CSIP Exam 2006 Moed B
Time of Exam: 2 hours

Choose 2 questions from the following:

1. In the problem of *shadow removal* we are given as input an image $I(i, j)$ and calculate its gradient field $I_x(i, j) = I(i + 1, j) - I(i - 1, j)$, $I_y(i, j) = I(i, j + 1) - I(i, j - 1)$. Elements in the gradient field are then passed through a scalar, nonlinear function, giving a new gradient field $\hat{I}_x(i, j) = f(I_x(i, j))$, $\hat{I}_y(i, j) = f(I_y(i, j))$.

   We then search for an image $X^*(i, j)$ that minimizes:

$$J(X) = \sum_{i,j} \Big(X(i + 1, j) - X(i - 1, j) - \hat{I}_x(i, j)\Big)^2 + \Big(X(i, j + 1) - X(i, j - 1) - \hat{I}_y(i, j)\Big)^2$$

   (a) Show that $X^* = \arg\min_X J(X)$ can be found by solving a sparse system of linear equations $Mx = b$ where M is a $N \times N$ matrix and N is the number of pixels. Give explicit equations for M and b. How many nonzero elements are there in each row of M ?

   (b) Assume we also want to bias the new image to be similar to a target image $I^t(i, j)$. That is we wish to minimize:

$$J_2(X) = J(X) + \lambda \sum_{i,j} \Big(X(i, j) - I^t(i, j)\Big)^2$$

   where λ is a known scalar.

   Show that $X^* = \arg\min_X J_2(X)$ can also be found by solving a sparse system of linear equations $Mx = b$ where M is a $N \times N$ matrix and N is the number of pixels. Give explicit equations for M and b. How many nonzero elements are there in each row of M ?

   (c) In the Gauss-Seidel numerical method for minimizing $J(X)$ we iteratively loop over all pixels in the image, and set each pixel value $X(i, j)$ to be the value that minimizes $J(X)$ *assuming all other pixels are fixed to their values in the previous iteration.* Give the explicit formula for updating the pixel value $X(i, j)$.

   Note: it is OK to use $\hat{I}_x, \hat{I}_y$ in the explicit formulas in this question.

2. In the *scaled orthographic* projection model, a 2D view $(x_i(t), y_i(t))^T$ of a 3D point $(X_i(0), Y_i(0), Z_i(0))^T$ is calculated according to the following equations:

$$\begin{pmatrix} X_i(t) \\ Y_i(t) \\ Z_i(t) \end{pmatrix} = R(t) \begin{pmatrix} X_i(0) \\ Y_i(0) \\ Z_i(0) \end{pmatrix}$$

and

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \frac{1}{\lambda(t)} \begin{pmatrix} X_i(t) \\ Y_i(t) \end{pmatrix}$$

where $\lambda(t)$ is a scalar and $R(t)$ is a 3×3 rotation matrix.

Suppose we are given F scaled orthographic views of N points. Define the matrix W by:

$$W = \begin{pmatrix} x_1(1) & x_2(1) & \cdots & x_N(1) \\ y_1(1) & y_2(1) & \cdots & y_N(1) \\ \cdots & & & \\ x_1(F) & x_2(F) & \cdots & x_N(F) \\ y_1(F) & y_2(F) & \cdots & y_N(F) \end{pmatrix}$$

(a) Show that the matrix W can be factorized into a product of matrices $W = MS$ where S is a $3 \times N$ and M is $2F \times 3$. What is the rank of W ?

(b) Show how to use the SVD to find rank 3 matrices $\hat{S}, \hat{M}$ such that $\hat{M}\hat{S} = W$. Is this decomposition into a product of rank 3 matrices unique ?

(c) Show that the matrix M obeys additional constraints beyond being rank 3. Show how to use these constraints to recover (up to an orthogonal transformation and a global scaling) the three dimensional structure S and the projection matrices M from $\hat{M}$,$\hat{S}$ using linear algebra.

3. Consider a classification dataset $\{x(t), y(t)\}_{t=1}^T$ in which $x(t) \in R^2$ and $y(t) \in \{-1, 1\}$. We are given the empirical means and covariances of the two classes:

$$\mu_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1)$$

$$\mu_- = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (2)$$

$$\Sigma_+ = \epsilon I \quad (3)$$

$$\Sigma_- = \epsilon I \quad (4)$$

$$(5)$$

Recall that any vector w defines a linear classifier $w^T x > 0$. Consider three classifiers $w_1 = (0, 1)$, $w_2 = (1, 1)$ and $w_3 = (1, 2)$.

(a) Recall that the Fisher criterion for w is given by $F(w) = \frac{(m_+(w)-m_-(w))^2}{V_+(w)+V_-(w)}$ where m_+, V_+ are the empirical means and variances of the projected classes:

$$m_+(w) = \frac{1}{N_+} \sum_{t:y(t)=+1} w^T x(t) \quad (6)$$

$$V_+(w) = \frac{1}{N_+} \sum_{t:y(t)=+1} (w^T x(t) - m_+)^2 \quad (7)$$

(and similarly for m_-,V_-). Calculate $F(w_i)$ for the three classifiers defined above. Which is the best ? Which is the worst ?

(b) What is the optimal w according to the Fisher criterion ? Find

$$w^* = \arg\max_w F(w)$$

(c) Recall that the training error of a classifier $w^T x > 0$ is given by:

$$Err(w) = \sum_t |y(t) - sign(w^T x(t))|$$

where $sign(x) = 1$ when $x > 0$,$sign(x) = -1$ for $x < 0$ and $sign(0) = 0$. Assume there are 10 training examples in each class, and $\epsilon = 1/100$. Show that the three classifiers defined above w_1, w_2, w_3 all satisfy $Err(w_i) = 0$.

Reminder: the empirical mean of a dataset $\{z(t)\}_{t=1}^T$ is $\mu = \frac{1}{T}\sum z(t)$ and the empirical covariance is $\frac{1}{T}\sum_t (z(t)-\mu)(z(t)-\mu)^T$.