Iterative Methods for Sparse Linear Systems

Sometimes we need to solve the linear equation $Ax = b$ for a very big and very sparse $A$. For example, the Poisson equation where only 5 entries of each row of the matrix $A$ are non-zero. Standard methods such as inverting the matrix $A$ (numerically unstable) or Gauss elimination do not take advantage of the sparsity of $A$. In this lesson we will discuss two iterative methods suitable for sparse linear systems: Jacobi and Gauss-Seidel.

**Jacobi**

The $i^{th}$ row of the equation $Ax = b$ is $\sum_j a_{ij}x_j = b_i$. This can be written as:

$$\sum_{j<i} a_{ij}x_j + a_{ii}x_i + \sum_{j>i} a_{ij}x_i = b_i \quad (1)$$

$$a_{ii}x_i = b_i - \sum_{j<i} a_{ij}x_j + \sum_{j>i} a_{ij}x_i \quad (2)$$

$$x_i = \frac{(b_i - \sum_{j<i} a_{ij}x_j + \sum_{j>i} a_{ij}x_i)}{a_{ii}} \quad (3)$$

This suggests the following iterations:

1. iterate until convergence

   (a) $$x_i^{(k+1)} = (b_i - \sum_{j<i} a_{ij}x_j^{(k)} - \sum_{j>i} a_{ij}x_i^{(k)})/a_{ii} \quad (4)$$

**Gauss-Seidel**

The Jacobi method does not use all the available information when updating $x_i^{(k+1)}$. It uses values from the $k^{th}$ iteration for all $x_j$, even for $j < i$ where $x_j^{(k+1)}$ is already known.

If we revise the Jacobi iteration so that we always use the most current estimate of the exact $x_i$ then we obtain the Gauss-Seidel iteration:
1. iterate until convergence

\[(a)\]

\[x_i^{(k+1)} = (b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_i^{(k)})/a_{ii} \quad (5)\]

**Convergence**

The Jacobi and Gauss-Seidel methods can be written in matrix form as follows:

Let \( A = L + D + U \) where

\[
L = \begin{pmatrix}
0 & \ldots & 0 & 0 \\
a_{21} & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
a_{n1} & \ldots & a_{n,n-1} & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
a_{11} & 0 & \ldots & 0 \\
0 & a_{22} & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & a_{nn}
\end{pmatrix},
\]

and \( U = \begin{pmatrix}
0 & a_{12} & \ldots & a_{1n} \\
0 & 0 & \ldots & a_{2n} \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & a_{nn}
\end{pmatrix} \).

Using these notations the Jacobi method is:

\[M_J x^{(k+1)} = N_J x^{(k)} + b \quad (6)\]

where \( M_J = D \) and \( N_J = -(L + U) \). The Gauss-Seidel method is:

\[M_G x^{(k+1)} = N_G x^{(k)} + b \quad (7)\]

where \( M_G = D + L \) and \( N_G = -U \).

In general, we are discussing methods of the form

\[M x^{(k+1)} = N x^{(k)} + b \quad (8)\]

where \( A = M - N \).

**Theorem:** Suppose \( b \in \mathbb{R}^n \) and \( A = M - N \in \mathbb{R}^{n \times n} \) is nonsingular. If \( M \) is nonsingular and the spectral radius of \( M^{-1}N \) satisfies the inequality \( \max \lambda(M^{-1}N) < 1 \), then the iterates \( x^{(k)} \) defined by \( M x^{(k+1)} = B x^{(k)} + b \) converge to \( x = A^{-1}b \) for any starting vector \( x^{(0)} \).

**Proof:** Let \( e^{(k)} = x^{(k)} - x \) denote the error in the \( k^{th} \) iterate. Since \( M x = N x + b \) it follows that \( M(x^{(k+1)} - x) = N(x^{(k)} - x) \), and thus, the error in \( x^{(k+1)} \) is given by \( e^{(k+1)} = M^{-1}N e^{(k)} = (M^{-1}N)^{k+1} e^{(0)} \). This resembles the power method but without the normalization. Notice that we also want \( e^{(k)} \) to converge to 0, not just convergence. In the case where \( M^{-1}N \) is diagonalizable, a calculation similar to the one we used in the proof of convergence for the power method proves the claim (we will not discuss other cases).
Example: Gauss Seidel for the Poisson Equation

In the case of the Poisson equation,

\[
\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = \frac{\partial \hat{r}_x}{\partial x} + \frac{\partial \hat{r}_y}{\partial y}
\]  

(9)

the \( k^{th} \) iteration of the Gauss Seidel method takes the form:

1. for \( j=1..M \)
   
   (a) for \( i=1..N \)
   
   i. \( G(i,j) = (F(i,j)+G(i-1,j)+G(i+1,j)+G(i,j-1)+G(i,j+1))/4 \)

   (b) end

2. end