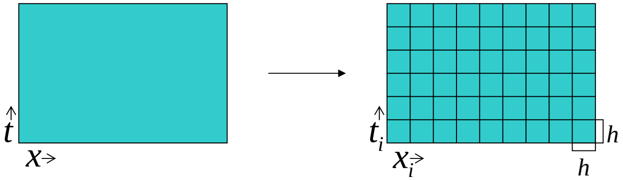
Part II - Numerical Solution

Numerical Methods of Solution

- Such sets of differential equations are usually impossible to solve exactly (analytically). So approximations are needed.
- **Finite Differences Method**:
  Using discrete functions representations, derivatives can be approximated using differences.

Numerical Methods of Solution

- A one dimensional example for parabolic equations: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ whereas $u = u(t,x) \to \Re$



- The domain's discretization is illustrated above with some interval h for both x,t directions.

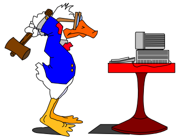
Numerical Methods of Solution

⌘ Approximating the first derivative using Taylor poly.:

$$f(t+h,x) = f(t,x) + \frac{\partial f}{\partial t}(t,x)h - \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,x)h^2 + o(h^3)$$

$$f(t-h,x) = f(t,x) - \frac{\partial f}{\partial t}(t,x)h - \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,x)h^2 + o(h^3)$$

$$f(t+h,x) - f(t-h,x) = 2\frac{\partial f}{\partial t}(t,x)h + o(h^3)$$



$$\frac{\partial f}{\partial t}(t,x) = \frac{f(t+h,x) - f(t-h,x)}{2h} + o(h^2)$$

⌘ And in our discrete domain:

$$\frac{\partial f}{\partial t}(t_i,x_j) \approx \frac{f(t_{i+1},x_j) - f(t_{i-1},x_j)}{2h} \approx \frac{f(t_{i+1},x_j) - f(t_i,x_j)}{h}$$

Numerical Methods of Solution

⌘ Approximating the second derivative using Taylor poly.:

$$f(t+h,x) = f(t,x) + \frac{\partial f}{\partial t}(t,x)h + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,x)h^2 + o(h^3)$$

$$f(t-h,x) = f(t,x) - \frac{\partial f}{\partial t}(t,x)h + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,x)h^2 + o(h^3)$$

$$f(t+h,x) + f(t-h,x) = 2f(t,x) + \frac{\partial^2 f}{\partial t^2}(t,x)h^2 + o(h^3)$$

$$\frac{\partial^2 f}{\partial t^2}(t,x) = \frac{f(t+h,x) - 2f(t,x) + f(t-h,x)}{h^2} + o(h^2)$$

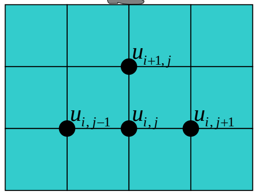
⌘ And in our discrete domain:

$$\frac{\partial^2 f}{\partial t^2}(t_i,x_j) \approx \frac{f(t_{i+1},x_j) - 2f(t_i,x_j) + f(t_{i-1},x_j)}{h^2}$$

Numerical Methods of Solution

⌘ Finally, we get an discrete set of linear equations for the discrete variables:

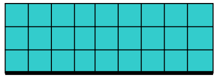
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{u_{i+1,j} - u_{i,j}}{h} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$



whereas $u_{i,j} = u(t_i, x_j)$ we get:

$$u_{i+1,j} = \frac{1}{h}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) + u_{i,j}$$

⌘ This can be solved using one iteration, given the boundary conditions. (i.e. $u_{0,j}\ \forall j$).



Fractional Stepping

⌘ Given equations of the form:

$$\frac{\partial u}{\partial t} = A(u,x,t) + B(u,x,t)$$

⌘ We can decompose it into several steps:

$$\frac{\partial u^*}{\partial t} = A(u,x,t)$$

$$\frac{\partial u}{\partial t} = B(u,x,t)$$

$$\longrightarrow$$

$$u^* = u^n + \Delta t A(u^n, x, t)$$

$$u^{n+1} = u^* + \Delta t B(u^{n/*}, x, t)$$

Computer Graphics rendering

- So far we have the velocity field u calculated. How can we use it for the visual animation?
- How fine can u be in case of 3D? u is a 3D vector field, this means that for every 3 coordinates vector u matches 3 dimensional vector (fractional vector!).
- Simple calculation shows that for 100 by 100 by 100 resolution we need to perform million operations for each time step in our animation (and will consume 6-12MB's).
- Therefore we will have to use lower volume resolution comparing with the output image details level.

