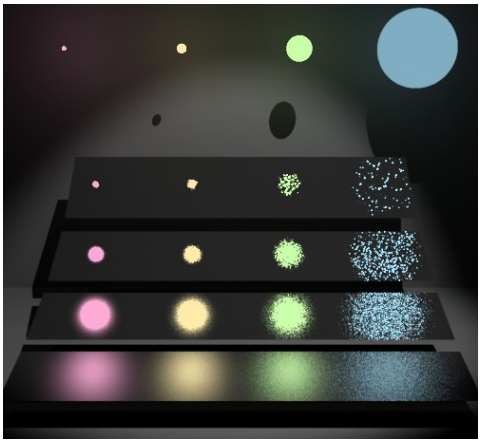
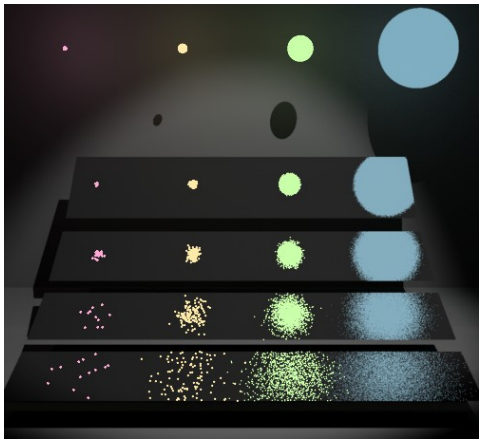
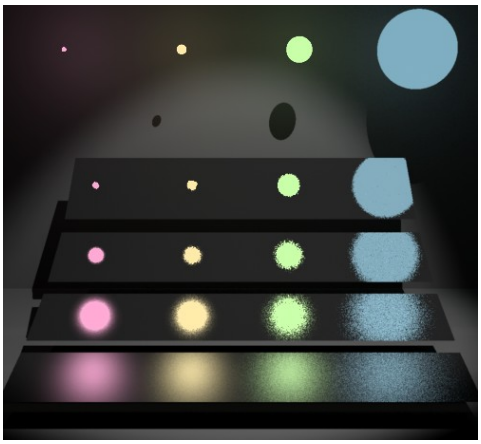
Sampling the light sources:



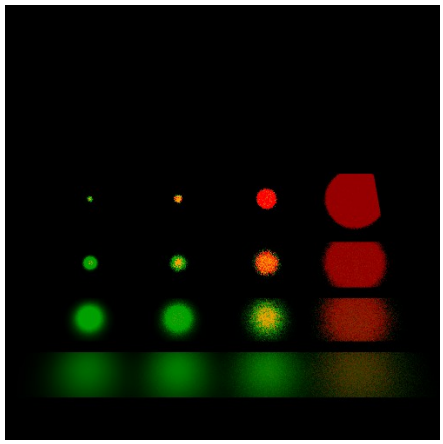
Sampling the BRDF:



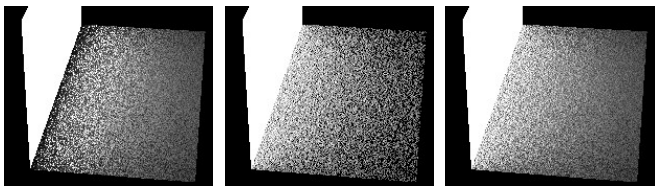
Combining the two strategies:



Combining the two strategies:



Another Example



Sampling the light source | Sampling the hemisphere | A weighted combination

Combined Estimator

- Given n sampling strategies $p_1,\ldots,p_n$ take n_i samples from each distribution:

$$F = \sum_{i=1}^{n} \frac{1}{n_i} \sum_{j=1}^{n_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

- The w_i are weighting functions $\sum_{i=1}^{n} w_i(x) = 1$

Unbiased?

- Yes:

$$\begin{aligned} E[F] &= \sum_{i=1}^{n} \frac{1}{n_i} n_i \int \frac{w_i(x) f(x)}{p_i(x)} p_i(x)\, dx \\ &= \sum_{i=1}^{n} \int w_i(x) f(x)\, dx \\ &= \int f(x)\, dx \end{aligned}$$

Balance Heuristic

- Define weighting functions as:

$$\hat{w}_i(x) = \frac{c_i p_i(x)}{\sum c_j p_j(x)}$$

- Where $n_i = c_i N$
- This heuristic can be shown to be nearly optimal.

Optimality Theorem

- Let $w_1,\ldots,w_n$ be any non-negative functions with $\sum_i w_i = 1$, and let F be the corresponding combined estimator. Then:

$$V[\hat{F}] \le V[F] + \left(\frac{1}{\min_i n_i} - \frac{1}{\sum_i n_i} \right) \mathcal{F}^2$$

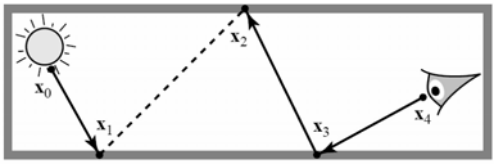
Additional Heuristics

- The cutoff heuristic:

$$w_i = \begin{cases} 0 & \text{if } p_i < \alpha p_{\max} \\ \dfrac{p_i}{\sum_j \left\{ p_j \middle| p_j \ge \alpha p_{\max} \right\}} \end{cases}$$

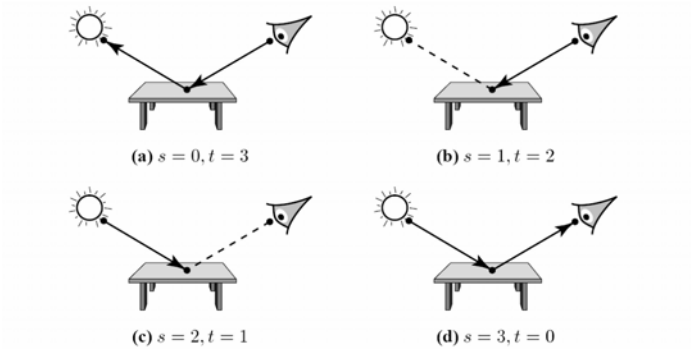
- The power heuristic: $w_i = \dfrac{p_i^{\beta}}{\sum_j p_j^{\beta}}$

Bi-Directional Path Tracing



- To sample a path of length k:
  - Trace a path with s vertices from the source;
  - Trace an t = (k+1-s) vertex path from the eye;
  - Deterministically connect the two paths.

Example: Paths of length 2



Combined result (25)

Standard path tracing (56)