Lecture 14
SVD Applications

EE263 Autumn 2004

- general pseudo-inverse
- full SVD
- image of unit ball under linear transformation
- SVD in estimation/inversion
- sensitivity of linear equations to data error
- low rank approximation via SVD

General pseudo-inverse

if A has SVD $A = U\Sigma V^T$,

$$A^\dagger = V\Sigma^{-1}U^T$$

is the *pseudo-inverse* or *Moore-Penrose inverse* of A

if A is skinny and full rank,

$$A^\dagger = (A^TA)^{-1}A^T$$

gives the least-squares solution $x_{\mathrm{ls}} = A^\dagger y$

if A is fat and full rank,

$$A^\dagger = A^T(AA^T)^{-1}$$

gives the least-norm solution $x_{\mathrm{ln}} = A^\dagger y$

in general case:

$$X_{\mathrm{ls}} = \{\ z \mid \|Az - y\| = \min_w \ \|Aw - y\|\ \}$$

is set of least-squares solutions

$x_{\mathrm{pinv}} = A^\dagger y \in X_{\mathrm{ls}}$ has minimum norm on X_{ls}, $i.e.$, x_{pinv} is the minimum-norm, least-squares solution

Pseudo-inverse via regularization

for $\mu > 0$, let x_μ be (unique) minimizer of

$$\|Ax - y\|^2 + \mu\|x\|^2$$

$i.e.$,

$$x_\mu = \left(A^T A + \mu I\right)^{-1} A^T y$$

here, $A^T A + \mu I > 0$ and so is invertible

then we have $\lim_{\mu \to 0} x_\mu = A^\dagger y$

in fact, we have $\lim_{\mu \to 0} \left(A^T A + \mu I\right)^{-1} A^T = A^\dagger$

(check this!)

Full SVD

SVD of $A \in \mathbf{R}^{m\times n}$ with $\mathbf{Rank}(A) = r$:

$$A = U_1\Sigma_1 V_1^T = \left[\, u_1 \;\cdots\; u_r \,\right] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$$

- find $U_2 \in \mathbf{R}^{m\times(m-r)}$, $V_2 \in \mathbf{R}^{n\times(n-r)}$ s.t. $U = [U_1\ U_2] \in \mathbf{R}^{m\times m}$ and $V = [V_1\ V_2] \in \mathbf{R}^{n\times n}$ are orthogonal
- add zero rows/cols to Σ_1 to form $\Sigma \in \mathbf{R}^{m\times n}$:

$$\Sigma = \left[\begin{array}{c|c} \Sigma_1 & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right]$$

then we have

$$A = U_1\Sigma_1 V_1^T = \left[\, U_1 \,\middle|\, U_2 \,\right] \left[\begin{array}{c|c} \Sigma_1 & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right] \left[\begin{array}{c} V_1^T \\ \hline V_2^T \end{array}\right]$$

$i.e.$:

$$A = U\Sigma V^T$$

called *full SVD* of A

Image of unit ball under linear transformation

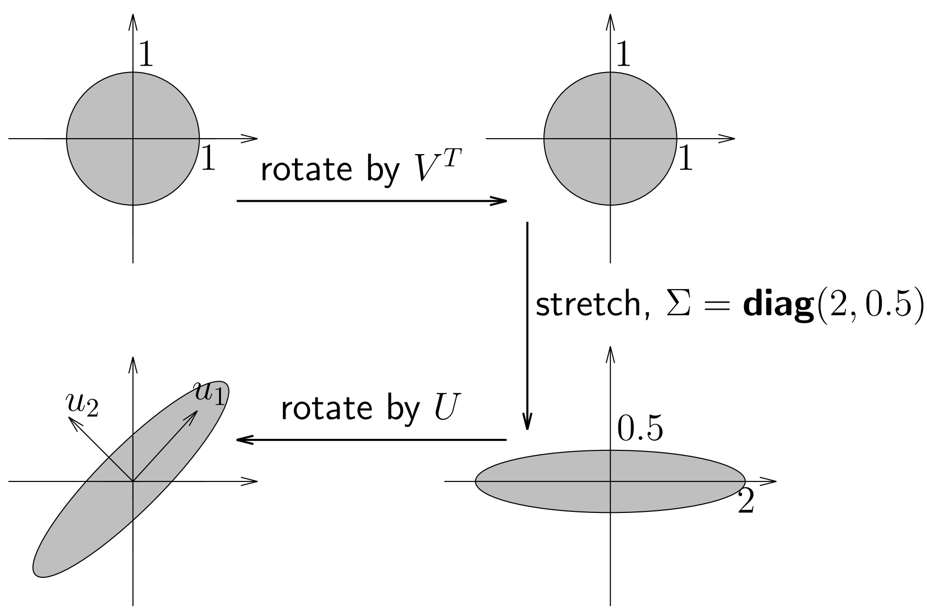
full SVD:

$$A = U\Sigma V^T$$

gives intepretation of $y = Ax$:

- rotate (by V^T)
- stretch along axes by σ_i ($\sigma_i = 0$ for $i > r$)
- zero-pad (if $m > n$) or truncate (if $m < n$) to get m-vector
- rotate (by U)

application: image of unit ball under A



$\{Ax \mid \|x\| \le 1\}$ is *ellipsoid* with principal axes $\sigma_i u_i$.

