Lecture Notes:
Radiance, BRDF, and the Rendering Equation

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1 Radiance

We are interested in quantifying light energy in a 3D scene.

Energy can be transported in space using three different processes:

1. conduction
2. convection
3. radiation

Of these processes, the relevant to light is RADIATION. The relevant physical theory is the theory of RADIANT ENERGY TRANSFER.

There is a field called RADIOMETRY that is the science of measuring radiant energy transfers.

First of all, what is the ENERGY OF LIGHT?

- Light can be thought of as flow of photons;

- Each photon has some energy associated with it. This energy is $h\nu$, where h is Planck's constant and ν is the frequency of the photon. $\nu = c/\lambda$, where c is the speed of light and λ is the wavelength.

- Suppose all photons are traveling in the same direction. Place an imaginary surface perpendicular to that direction, and count all the photons passing through this surface over some period of time. The light energy received by this surface is simply the sum of the energies of all these photons. Like all forms of energy, radiant energy can be measured in joules (J).

- Light moves so fast, that we're more interested in the equilibrium of light in a 3D scene. Thus, it is more appropriate to reason in terms of *flux*, the radiant energy power (energy per second), denoted as Φ. The units of flux are the usual units of power — watts (W).

- In computer graphics, we are often interested in the flux per unit area. This quantity is called *flux density*, and the units are (W/m^2). (For instance, if we want to compare the brightness of two areas).

BUT,

- Photons do not all travel in the same direction. They bounce everywhere in all directions. Thus, to completely describe the distribution of light energy in a 3D scene, we need some function that will tell us what the flux density is at every point in the scene, AND in every possible direction. This leads to the definition of yet another quantity — *radiance*.

Let $x = (x, y, z)$ be a point in 3D, and $\omega = (\theta, \phi)$ a direction. We define the radiance $L(x, \omega)$ as the flux at x, in direction ω, per unit area perpendicular to ω, per unit solid angle.

This definition can be formally expressed as follows:

$$d^2\Phi = L(x, \omega) \cos\theta \, dx \, d\omega$$

Important property of radiance: For any two mutually visible points x and y (separated by vacuum) the radiance leaving x in the direction of y is the same as the radiance impinging on y from the direction of x.

Question1: Doesn't this contradict the attenuated effect of light sources as a quadratic function of the distance?

Question2: What is the flux density arriving at a differential area receiver from a finite area light source with known emitted radiance?

NOTE: there is no attenuation of radiance with distance, because radiance is a per unit solid angle quantity. In practice, real light sources have finite areas, and thus they subtend a finite solid angle. This solid angle diminishes quadratically with distance, and so does the flux arriving from the light source.

2 BRDF

How can we quantitatively describe the manner in which surfaces reflect incident light? The key concept here is the *BRDF (Bidirectional Reflectance Distribution Function)*.

The BRDF is a function that for every pair of directions (ω_i, ω_r) specifies the ratio between the radiance reflected in the direction ω_r and the differential flux density incident from direction ω_i.

The function is denoted as ρ_{bd} or f_r.

Definition:

$$f_r(x, \omega_i \rightarrow \omega_r) \equiv \frac{L(x, \omega_r)}{L_i(x, \omega_i) \cos\theta_i \, d\omega}$$

What are the units of the BRDF?

Where do we get these functions from?

- Theoretical models yield analytical formulas;
- Physical measurements yield a discrete set of samples from which a continuous BRDF can be interpolated.

Note: all these functions have a dependency on the wavelength that I am omitting for the sake of notation simplicity.

Does the BRDF suffice to describe everything that happens to light as it hits a surface?

- reflection

- transmission (possibly with refraction)

- absorption

- fluorescence (change of wavelength)

- phosforescence (time delay)

Properties:

- Can take any value in [0, infinity]
- Must integrate to something less than 1 because of energy conservation.
- Helmholtz reciprocity principle:

$$f_r(\omega_1 \rightarrow \omega_2) = f_r(\omega_2 \rightarrow \omega_1)$$

Question: Does the reciprocity principle contradict the existance of one-way windows?

Anisotropic vs Isotropic BRDFs (4 vs 3 degrees of freedom)

3 Reflection Equations

Having defined radiance and BRDFs we can now start writing equations that describe light reflection off surfaces:

Suppose we have a differential area light source illuminating point x on some receiving surface. What is the radiance reflected from the receiving point x into some direction ω_r?

$$L(x,\omega_r) = f_r(x,\omega_i \rightarrow \omega_r)L_i(x,\omega_i)\cos\theta_i d\omega$$

This equation follows directly from the definition of the BRDF.

What about a finite area light source S? Let $\Omega(S)$ denote the solid angle subtended by the light source. We must integrate the incoming flux over this solid angle:

$$L(x,\omega_r) = \int_{\omega\in\Omega(S)} f_r(x,\omega_{xy} \to \omega_r)\, L_i(x,\omega)\, \cos\theta_x\, d\omega$$

Remember the property of radiance? We can replace $L_i(x,\omega)$ by $L(y,\omega_{yx})$

What is the $d\omega$ corresponding to a differential area of the source centered at y?

$$d\omega = \frac{\cos\theta_y\, dy}{\|x-y\|^2}$$

Yes, but what about occlusion? Introduce a $V(x,y)$ term (0 if x and y are occluded from each other, and 1 otherwise).

Let's define

$$G(x,y) = \frac{\cos\theta_x\, \cos\theta_y\, V(x,y)}{\|x-y\|^2}$$

Note that this term depends only on the relative geometric relationship between x and y.

