Multivariate Normal Distribution

1. Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in R^N$. The expectation of x is $E(x) = \begin{bmatrix} E(x_1) \\ \vdots \\ E(x_N) \end{bmatrix} \in R^N$.
   The covariance matrix of x is $Cov(x) = E((x-\mu)(x-\mu)^T) \in R^{N \times N}$ where $\mu = E(x) \in R^N$. In other words, the entries of the covariance matrix are $Cov(x)_{i,j} = Cov(x_i, x_j)$. Notice that the covariance matrix is symmetric and positive definite.

2. Let $x \in R$. $x \sim N(\mu, \sigma^2)$ (x is normally distributed with parameters μ and σ^2) if
   $$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$
   where $p(x)$ is the probability density function of x. μ is the expectation of x and σ^2 is the variance of x.

3. Let $x \in R^N$. $x \sim N(\mu, \Sigma)$ (x is normally distributed with parameters μ and Σ) if
   $$p(x) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{1/2}} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}} \tag{2}$$
   $\mu \in R^N$ is the mean and $\Sigma \in R^{N \times N}$ is symmetric and positive definite. Σ is the covariance matrix of x, i.e. $\Sigma_{i,j} = Cov(x_i, x_j)$.

4. Let $x_1, \ldots, x_N \in R$ be independent random variables, $x_i \sim N(\mu_i, \sigma_i^2)$.
   Let $x = (x_1, \ldots, x_N)^T \in R^N$.
   The joint distribution $p(x) = p(x_1, \ldots, x_N) = \prod_{i=1}^N p(x_i)$ can be written as $p(x) \sim N(\mu, \Sigma)$ where:
   $\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \ldots & 0 \\ 0 & \sigma_2^2 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & \sigma_N^2 \end{bmatrix}$

5. Let $x \in R^N$ as in 3, and let $y = Ax$ where $A \in R^{N \times N}$ is a regular matrix. Then $y \sim N(A\mu, A\Sigma A^T)$. It is easy to see this from

$$p(y) = \frac{p(x(y))}{|\frac{\partial y}{\partial x}|} = \frac{p(x = A^{-1}y)}{|A|}$$

(We saw this in class for a rotation matrix U (for which the determinant is 1 $|U| = 1$); The derivation is almost the same for a regular A).