Iterative Methods for Sparse Linear Systems

Sometimes we need to solve the linear equation $Ax = b$ for a very big and very sparse A. For example, the Poisson equation where only 5 entries of each row of the matrix A are non-zero. Standard methods such as inverting the matrix A (numerically unstable) or Guass elimination do not take advantage of the sparsity of A. In this lesson we will discuss two iterative methods suitable for sparse linear systems: Jacobi and Gauss-Seidel.

Jacobi

The i^{th} row of the equation $Ax = b$ is $\sum_j a_{ij}x_j = b_i$. This can be written as:

$$\sum_{j<i} a_{ij}x_j + a_{ii}x_i + \sum_{j>i} a_{ij}x_i = b_i \tag{1}$$

$$a_{ii}x_i = b_i - \sum_{j<i} a_{ij}x_j + \sum_{j>i} a_{ij}x_i \tag{2}$$

$$x_i = (b_i - \sum_{j<i} a_{ij}x_j + \sum_{j>i} a_{ij}x_i)/a_{ii} \tag{3}$$

This suggests the following iterations:

1. iterate until convergence

   (a)
   $$x_i^{(k+1)} = (b_i - \sum_{j<i} a_{ij}x_j^{(k)} - \sum_{j>i} a_{ij}x_i^{(k)})/a_{ii} \tag{4}$$

Gauss-Seidel

The Jacobi method does not use all the available information when updating $x_i^{(k+1)}$. It uses values from the k^{th} iteration for all x_j, even for $j < i$ where $x_j^{(k+1)}$ is already known.

If we revise the Jacobi iteration so that we always use the most current estimate of the exact x_i then we obtain the Gauss-Seidel iteration:

1. iterate until convergence

   (a)
$$x_i^{(k+1)} = (b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_i^{(k)})/a_{ii} \tag{5}$$

Convergence

The Jacobi and Gauss-Seidel methods can be written in matrix form as follows:

Let $A = L+D+U$ where $L = \begin{pmatrix} 0 & \dots & 0 & 0 \\ a_{21} & \dots & 0 & 0 \\ \vdots & & & \\ a_{n1} & \dots & a_{n,n-1} & 0 \end{pmatrix}$, $D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$ and $U = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$.

Using these notations the Jacobi method is:

$$M_J x^{(k+1)} = N_J x^{(k)} + b \tag{6}$$

where $M_J = D$ and $N_J = -(L+U)$. The Gauss-Seidel method is:

$$M_G x^{(k+1)} = N_G^{(k)} x + b \tag{7}$$

where $M_G = D + L$ and $N_G = -U$.

In general, we are discussing methods of the form

$$M x^{(k+1)} = N x^{(k)} + b \tag{8}$$

where $A = M - N$.

Theorem: Suppose $b \in R^n$ and $A = M - N \in R^{n \times n}$ is nonsingular. If M is nonsingular and the spectral radius of $M^{-1}N$ satisfies the inequality $\max \lambda(M^{-1}N) < 1$, then the iterates $x^{(k)}$ defined by $Mx^{(k+1)} = Bx^{(k)} + b$ converge to $x = A^{-1}b$ for any starting vector $x^{(0)}$.

Proof: Let $e^{(k)} = x^{(k)} - x$ denote the error in the k^{th} iterate. Since $Mx = Nx + b$ it follows that $M(x^{(k+1)} - x) = N(x^(k) = x)$, and thus, the error in $x^{(k+1)}$ is given by $e^{(k+1)} = M^{-1}Ne^{(k)} = (M^{-1}N)^{k+1}e^{(0)}$. This resembles the power method but *without* the normalization. Notice that we also want $e^{(k)}$ to converge to 0, not just convergence. In the case where $M^{-1}N$ is diagonalizable, a calculation similar to the one we used in the proof of convergence for the power method proves the claim (we will not discuss other cases).

Example: Gauss Seidel for the Poisson Equation

In the case of the Poisson equation,

$$\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = \frac{\partial \hat{r}_x}{\partial x} + \frac{\partial \hat{r}_y}{\partial y} \tag{9}$$

the k^{th} iteration of the Gauss Seidel method takes the form:

1. for j=1..M
   (a) for i=1..N
       i. G(i,j) = (F(i,j)+G(i-1,j)+G(i+1,j)+ G(i,j-1)+G(i,j+1))/4
   (b) end
2. end