Sampling the light sources:

Sampling the BRDF:

Combining the two strategies:

Another Example

Combined Estimator

- Given $n$ sampling strategies $p_1, \ldots, p_n$, take $n_i$ samples from each distribution:

  $$F = \sum_{i=1}^{n} \frac{1}{n_i} \sum_{j=1}^{n_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

- The $w_i$ are weighting functions $\sum_{i=1}^{n} w_i(x) = 1$
Unbiased?

- Yes:

\[
E[F] = \sum_{i=1}^{n} \frac{1}{n_i} \int \frac{w_i(x) f(x)}{p_i(x)} p_i(x) \, dx
\]
\[
= \sum_{i=1}^{n} \int w_i(x) f(x) \, dx
\]
\[
= \int f(x) \, dx
\]

Balance Heuristic

- Define weighting functions as:

\[
\hat{w}_i(x) = \frac{c_i p_i(x)}{\sum c_j p_j(x)}
\]
- Where \( n_t = c_t N \)
- This heuristic can be shown to be nearly optimal.

Optimality Theorem

- Let \( w_1, \ldots, w_n \) be any non-negative functions with \( \sum_i w_i = 1 \), and let \( F \) be the corresponding combined estimator. Then:

\[
V[\hat{F}] \leq V[F] + \left( \frac{1}{\min_i n_i} - \frac{1}{\sum_i n_i} \right) \frac{F^2}{n}
\]

Additional Heuristics

- The cutoff heuristic:

\[
w_i = \begin{cases} 
0 & \text{if } p_i < \alpha p_{\max} \\
\frac{p_i}{\sum_j \left| p_j \right| p_j \geq \alpha p_{\max}} & \text{otherwise}
\end{cases}
\]
- The power heuristic:

\[
w_i = \frac{p_i^\beta}{\sum_j p_j^\beta}
\]

Bi-Directional Path Tracing

- To sample a path of length \( k \):
  - Trace a path with \( s \) vertices from the source;
  - Trace an \( t = (k+1-s) \) vertex path from the eye;
  - Deterministically connect the two paths.

Example: Paths of length 2

- (a) \( s = 0, t = 3 \)
- (b) \( s = 1, t = 2 \)
- (c) \( s = 2, t = 1 \)
- (d) \( s = 3, t = 0 \)
Combined result (25)

Standard path tracing (56)