Fictitious Play

with a Continuum of Anonymous Players

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Agenda

- Background
  - Games in normal form
  - Fictitious Play (FP)
  - Auctions as typed games
- FP for typed games
- Continuum of Anonymous Players (CAPs)
- FP for CAPs
  - Linear structure of utilities
Games in normal form

Game is a tuple $< N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N}>$

- $N$ is a set of player tags
- $A_i$ is a set of actions available to player $i \in N$
- $u_i : \bigotimes_{i \in N} A_i \rightarrow \mathbb{R}$ is the utility function of player $i \in N$
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- Game play
  - All players simultaneously choose \( a_i \in A_i \)
  - The actions are combined into a joint action profile \( a = (a_i)_{i \in N} \)
  - Each player receives utility \( u_i(a) \)
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- Game’s purpose
  - Each player wishes to maximise its utility
Game’s Purpose: Maximise how?

MaxMin solution: be ready for the worst

\[ a^*_i = \arg \max_{a_i} \min_{a_{-i}} u_i(a_i, a_{-i}) \]
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  \[ a^*_i = \arg\max_{a_i} \min_{a_{-i}} u_i(a_i, a_{-i}) \]

- Nash equilibrium \( a^* \): no (single) player can do better
  \[ \forall i \in N \quad \forall a_i \in A_i \quad u_i(a^*) \geq u_i(a_i, a^*_i) \]
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- Mixed Nash equilibrium \( \pi^* \): choose a good (rigged) dice
  Actions are selected at random \( \pi_i \in \Delta(A_i) \)
  Joint mixed profile \( \pi = (\pi_i)_{i \in N} : \pi(a) = \prod_{i \in N} \pi_i(a_i) \)
  \[ \forall i \in N \quad \forall \pi_i \in \Delta(A_i) \quad E_{\pi^*}[u_i] \geq E_{(\pi_i, \pi^*_{-i})}[u_i] \]
Find Mixed Nash: Fictitious Play

- Idea – Statisticians’ Fun
  - Assume fixed (unknown) $\pi_{-i}$
  - Play the game repeatedly
  - Form (online) history estimate $\pi_{-i} = \sum_{k=1}^{t} \delta(a^t_i)$
  - Choose actions to optimise against $\pi_{-i}$
Find Mixed Nash: Fictitious Play

- Repeat until convergence
- Policy estimates at time $t$ are $\{\pi_i^t\}_{i \in N}$
- Choose $a_i^{t+1} = \arg \max_{a_i \in A_i} u_i(a_i, \pi_{-i})$
- Update $\pi_i^{t+1} = \frac{t-1}{t} \pi_i^t + \frac{1}{t} \delta(a_i^{t+1})$
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Twofold convergence

In action selection to Nash \( a^* \):

\( \exists T > 0 \) and \( a^* \) s.t. \( \forall t > T \) \( a^t = a^* \)

In beliefs to mixed Nash \( \pi^* \):

\( \exists \pi^* = \lim_{t \to \infty} \pi^t \)
FP Convergence: mixed Nash

- Two players
  - In zero sum games
  - In $2 \times K$ games
- $N$ players
  - Identical interest games
Sealed Second Price Auctions

Protocol:

- \( N \) players wish to acquire an item
- Each submits a bid \( b_i \in A_i \)
- The winner is \( i = \arg \max_{i \in N} b_i \)
- The looser pay nothing
- The winner pays \( c = \max_{j \neq i \in N} b_j \)
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Questions:
- Why would players $i, j \in N$ bid $b_i \neq b_j$?
- Is there a significance to the identity of a bid?
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**Properties:**
- Players have types $\alpha_i \in T$: $u_{gen}(\alpha_i, a) \rightarrow \mathbb{R}$
- A distribution over $T$ is given
- Is there a significance to the identity of a bid?
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Properties:

- Players have types $\alpha_i \in T$: $u_{gen}(\alpha_i, a) \rightarrow \mathbb{R}$
- Utility depends on the set of anonymous bids
Typed games

Game setting $< N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N}, T, \sigma >$