We can plug $G(x,y)$ into the equation, and we get integration over the surface of the source, instead of integration over the solid angle of the source.

$$L(x,\omega_r) = \int_{y\in S} f_r(x,\omega_{xy} \to \omega_r)\, L_i(x,\omega_{xy})\, G(x,y)\, dy$$

Example: ideal diffuse reflection

In ideal diffuse reflection, incoming light is scattered uniformly in all outgoing directions. Thus, the BRDF does not depend on the outgoing direction ω_r. Because of reciprocity, this means that the BRDF is also independent of the incoming direction, so it is simply a constant

$$f_r(\omega_i \to \omega_r) = \rho$$

Note: in simple illumination models for computer graphics we use *diffuse reflectance* ρ_d, rather than BRDF. The diffuse reflectance is defined as the ratio between the total (hemispherical) outgoing and incoming flux densities. It can be shown that the diffuse BRDF and the diffuse relectance are related by a factor of π:

$$\rho_d = \pi\rho$$

Example: ideal mirror reflection

In the case of a perfect mirror, the angle between the normal and the reflected direction is equal to the angle between the normal and the incident direction, and the reflected direction lies in the plane defined bu the incident direction and the normal. Thus, if $\omega_r = (\theta_r, \phi_r)$ and $\omega_i = (\theta_i, \phi_i)$, it holds that:

$$\begin{aligned} \theta_r &= \theta_i \\ \phi_r &= \phi_i \pm \pi \\ L(x, \omega_r) &= L_i(x, \omega_i) \end{aligned}$$

An appropriate BRDF can be defined using Dirac's delta functions

$$f_r = 2\delta\left(\sin^2\theta_i - \sin^2\theta_r\right)\delta\left(\phi_i - \phi_r \pm \pi\right)$$

This BRDF is non-zero only for a pair mirror reflected directions. If we hold a particular outgoing direction (θ_r, ϕ_r) fixed and integrate the BRDF over the entire hemisphere of incoming directions we get:

$$\begin{aligned} L_r(\theta_r, \phi_r) &= \int_\Omega 2\delta\left(\sin^2\theta_i - \sin^2\theta_r\right)\delta\left(\phi_i - (\phi_r \pm \pi)\right) L_i(\theta_i, \phi_i)\cos\theta_i\, d\omega_i \\ &= \int_\Omega 2\delta\left(\sin^2\theta_i - \sin^2\theta_r\right)\delta\left(\phi_i - (\phi_r \pm \pi)\right) L_i(\theta_i, \phi_i)\cos\theta_i \sin\theta_i\, d\theta_i\, d\phi_i \\ &= \int 2\delta\left(\sin^2\theta_i - \sin^2\theta_r\right)\cos\theta_i \sin\theta_i\, d\theta_i \int \delta\left(\phi_i - (\phi_r \pm \pi)\right) L_i(\theta_i, \phi_i)\, d\phi_i \\ &= \int \delta\left(\sin^2\theta_i - \sin^2\theta_r\right) d\sin^2\theta_i\, L_i(\theta_i, \phi_i - \phi_r \pm \pi) \\ &= L_i(\theta_i, \phi_i - \phi_r \pm \pi) \end{aligned}$$

4 The Rendering Equation

What we want is to know the radiance $L(x, \omega)$ function at every point and every direction in the scene. Consider a point x on some surface in the scene. Where does the radiance leaving this point in a particular direction ω_r come from? It is a combination of the self-emitted radiance there, and the reflected radiance there. What is reflected? Everything that reaches the point from other surfaces in the environment:

$$L(x, \omega_r) = L_e(x, \omega_r) + \int_{\omega_i \in \Omega} f_r(x, \omega_i \to \omega_r)\, L_i(x, \omega_i)\cos\theta_x\, d\omega_i$$

Alternatively, we can write the reflected stuff as an integral over other points in the scene:

$$L(x, \omega_r) = L_e(x, \omega_r) + \int_{y \in \Gamma} f_r(x, \omega_{xy} \to \omega_r)\, L(y, \omega_{yx})\, G(x, y)\, dy$$

The above equations are integral equation, in which the unknown function L appears both on the left-hand-side of the equation, and under the integral on the right-hand-side. Equations of these type are called *Fredholm equations of the second kind*.

There is no analytical solution, in general, for Fredholm equations of the second kind. However, we can write down a formal solution. In order to do that, we must rewrite the equation using an *integral operator*. Let us define a *reflection operator* $\mathcal{R}$ as an operator that takes a radiance function and integrates its product with the BRDF everywhere in the scene:

$$(\mathcal{R}L)(x,\omega) = \int_{\omega_i \in \Omega} f_r(x,\omega_i \to \omega)\, L_i(x,\omega_i)\, \cos\theta_x\, d\omega_i$$

This is a linear operator, whose physical interpretation is to reflect the light in the scene once off all the surfaces. Using operator notation, the rendering equation can be rewritten as:

$$L = L_e + \mathcal{R}L.$$

Formally, such an equation has a solution:

$$\begin{aligned} L &= (I - \mathcal{R})^{-1} L_e \\ &= \sum_{n=0}^{\infty} \mathcal{R}^n L_e \end{aligned}$$

The resulting *Neumann series* is guaranteed to converge if the *spectral radius* of the operator is less than 1. The reflection operator can be shown to satisfy this requirement, since the amount of energy is decreased at every reflection.