Rotation matrices

1 Rotation in 2D

Rotation in the X-Y plane around (0,0) is a linear transformation. Rotation of α around (0,0) transforms the vector $[1\ 0]^T$ to $[\cos\alpha\ \ \sin\alpha]^T$. The vector $[0\ 1]^T$ is transformed to the vector $[-\sin\alpha\ \ \cos\alpha]^T$. Therefore the matrix describing this rotation is

$$R_\alpha(x) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} x \tag{1}$$

Rotation matrices are orthonormal ($RR^T = R^T R = I$, $\det(R) = 1$) or in other words they preserve lengths and angles.

Rotation around a point $c \neq (0,0)$ is not a linear transformation: it transforms $\vec{0} = (0,0)$ to a point other than $\vec{0} = (0,0)$. However, if we want to rotate around an arbitrary rotation center c, we can shift the plane by $-c$ such that the rotation center will be 0, then perform the rotation around $(0,0)$ and shift the plane back by $+c$:

$$R_{c,\alpha}(x) = R_\alpha(x - c) + c = R_\alpha(x) + (-R_\alpha(c) + c) \tag{2}$$

$R_{c,\alpha}$ is not a linear transformation, but it differs from the linear transformation R_α only in the addition of a constant. Transformations (like $R_{c,\alpha}$) of the form $T(x) = Ax + b$ are called Affine transformations.

2 Rotation in 3D

The 2D rotation in the X-Y plane we described in the previous section is a rotation in 3D around the Z axis. The rotation of points around the Z-axis does not depend on their Z value and points on the Z axis are not affected by this rotation (they are fixed points of the rotation). Therefore the rotation matrix around the Z axis takes a simple form: the submatrix corresponding to X-Y is identical to the 2D case, and the entries corresponding to the influence of Z on X and Y and vice versa are zeros:

$$R_{Z,\alpha} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Similarly, the rotation matrix around the Y axis is given by:

$$R_{Y,\alpha} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{4}$$

Rotation of a positive angle is defined to be counter clockwise around the positive direction of the axis, and this determines the arrangement of the entries in the matrix $R_{Y,\alpha}$.

Example: A rotation of 90 degrees around the Y axis transforms the X axis to the negative Z axis and is given by the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

The rotation matrix around the X axis is given by:

$$R_{X,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{5}$$

Any rotation in 3D (around an axis that passes through the origin) can be represented as a composition of rotations around the 3 canonical axes. This representation is not unique.

Notice that the order of multiplication of rotation matrices is important. For example, $R_{X,\pi/2}R_{Z,\pi/2} \neq R_{Z,\pi/2}R_{X,\pi/2}$ because when we apply $R_{Z,\pi/2}R_{X,\pi/2}$ to the X axis (the vector $[1\ 0\ 0]^T$), it is first unaffected by $R_{X,\pi/2}$ and then $R_{Z,\pi/2}$ transforms the X axis to the Y axis. On the other hand when we apply $R_{X,\pi/2}R_{Z,\pi/2}$ to the X axis. $R_{Z,\pi/2}$ first transforms the X axis to the Y axis and then it is further transformed to the positive Z axis by $R_{X,\pi/2}$.