Solving NS using Fractional stepping

Momentum Advection $$\frac{\partial u^1}{\partial t} = -u \cdot \nabla u$$

Momentum Diffusion $$\frac{\partial u^2}{\partial t} = \Delta u^1$$

Projection to Divergence-free space $$u^3 = u^2 + \nabla p \quad s.t. \quad \nabla \cdot u^3 = 0$$

Linear Advection equation

$$\frac{\partial u}{\partial t} = -u \cdot \nabla u$$

- Lets assume that *x* is some particles trajectory along *u(x,t)*.

$$\frac{d}{dt}(u(x(t),t)) = \frac{\partial u(x(t),t)}{\partial t} + \frac{\partial x(t)}{\partial t} \cdot \nabla u(x(t),t) =$$
$$-u \cdot \nabla u + u \cdot \nabla u = 0$$

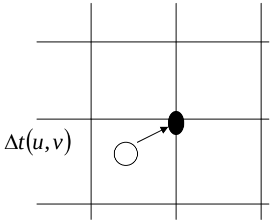
- *u* is fixed! Such trajectories are called characteristics curves of the equations.

Computer Graphics rendering

- This immediately calls for volume interpolation of the results of u.
- Next, we want to mark some of the space with a certain color and thus define a substance inside the volume. Such as milk stirred in coffee.
- This could be done in several ways:

Tracking characteristics curves

- Idea: lets “bring” *u*’s values from where it was a time step ago – upwinding.



- Bilinear Interpolation: (CFL,sur)

$$\alpha = \Delta t u, \beta = \Delta t v \quad u_{i,j}^{n+1} = u_{i,j}^{n} \cdot \alpha\beta + u_{i+1,j}^{n} \cdot (1-\alpha)\beta + u_{i,j+1}^{n} \cdot \alpha(1-\beta) + u_{i+1,j+1}^{n} \cdot (1-\alpha)(1-\beta)$$

Solving the Heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

- What do we know about the nature of this solution?
- Its *Green* function is known:

$$u(x,t) = \int K(x,\xi,t) u_0(\xi) d\xi$$

$$K(x,\xi,t) = \frac{1}{(4\pi)^{d/2}} \exp(-\|x-\xi\|^2 / 4t)$$

- As time evolves every point is receives material from its surrounding according Gaussian distribution with deviations *t.*

Solving the Heat equation

- If dt is large we might have to look at distant variables:
- Explicit:

$$u_i^{n+1} = u_i^{n} + \Delta t \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\Delta x^2} \quad \Rightarrow \quad \Delta t < \Delta x^2 / 2$$

- Implicit:

$$u_i^{n+1} = u_i^{n} + \Delta t \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

Fast Diffusion Solver

$$\frac{\partial^2}{\partial x} \sin(nx) = -n^2 \sin(nx)$$

- Fourier base diagonalizes the Laplacian,

$$Ax = b \quad F^{-1} D F = A \quad \Rightarrow \quad DFx = Fb$$

$$A = \left(I - \frac{\Delta t}{\Delta x^2} \right)$$

Projection step

- Helmotz-Hodge Decomposition Theorem:

$$w = u + \nabla p \quad s.t. \quad \nabla \cdot u = 0$$

- And *w* is the closest in L2 sense!
- But how can we decompose it?

$$\nabla \cdot w = \nabla \cdot u + \nabla \cdot \nabla p$$

$$\nabla \cdot w = \Delta p$$

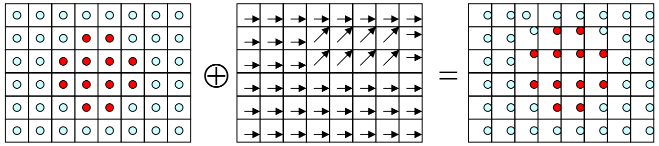
- Poisson Equation which can be solved via Fast Poisson solver.

$$\Delta p = \nabla \cdot u^{n/*}$$

$$u^{n+1} = u^{n/*} - \nabla p$$

Computer Graphics rendering

- We can define a sort of density field of that substance over the entire volume and update it using <u>u</u>.



- Note that here we saw a change of density in the fluid caused by <u>u</u>. According the equations such thing shouldn’t happen. Yet due to error caused by discretization such can happen.

