CSIP Exam 2006 Moed A
Time of Exam: 2 hours

Choose 3 questions from the following:

1. In the problem of *image alignment* we are given as input two images $I_1(i,j), I_2(i,j)$ that are related by an unknown translation vector $t = (\Delta i, \Delta j)$. The assumption is that $I_1(i+\Delta i, j+\Delta j) \approx I_2(i,j)$. The problem is to find the vector t. This is done by finding:
$$t^* = \arg\min_t \sum_{i,j} \left(a^T(i,j)t - b(i,j)\right)^2$$
where $a^T(i,j) = (I_1(i+1,j) - I_1(i-1,j), I_1(i,j+1) - I_1(i,j-1))$, $b(i,j) = (I_1(i,j) - I_2(i,j))$.

    (a) Show that t^* can be found by solving a linear equation $Mt = c$. Where M is a 2×2 matrix. Give the explicit form of M and c. [11]

    (b) Suppose we are given an additional input mask $A(i,j)$ which weights the importance of the (i,j) pixel in determining the alignment. We wish to find:
    $$t^* = \arg\min_t \sum_{i,j} A(i,j) \left(a^T(i,j)t - b(i,j)\right)^2$$
    where $A(i,j)$ is a positive scalar defined on each pixel. Show that t^* can again be found by solving a linear equation. Give the explicit form of M and c in this case [11].

    (c) Sometimes we wish to add an additional cost that penalizes for large translations. This is done by minimizing:
    $$t^* = \arg\min_t \sum_{i,j} A(i,j) \left(a^T(i,j)t - b(i,j)\right)^2 + \lambda \|t\|^2$$
    where λ is a scalar. Show that t^* can again be found by solving a linear equation. Give the explicit form of M and c in this case. [11]

2. In the *matched filter* approach to signal detection we are given as input a signal vector $x \in \mathcal{R}^{\mathcal{N}}$ and a set of template vectors $\{t_i \in R^N\}_{i=1}^K$. We find the template that best fits the signal by:
$$i^* = \arg\max_i t_i^T x$$

(a) Assume $N = 10,000$ and $K = 1,000$, how many multiplications are needed to calculate $t_i^T x$ for all i ? [5]

(b) Show that if $t_i = \alpha(i)b_1 + \beta(i)b_2$ where $\alpha(i), \beta(i)$ are scalars then it is possible to calculate $t_i^T x$ for all i using $22,000$ multiplications. [10]

(c) Show how to find $b_1, b_2, \alpha(i), \beta(i)$ such that $\|b_1\| = \|b_2\| = 1$ and $b_1^T b_2 = 0$ that minimize:

$$J(b_1, b_2) = \sum_i \|t_i - (\alpha(i)b_1 + \beta(i)b_2)\|^2$$

[12]

(d) Is it possible to find b_1, b_2 without calculating the eigenvectors of a $10,000 \times 10,000$ matrix ? [6]

3. In the spectral approach to image segmentation, we define a weighted graph whose vertices correspond to pixels and weights between neighboring pixels $W(i,j)$.

Suppose we define:

$$W(i,j) = \begin{pmatrix} e^{-\frac{1}{2}(I(i)-I(j))^2} & , |I(i) - I(j)| < 3 \\ 0 & , |I(i) - I(j)| >= 3 \end{pmatrix}$$

where $I(i)$ is the intensity at pixel i.

(a) Show that for any vector x the following holds:

$$\sum_{i,j} W(i,j)\,(x(i) - x(j))^2 = 2x^T Lx$$

where $L = D - W$ and D is a diagonal matrix whose diagonal elements $D(i,j) = \sum_j W(i,j)$ [12]

(b) Suppose the image can be divided into two non-overlapping segments A, B such that $A \cup B$ is the full image. Suppose furthermore that $I(i) \in [100, 255]$ for all $i \in A$ and $I(i) \in [0, 96]$ for all $i \in B$. Let x_A be an indicator vector for the segment A and x_B be an indicator vector for the segment B. Show that:

$$\frac{x_A^T L x_A}{x_A^T x_A} + \frac{x_B^T L x_B}{x_B^T x_B} = \min_{x_1, x_2 : x_1^T x_2 = 0} \frac{x_1^T L x_1}{x_1^T x_1} + \frac{x_2^T L x_2}{x_2^T x_2}$$

[14]

(c) Is the minimum in the previous question unique ? [9]

4. In the *Hub and Authority* model of the world wide web, each web page p is given two scores: a hub score $x(p)$ and an authority score $y(p)$. These scores are related by the following equations:

$$x(p) = \alpha \sum_{q:link(q,p)} y(q) \quad (1)$$

$$y(p) = \beta \sum_{q:link(p,q)} x(q) \quad (2)$$

where $link(p,q)$ means that web page p contains a link to web page q and α, β are scalars.

(a) Define an $N \times N$ matrix $A(i,j)$ where N is the number of web pages on the web and $A(i,j) = 1$ if page i links to page j. Show that $A^T A$ and AA^T have the same eigenvalues but with different eigenvectors. [10]

(b) Show that if x is an eigenvector of $A^T A$ and y is an eigenvector of AA^T with the same eigenvalue, then x, y satisfy the hub and authority equations (equations 1,2). [10]

(c) Consider the following iterative algorithm:

$$
\begin{aligned}
x &\leftarrow Ay \qquad (3) \\
x &\leftarrow \frac{x}{\|x\|_1} \qquad (4) \\
y &\leftarrow A^T x \qquad (5) \\
y &\leftarrow \frac{y}{\|y\|_1} \qquad (6) \\
& \qquad (7)
\end{aligned}
$$

Let u_1 be the eigenvector of $A^T A$ with maximal eigenvalue, and v_1 be the eigenvector of AA^T with maximal eigenvalue. Give sufficient conditions for (x, y) in the iterative algorithm to converge to (u_1, v_1). Prove convergence under these conditions. [13]