- $T$ is the space of private types
- $\sigma$ is a distribution over $T$
- $u_i : T \times \bigotimes_{i \in N} A_i \rightarrow \mathbb{R}$
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- **Game play:**
  - $\alpha_i \in T$ is sampled from $\sigma$ for each player,
  - Utilities are fixed $u_i(a) = u_i(\alpha_i, a)$
  - Normal form game starts
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- Solution concept:
  - Policy is conditioned on type \( \pi_i : T \rightarrow \Delta(A_i) \)
  - Equilibrium is now in expectation over \( T \) as well
    \[ E_{\pi^*, \sigma}[u_i] \geq E_{(\pi_i, \pi^*_i), \sigma}[u_i] \]
Auctions vs. Typed games

- Auctions are typed games
Auctions vs. Typed games

- Auctions are typed games, but also commonly:
  - Anonymous
  - $A_i$ is discrete and finite
  - Money is finite and discrete
  - $T$ is continuous
  - Evaluations may be in fractions of a pence
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- Type and player tag are a name for an utility function
Auctions and CAPs

- Continuum of Anonymous Players (CAPs) captures a vast range of auction problems
  - First and second price auctions
  - Multiple simultaneous or sequential
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- Algorithm to compute an equilibrium in CAPs is a generic solution
  - Previous algorithms have narrow specialisation
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- Fictitious Play algorithm for CAPs
CAPs

Anonymous games
- Given a set of actions \( \{a_i\}_{i \in I} \) taken by the opponents
- A player's utility depends on the composition of the set, but not on the identity of opponents

Continuum of players
- The set of players in the game is infinite, the cardinality of the set is \( \aleph \)
- A finite set of players is randomly selected from a cardinality \( \aleph \) set.
Example: Private Value Auctions

- An item is auctioned to $n$ players
- Each player has a private value for the item
  - The value is not known to opponents
  - Value obtained by sampling from a common distribution
    - E.g. uniform over a $[0, 1]$ range.
Example: Private Value Auctions

- An item is auctioned to \( n \) players
- Utility from winning
  - Depends on bids composition, not bidders identity
- Probability of winning
  - Engendered by probability distributions of opponent bids
  - Not specific choices of bidders
CAPs (cont)

- $A$ is a compact metric space
- $\mathcal{M}$ is the space of probability distributions over $A$
- $\mathcal{U}$ is the space of continuous functions $u : A \times \mathcal{M} \rightarrow \mathbb{R}$
  - This is the set of possible players
  - For auctions this is utility given private value
    - Winning by placing a bid $b \in A$
    - Against opponent bids distributed by $\omega b \in \mathcal{M}$

- Anonymous game with continuum of players is then characterised by a distribution $\mu$ over the space $\mathcal{U}$.
  - For auctions this corresponds to the distribution of private values
Cournot-Nash Equilibrium (CNE)

- A distribution \( \tau \) over \( \mathcal{U} \times A \) so that
  - \( \mathcal{U} = \mu \)
  - \( \tau \{ (u, a) \mid u(a, \tau_A) \geq u(A, \tau_A) \} = 1 \)

- If there is indeed a continuum of players
  - Single player action choice will not modify \( \tau_A \)
  - Any modification away from the CNE can only reduce utility
Cournot-Nash Equilibrium (CNE)

- A distribution $\tau$ over $\mathcal{U} \times A$ so that
  - $\tau \mathcal{U} = \mu$
  - $\tau\{(u, a) | u(a, \tau_A) \geq u(A, \tau_A)\} = 1$

- For auctions the continuum of players is virtual
  - Player faces unknown values of opponents
  - Player’s response is computed with respect to continuum of private values
  - Distribution over the range of values
Existence of CNE

- **Theorem** (Mas-Colell, 84):
  If $\mu$ is atomless and $A$ is discrete and finite, then there exists a pure CNE.