Computer Graphics rendering

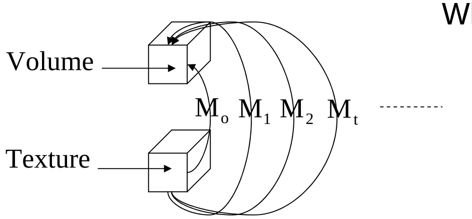
- Volume rendering is a widely researched subject.
- The main problem with this process is the big amounts of data involved - the 3D arrays representing the volume.
- Many methods are suggested for achieving a better speed results.
- Here we will see two fundamental approaches:

Computer Graphics rendering

- Evaluation using line integrals.
- Since we are dealing with semi-transparent substances such as gases and liquids we have to compute the additive influence of every fluid particle on the rendered image.
- Here, each pixel in the resulting image is the line integral of the fluid density oven the projected ray.

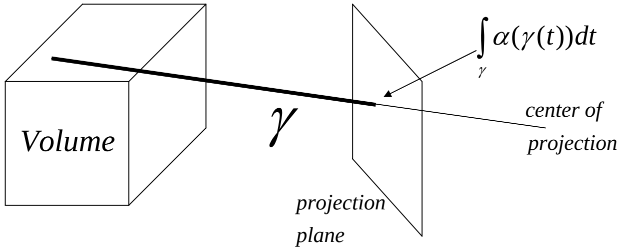
Computer Graphics rendering

- Or we can use volume textures and map them to the volume using a mapping which is updated by u (i.e. the mapping will accumulate the movement of the fluid given by u).
- Given an initial mapping M:V->T, M is updated over time by u. Lets say that M is given explicitly by a table (vector field). The update would be M(t)=M(t-1)-u(t) Where as M(0)=M.



Computer Graphics rendering

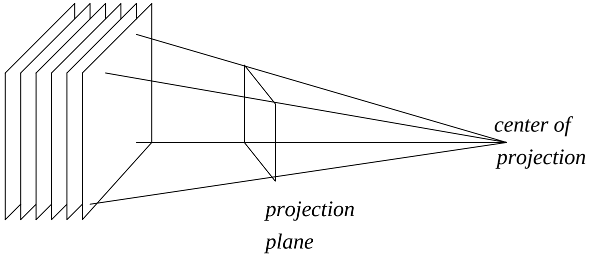
- This line-ray integral is computed for any point in the projection plane.



Computer Graphics rendering

- Since we don’t have full occlusions we may additively project the volume into the viewing plane using forward mapping of the voxels to the image.
- Or by performing projective transformations for the volume’s 2D layers while adding them together.

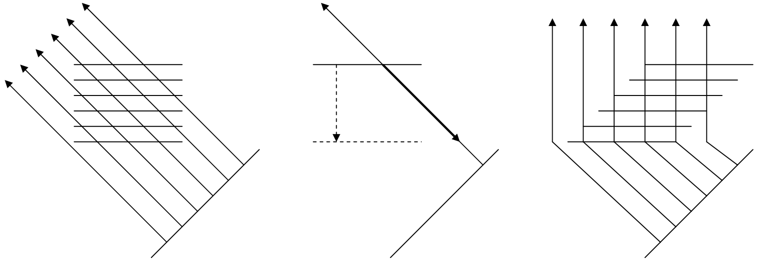
Computer Graphics rendering



- Each layer is projected onto the projection plane using homography in an additive composition.

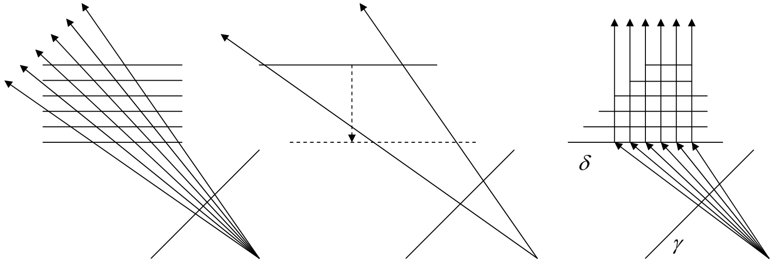
Computer Graphics rendering

- An optimized version is call the **shear-warp factorization**:
- The case of a parallel projection:



Computer Graphics rendering

- The case of a perspective projection:



- And one plane to plane perspective (homography) transformation is needed between δ and γ.

Liquids animations

- Animations by Jos Stam, Ronald Fedkiw

Liquids animations

THE END