SVD in estimation/inversion

suppose $y = Ax + v$, where

- $y \in \mathbf{R}^m$ is measurement
- $x \in \mathbf{R}^n$ is vector to be estimated
- v is a measurement noise or error

'norm-bound' model of noise: we assume $\|v\| \leq \alpha$
but otherwise know nothing about v
(α gives max norm of noise)

consider estimator $\hat{x} = By$, with $BA = I$ (*i.e.*, unbiased)

estimation or inversion error is $\tilde{x} = \hat{x} - x = Bv$

set of possible estimation errors is ellipsoid

$$\tilde{x} \in \mathcal{E}_{\rm unc} = \{\ Bv \mid \|v\| \leq \alpha\ \}$$

so $x = \hat{x} - \tilde{x} \in \hat{x} + \mathcal{E}_{\rm unc}$

$\mathcal{E}_{\rm unc}$ is 'uncertainty ellipsoid' for x
'good' estimator has 'small' $\mathcal{E}_{\rm unc}$
(with $BA = I$, of course)

semiaxes of $\mathcal{E}_{\rm unc}$ are $\alpha\sigma_i u_i$
(singular values & vectors of B)

$e.g.$, maximum norm of error is $\alpha\|B\|$, $i.e.$,

$$\|\hat{x} - x\| \leq \alpha\|B\|$$

optimality of least-squares: suppose $BA = I$ is any estimator, and $B_{\rm ls} = A^\dagger$ is the least-squares estimator

then:

- $B_{\rm ls}B_{\rm ls}^T \leq BB^T$
- $\sigma_i(B_{\rm ls}) \leq \sigma_i(B)$, $i = 1, \ldots, n$
- in particular $\|B_{\rm ls}\| \leq \|B\|$
- $\mathcal{E}_{\rm ls} \subseteq \mathcal{E}$

$i.e.$, the least-squares estimator gives the *smallest* uncertainty ellipsoid

Example: navigation using range measurements (lect. 4)

we have $y_i = -k_i^T x + v_i$ for $i = 1, \ldots, 4$; assume

$$A_1 = -[k_1\ k_2]^T, \quad A_2 = -[k_1\ k_2\ k_3\ k_4]^T$$

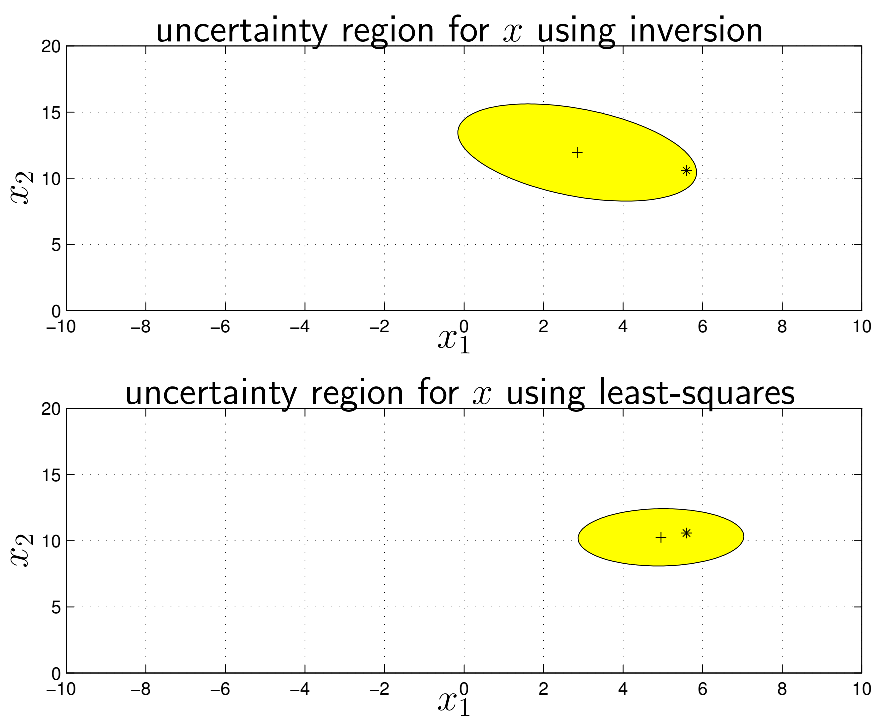
using first two measurements and inverting:

$$\hat{x} = A_1^{-1} \left[\, y_1 \ \ y_2 \,\right]^T$$

using all four measurements and least-squares:

$$\hat{x} = A_2^{\dagger} \left[\, y_1 \ \ y_2 \ \ y_3 \ \ y_4 \,\right]^T$$

uncertainty regions (with $\alpha = 1$):



proof of optimality property:

