Statistical Signal Processing
Probability and Stochastic Processes Review

1 Mean and Variance

	Continuous	Discrete
mean	$E(\mathbf{x}) = \int_{-\infty}^{\infty} x f(x) dx$	$E(\mathbf{x}) = \sum_i x_i \Pr\{\mathbf{x} = x_i\}$
conditional mean	$E(\mathbf{x}\|\mathcal{M}) = \int_{-\infty}^{\infty} x f(x\|\mathcal{M}) dx$	$E(\mathbf{x}\|\mathcal{M}) = \sum_i x_i \Pr\{\mathbf{x} = x_i \| \| \mathcal{M}\}$
mean of function	$E\{g(\mathbf{x})\} = \int_{-\infty}^{\infty} g(x) f(x) dx$	$E\{g(\mathbf{x})\} = \sum_i g(x_i) \Pr\{\mathbf{x} = x_i\}$
variance	$\text{var}(\mathbf{x}) = \int_{-\infty}^{\infty} [x - E(\mathbf{x})]^2 f(x) dx$	$\text{var}(\mathbf{x}) = \sum_i [x_i - E(\mathbf{x})]^2 \Pr\{\mathbf{x} = x_i\}$

Denote $\eta = E(\mathbf{x})$ and $\sigma^2 = \text{var}(\mathbf{x})$:

$$\sigma^2 = E(\mathbf{x}^2) - E^2(\mathbf{x}) \quad (1)$$

$$E(a\mathbf{x} + b) = aE(\mathbf{x}) + b \quad (2)$$

$$\text{var}(a\mathbf{x} + b) = a^2 \text{var}(\mathbf{x}) \quad (3)$$

2 Two Random Variables

Covariance The covariance C or C_{xy} of two random variables $\mathbf{x}$ and $\mathbf{y}$ is by definition the number

$$C = E\{[\mathbf{x} - E(\mathbf{x})][\mathbf{y} - E(\mathbf{y})]\} = E(\mathbf{xy}) - E(\mathbf{x})E(\mathbf{y})$$

Uncorrelated	$C = 0 \Rightarrow E(\mathbf{xy}) = E(\mathbf{x})E(\mathbf{y})$
Orthogonal	$E(\mathbf{xy}) = 0$
Independent	$f(x, y) = f_x(x)f_y(y)$

$$E(\mathbf{x} \pm \mathbf{y}) = E(\mathbf{x}) \pm E(\mathbf{y}) \tag{4}$$

$$\text{var}(\mathbf{x} \pm \mathbf{y}) = \text{var}(\mathbf{x}) + \text{var}(\mathbf{y}) \pm 2C_{xy} \tag{5}$$

3 Transformations

We wish to determine the density of $\mathbf{y} = g(\mathbf{x})$ in terms of the density of $\mathbf{x}$.

To find $f_y(y)$ for a specific y, we solve the equation $y = g(x)$. Denoting its *real* roots by x_n

$$y = g(x_1) = \cdots = g(x_n)$$

Then

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + \cdots + \frac{f_x(x_n)}{|g'(x_n)|}$$

where $g'(x)$ is the derivative of $g(x)$.

4 Stochastic Processes

A stochastic process is a non-countable infinity of random variables, one for each t. For a specific t, $\mathbf{x}(t)$ is a random variable with distribution

$$F(x, t) = \Pr\{\mathbf{x}(t) \leq x\}$$

and density function

$$f(x, t) = \frac{\partial F(x, t)}{\partial x}$$

Mean The mean $\eta(t)$ of $\mathbf{x}(t)$ is the expected value of the random variable $\mathbf{x}(t)$:

$$\eta(t) = E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f(x, t)\, dx$$

Autocorrelation The autocorrelation $R(t_1, t_2)$ of $\mathbf{x}(t)$ is the expected value of the product $\mathbf{x}(t_1)\mathbf{x}(t_2)$:

$$R(t_1, t_2) = E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2)\, dx_1\, dx_2.$$

The *autocovariance* $C(t_1, t_2)$ of $\mathbf{x}(t)$ is the covariance of the random variables $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$:

$$C(t_1, t_2) = R(t_1, t_2) - \eta(t_1)\eta(t_2)$$

Concerning discrete processes the autocorrelation and the autocovariance of $\mathbf{x}[n]$ are given by

$$R[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{x}[n_2]\} \qquad C[n_1, n_2] = R[n_1, n_2] - \eta[n_1]\eta[n_2]$$

Stationary Processes

A stochastic process $\mathbf{x}(t)$ is called *wide-sense stationary* if its mean is constant

$$E\{\mathbf{x}(t)\} = \eta$$

and its autocorrelation depends only on $\tau = t_1 - t_2$:

$$E\{\mathbf{x}(t + \tau)\mathbf{x}(t)\} = R(\tau).$$

For the discrete case:

$$\eta[n] = \eta$$

and

$$R(n + m, n) = E\{\mathbf{x}[n + m]\mathbf{x}[n]\} = R[m].$$

The Power Spectrum

The power spectrum (or spectral density) of a stationary process $\mathbf{x}(t)$ is the Fourier transform $S(\omega)$ of its autocorrelation $R(\tau)$:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\, d\tau.$$

Concerning discrete processes

$$S(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R[m]e^{-jm\omega}.$$

White Noise

We shall say that process $\boldsymbol{\nu}(t)$ is white noise if its values $\boldsymbol{\nu}(t_i)$ and $\boldsymbol{\nu}(t_j)$ are uncorrelated for every t_i and $t_j \neq t_i$:

$$C(t_i, t_j) = 0 \qquad t_i \neq t_j$$

The autocovariance of the white-noise process must be of the form

$$C(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \qquad q(t_1) \geq 0$$

The discrete case:

$$R[n_1, n_2] = q[n_1]\delta[n_1 - n_2]$$

The power spectrum of white noise is the constant function.