CSC 2411H - Assignment 3

Due March 26, 2009

1. Let $E = \{x : (x-c)^t Q^{-1}(x-c) \leq 1\}$ be an ellipsoid enclosing a (nonempty) polytope P with smallest possible volume. Is it possible that c (the centre of E) does not belong to P? Either give an example of such a polytope and its smallest enclosing ellipsoid or prove that it is impossible.

2. Recall that one of the steps of the ellipsoid algorithm was to define a large enough cube $B = [-K, K]^n$ such that if $P = \{x : Ax < b\}$ is nonempty then also $P \cap B$ is nonempty and moreover there is a lower bound on its volume $2^{-\Omega(L)}$ where L is the length of the description of A. Remember that in class such a construction and proof were given for the case where we assume that P is bounded. Your goal is to prove the existence of such a box for P that is not necessarily bounded.

3. Consider the following optimization problem: Let n be even and let c be a positive vector in $\mathbb{R}^n$, find

$$\min\{\langle c, x\rangle \mid x \geq 0 \text{ and } \sum_{i \in S} x_i \geq 1 \quad \forall S \subset \{1, 2, \ldots, n\} \text{ of size } n/2\}.$$

   This is an LP with exponentially (in n) many constraints.

   (a) Show that the Ellipsoid algorithm can be used to get a poly(n)-time algorithm for the problem.

   (b) Show a simpler ($O(n \log n)$ time) algorithm for the problem. Hint: It may be useful to first find a simpler LP for the problem. You can then use facts from the theory of polynomial to get a simple algorithm that doesn't actually solve an LP (obviously such an algorithm is too costly).

4. Recall that in an interior point algorithm we require a potential function that takes into account the objective function as well as the distance from the boundary of the feasible set. In Ye's Primal-dual interior point algorithm we have x the primal variables, and y and s the dual variables, where s is the slack variables. We have also seen that if we have a primal-dual pair of solutions then $x \cdot s$ is an upper bound on the gap between the current value of the objective function and its optimum. Our goal was then to minimize $x \cdot s$ (and recall $x, s \geq 0$ always.)

   The potential function that we discussed was

$$G(x, s) = (n + \sqrt{n}) \ln(x \cdot s) - \sum_{i=1}^{n} \ln(x_i s_i)$$

and we showed the critical property that we can upper bound $x \cdot s$ in terms of $G(x, s)$. Explain what would go wrong if instead of G we will take

$$H(x, s) = n \ln(x \cdot s) - \sum_{i=1}^{n} \ln(x_i s_i).$$

5. (a) Let A be a totally unimodular matrix and let l_1, l_2, u_1 and u_2 be integral vectors. Show that the LP relaxation of the following Integer Program is *exact* (i.e. all its vertices are integral).

$$\begin{array}{ll} & \min\langle x, c\rangle \\ \text{s.t.} & \\ & l_1 \leq Ax \leq u_1 \\ & l_2 \leq x \leq u_2 \\ & x \in \mathbb{Z}^m \end{array} .$$

   (b) Let A be an $m \times n$ integer matrix of rank m. Show that A is unimodular (that is, the determinant of all $m \times m$ submatrices of A is $-1, 0$ or 1) if and only if for every integral vector b the vertices of the polyhedron

$$P = \{x \mid x \geq 0, Ax = b\}$$

   are all integral.