- Pure CNE is characterised by a function $h : \mathcal{U} \rightarrow A$
  - Inherently symmetric: players with the same type choose the same action
Fictitious Play

- Iterative process of adaptation
  - Given a history of opponents play
  - Compute a frequency of action for each player
  - Compute and apply best response to that frequency

- Properties:
  - If best response converges, then to a pure Nash
  - If frequency estimate converges, then to a mixed Nash equilibrium
Fictitious Play

- Iterative process of adaptation
  - Given a history of opponents play
  - Compute a frequency of action for each player
  - Compute and apply best response to that frequency

- Properties:
  - For anonymous continuum games the procedure becomes symmetric
  - If frequency converges, a pure CNE can be obtained from it
FP in CAPs

- Frequency of actions (for all types)
  - A distribution $\tau$ over $\mathcal{U} \times A$

- FP computes best response
  - A function $h : \mathcal{U} \rightarrow A$
  - Inherent $\tau_h(u, a) = \mu(h^{-1}(a) \cap u)$

- Updating frequency estimate
  - $\tau = \alpha \ast \tau + (1 - \alpha) \ast \tau_h$
FP in CAPs

- Best response computation depends only on $\tau_A$, the marginal distribution
- The algorithm can be simplified
  - Frequency of actions $\tau_A$
  - Compute best response $h : \mathcal{U} \rightarrow A$
  - Compute $\tau_{h,A}(a) = \mu(h^{-1}(a))$
  - Update $\tau_A = \alpha \star \tau_A + (1 - \alpha) \star \tau_{h,A}$
- $\tau_A$ can be transformed into a complete equilibrium
FP in CAPs

- Best response computation depends only on $\tau_A$, the marginal distribution.
- The algorithm can be simplified:
  - Frequency of actions $\tau_A$
  - Compute best response $h : \mathcal{U} \rightarrow A$
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  - Update $\tau_A = \alpha \ast \tau_A + (1 - \alpha) \ast \tau_{h,A}$
- $\tau_A$ can be transformed into a complete equilibrium.
If $U$ is linear, single dimensional (e.g. in auctions)

One can associate a function

$$u_{gen} : T \times A \times M \rightarrow R \text{ linear in type } T$$

Given a fixed $\tau_A$, utility of an action $a \in A$ becomes linear in type

Best response is:

$$h(\lambda \in T) = \arg \max_{a \in A} u_{gen}(\lambda, a, \tau_A)$$
Best Response Computation

Best response is: \( h(\lambda \in T') = \arg \max_{a \in A} u_{gen}(\lambda, a, \tau_A) \)
Best Response Computation

Best response is: \( h(\lambda \in T') = \arg \max_{a \in A} u_{gen}(\lambda, a, \tau_A) \)

- Set of distinct intervals \( I \)
- For any \( \alpha \in I, \text{if } \lambda_1, \lambda_2 \in \alpha \subset T \text{ then } h(\lambda_1) = h(\lambda_2) \)
- For any \( \alpha_1 \neq \alpha_2 \in I, \text{if } \lambda_i \in \alpha_i \text{ then } h(\lambda_1) \neq h(\lambda_2) \)
Best Response Computation

- **Best response is:**
  \[ h(\lambda \in T) = \arg \max_{a \in A} u_{gen}(\lambda, a, \tau_A) \]

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- For any \( \alpha \in I, \) if \( \lambda_1, \lambda_2 \in \alpha \subset T \) then \( h(\lambda_1) = h(\lambda_2) \)
- For any \( \alpha_1 \neq \alpha_2 \in I, \) if \( \lambda_i \in \alpha_i \) then \( h(\lambda_1) \neq h(\lambda_2) \)

- **Best response is piece-wise linear**
- Efficient computation of \( \tau_{h,A} \)
FP in CAPs: Results

- Empirically converges in all auction experiments
  - Equilibrium for simultaneous first price auctions with continuous private type
- Empirical convergence rate appears to be exponential
  - Surprise: in finite games FP converges slowly
Present and Future Work

- Efficient best response computation for manifold $U$
- Linearity of $u_{gen}$ is type is not essential
- Counterexample exists for general convergence of FP
  - In auction based experiments it converges – why?
  - Necessary and sufficient conditions for FP to converge in CAPs
Questions