suppose $A \in \mathbf{R}^{m\times n}$, $m > n$, is full rank

SVD: $A = U\Sigma V^T$, with V orthogonal

$B_{\rm ls} = A^\dagger = V\Sigma^{-1}U^T$, and B satisfies $BA = I$

define $Z = B - B_{\rm ls}$, so $B = B_{\rm ls} + Z$

then $ZA = ZU\Sigma V^T = 0$, so $ZU = 0$
(multiply by $V\Sigma^{-1}$ on right)

therefore

$$\begin{aligned} BB^T &= (B_{\rm ls}+Z)(B_{\rm ls}+Z)^T \\ &= B_{\rm ls}B_{\rm ls}^T + B_{\rm ls}Z^T + ZB_{\rm ls}^T + ZZ^T \\ &= B_{\rm ls}B_{\rm ls}^T + ZZ^T \\ &\geq B_{\rm ls}B_{\rm ls}^T \end{aligned}$$

using $ZB_{\rm ls}^T = (ZU)\Sigma^{-1}V^T = 0$

Sensitivity of linear equations to data error

consider $y = Ax$, $A \in \mathbf{R}^{n \times n}$ invertible

of course $x = A^{-1}y$

suppose we have an error or noise in y, $i.e.$, y becomes $y + \delta y$

then x becomes $x + \delta x$ with $\delta x = A^{-1}\delta y$

hence we have

$$\|\delta x\| = \|A^{-1}\delta y\| \leq \|A^{-1}\|\|\delta y\|$$

if $\|A^{-1}\|$ is large,

- small errors in y can lead to large errors in x
- can't solve for x given y (with small errors)
- hence, A can be considered singular in practice

a more refined analysis uses *relative* instead of *absolute* errors in x and y

since $y = Ax$, we also have $\|y\| \leq \|A\|\|x\|$, hence

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta y\|}{\|y\|}$$

$$\kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}(A)/\sigma_{\min}(A)$$

is called the *condition number* of A

we have:

relative error in solution x
$\leq$ condition number $\cdot$ relative error in data y

or, in terms of # bits of guaranteed accuracy:

bits in solution $\approx$ # bits in data $- \log_2 \kappa$

we say

- A is well conditioned if κ is small
- A is poorly conditioned if κ is large

(definition of 'small' and 'large' depend on application)

same analysis holds for least-squares solutions with A nonsquare, $\kappa = \sigma_{\max}(A)/\sigma_{\min}(A)$

Low rank approximations

suppose $A \in \mathbf{R}^{m \times n}$, $\mathbf{Rank}(A) = r$, with SVD

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

we seek matrix $\hat{A}$, $\mathbf{Rank}(\hat{A}) \leq p < r$, s.t. $\hat{A} \approx A$ in the sense that

$$\|A - \hat{A}\|$$

is minimized

solution: the optimal rank p approximator is

$$\hat{A} = \sum_{i=1}^{p} \sigma_i u_i v_i^T$$

- hence $\|A - \hat{A}\| = \left\| \sum_{i=p+1}^{r} \sigma_i u_i v_i^T \right\| = \sigma_{p+1}$
- interpretation: SVD dyads $u_i v_i^T$ are ranked in order of 'importance'; take p to get rank p approximant

proof: suppose $\mathbf{Rank}(B) \le p$

then $\dim \mathcal{N}(B) \ge n - p$

also, $\dim \operatorname{span}\{v_1, \ldots, v_{p+1}\} = p + 1$

hence, the two subspaces intersect, *i.e.*, there is a unit vector $z \in \mathbf{R}^n$ s.t.

$$Bz = 0, \quad z \in \operatorname{span}\{v_1, \ldots, v_{p+1}\}$$

$$(A - B)z = Az = \sum_{i=1}^{p+1} \sigma_i u_i v_i^T z$$

$$\|(A - B)z\|^2 = \sum_{i=1}^{p+1} \sigma_i^2 (v_i^T z)^2 \ge \sigma_{p+1}^2 \|z\|^2$$

hence $\|A - B\| \ge \sigma_{p+1} = \|A - \hat{A}\|$

Distance to singularity

another interpretation of σ_i:

$$\sigma_i = \min\{\ \|A - B\| \ |\ \mathbf{Rank}(B) \leq i - 1\ \}$$

$i.e.$, the distance (measured by matrix norm) to the nearest rank $i - 1$ matrix

for example, if $A \in \mathbf{R}^{n \times n}$, $\sigma_{\min}$ is distance to nearest singular matrix

hence, small $\sigma_{\min}$ means A is near to a singular matrix

application: model simplification

suppose $y = Ax + v$, where

- $A \in \mathbf{R}^{100\times 30}$ has SVs

$$10,\ 7,\ 2,\ 0.5,\ 0.01,\ \ \ldots, 0.0001$$

- $\|x\|$ is on the order of 1
- unknown error or noise v has norm on the order of 0.1

then the terms $\sigma_i u_i v_i^T x$, for $i = 5, \ldots, 100$, are substantially smaller than the noise term v

simplified model:

$$y = \sum_{i=1}^{4} \sigma_i u_i v_i^T x + v$$