1 Mean and Variance

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Discrete</th>
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</thead>
<tbody>
<tr>
<td>mean</td>
<td>$E(x) = \int_{-\infty}^{\infty} xf(x) , dx$</td>
<td>$E(x) = \sum_i x_i \Pr{x = x_i}$</td>
</tr>
<tr>
<td>conditional mean</td>
<td>$E(x</td>
<td>M) = \int_{-\infty}^{\infty} xf(x</td>
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<tr>
<td>mean of function</td>
<td>$E{g(x)} = \int_{-\infty}^{\infty} g(x)f(x) , dx$</td>
<td>$E{g(x)} = \sum_i g(x_i) \Pr{x = x_i}$</td>
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<tr>
<td>variance</td>
<td>$\text{var}(x) = \int_{-\infty}^{\infty} [x - E(x)]^2 f(x) , dx$</td>
<td>$\text{var}(x) = \sum_i [x_i - E(x)]^2 \Pr{x = x_i}$</td>
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Denote $\eta = E(x)$ and $\sigma^2 = \text{var}(x)$:

$$\sigma^2 = E(x^2) - E^2(x)$$

$$E(ax + b) = aE(x) + b$$

$$\text{var}(ax + b) = a^2 \text{var}(x)$$

2 Two Random Variables

**Covariance** The covariance $C$ or $C_{xy}$ of two random variables $x$ and $y$ is by definition the number

$$C = E\{[x - E(x)][y - E(y)]\} = E(xy) - E(x)E(y)$$
<table>
<thead>
<tr>
<th>Uncorrelated</th>
<th>$C = 0 \Rightarrow E(xy) = E(x)E(y)$</th>
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</thead>
<tbody>
<tr>
<td>Orthogonal</td>
<td>$E(xy) = 0$</td>
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<tr>
<td>Independent</td>
<td>$f(x,y) = f_x(x)f_y(y)$</td>
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</table>

$$E(x \pm y) = E(x) \pm E(y) \quad (4)$$
$$\text{var}(x \pm y) = \text{var}(x) + \text{var}(y) \pm 2C_{xy} \quad (5)$$

3 Transformations

We wish to determine the density of $y = g(x)$ in terms of the density of $x$.

To find $f_y(y)$ for a specific $y$, we solve the equation $y = g(x)$. Denoting its real roots by $x_n$

$$y = g(x_1) = \cdots = g(x_n)$$

Then

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + \cdots + \frac{f_x(x_n)}{|g'(x_n)|}$$

where $g'(x)$ is the derivative of $g(x)$.

4 Stochastic Processes

A stochastic process is a non-countable infinity of random variables, one for each $t$. For a specific $t$, $x(t)$ is a random variable with distribution

$$F(x, t) = \Pr \{x(t) \leq x\}$$

and density function

$$f(x, t) = \frac{\partial F(x, t)}{\partial x}$$

$Mean$ The mean $\eta(t)$ of $x(t)$ is the expected value of the random variable $x(t)$:

$$\eta(t) = E[x(t)] = \int_{-\infty}^{\infty} x f(x, t) \, dx$$
**Autocorrelation** The autocorrelation $R(t_1, t_2)$ of $x(t)$ is the expected value of the product $x(t_1)x(t_2)$:

$$R(t_1, t_2) = E\{x(t_1)x(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2f(x_1, x_2; t_1, t_2)\,dx_1\,dx_2.$$  

The **autocovariance** $C(t_1, t_2)$ of $x(t)$ is the covariance of the random variables $x(t_1)$ and $x(t_2)$:

$$C(t_1, t_2) = R(t_1, t_2) - \eta(t_1)\eta(t_2)$$

Concerning discrete processes the autocorrelation and the autocovariance of $x[n]$ are given by

$$R[n_1, n_2] = E\{x[n_1]x[n_2]\} \quad C[n_1, n_2] = R[n_1, n_2] - \eta[n_1]\eta[n_2]$$

**Stationary Processes**

A stochastic process $x(t)$ is called **wide-sense stationary** if its mean is constant

$$E\{x(t)\} = \eta$$

and its autocorrelation depends only on $\tau = t_1 - t_2$:

$$E\{x(t+\tau)x(t)\} = R(\tau).$$

For the discrete case:

$$\eta[n] = \eta$$

and

$$R(n+m,n) = E\{x[n+m]x[n]\} = R[m].$$

**The Power Spectrum**

The power spectrum (or spectral density) of a stationary process $x(t)$ is the Fourier transform $S(\omega)$ of its autocorrelation $R(\tau)$:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau.$$  

Concerning discrete processes

$$S(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R[m]e^{-jm\omega}.$$  

3
White Noise

We shall say that process $\nu(t)$ is white noise if its values $\nu(t_i)$ and $\nu(t_j)$ are uncorrelated for every $t_i$ and $t_j \neq t_i$:

$$C(t_i, t_j) = 0 \quad t_i \neq t_j$$

The autocovariance of the white-noise process must be of the form

$$C(t_1, t_2) = q(t_1) \delta(t_1 - t_2) \quad q(t_1) \geq 0$$

The discrete case:

$$R[n_1, n_2] = q[n_1] \delta[n_1 - n_2]$$

The power spectrum of white noise is the constant function.